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# New generalized numerical radius inequalities for Hilbert space operators

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**Abstract**

We define generalized real and imaginary parts of an operator, as well as the generalized $w_{h,g}(\cdot)$ numerical radius, which reduces to the $t$-weighted numerical radius for suitable functions $h, g$. Usual properties regarding the new numerical radius are shown, as well as various inequalities concerning the ratio between $w_{h,g}(\cdot)$ and $w(\cdot)$. In the last section, we give operator matrix inequalities, which generalize the standard numerical radius inequalities, and in one case it is shown that the inequality obtained is sharper than the inequality given by Ammar et al. (Kyungpook Math. J. 65(1):63–75, 2025, Theorem 2.13) for specific operator matrices.

**Keywords:** Numerical radius; Operator norm; Inequality

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## 1 Introduction

Let $\mathcal{H}$ denote a complex Hilbert space with an inner product $\langle . \rangle$. An operator $A$ is said to be positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. The numerical range and numerical radius are very useful concepts in Hilbert space operator inequalities. Specifically, the numerical range is defined as $W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$ and the numerical radius is given by $w(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}$. It is well known that $w(\cdot)$ forms a norm on the $C^*$-algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators, and this norm is equivalent to the standard operator norm $\|\cdot\|$. This relationship is precisely characterized by the two-sided sharp inequality

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|,$$

where equality holds under specific conditions (e.g., $A^2 = 0$ for the first inequality and $A$ being normal for the second).

Seminal work in the field includes Kato's celebrated generalization of the Schwarz inequality in 1952 [22], which says that for any $T \in \mathcal{B}(\mathcal{H})$ and $\alpha \in [0, 1]$, we have

$$|\langle Tx, y\rangle|^2 \leq \langle (T^*T)^{\alpha} x, x\rangle \langle (TT^*)^{1-\alpha} y, y\rangle \tag{1.1}$$

for all $x, y \in \mathcal{H}$. Building upon this, Furuta in 1994 [19] further generalized the result to the form

$$|\langle T|T|^{\alpha+\beta-1}x, y\rangle|^2 \le \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2\beta}x, x\rangle$$

for $x, y \in \mathcal{H}$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \ge 1$.

Another fundamental aspect is the power inequality,

$$w(A^n) \le w^n(A),$$

valid for any $A \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$.

Over the years, numerous researchers have focused on refining and extending these inequalities. Kittaneh [24] notably improved the upper bound of $w(A) \le \|A\|$ by demonstrating that

$$w(A) \le \frac{1}{2}(\|A\| + \|A^2\|^{1/2}).$$

Subsequent investigations by Kittaneh and Moradi [27], as well as El-Haddad and Kittaneh [18], led to further significant inequalities, such as

$$w^2(A) \le \frac{1}{6}\||A|^2 + |A^*|^2\| + \frac{1}{3}w(A)\||A| + |A^*|\|,$$

and generalizations involving powers of operators like

$$w^{2l}(A) \le \frac{1}{2}\||A|^{2l} + |A^*|^{2l}\|$$

for $l \ge 1$. More recently, Nayak [29] and Alomari [4] have continued to advance these refinements, contributing to a rich body of literature that seeks to establish sharper bounds and broader contexts for numerical radius inequalities.

Motivated by these ongoing developments, the current work aims to provide a novel and versatile framework for numerical radius inequalities. We introduce generalized real and imaginary parts of an operator, $\mathrm{Re}_{h,g}(A)$ and $\mathrm{Im}_{h,g}(A)$, defined by functions $h$ and $g$ from a specified class $Q$. This foundation allows us to construct a new generalized numerical radius, $w_{h,g}(A)$, and rigorously proves its properties as a norm on $\mathcal{B}(\mathcal{H})$ under appropriate conditions. Our framework not only encompasses existing definitions and generalizations but also yields new identities and refined bounds for $w_{h,g}(A)$, including a new identity for $(\mathrm{Re}_{h,g}(A))^2 - \mathrm{Re}_{h,g}(A^2)$, which for particular functions $h, g$ reduce to the already established identity given by Conde et al. [15]. The inherent adaptability of $w_{h,g}(A)$ through the functions $h$ and $g$ suggests promising applications in the analysis of dynamical systems, where operator properties may evolve over time.

For further relevant research, we refer to the recent publications concerning inequalities pertaining to numerical radii [1, 3, 5, 7–9, 11, 13, 19–21, 24, 26–28, 34–36] and the books with extensive references [12, 17]. A generalization of the Cartesian decomposition was introduced [31]. The weighted real and imaginary parts of $A$ are defined by

$$\mathrm{Re}_t(A) = tA + (1-t)A^* \text{ and } \mathrm{Im}_t(A) = \frac{(1-t)A - tA^*}{i}, t \in [0, 1].$$

It is obvious that for $t = \frac{1}{2}$, they reduce to the well-known $\operatorname{Re}(A)$ and $\operatorname{Im}(A)$, respectively. For $A \in \mathcal{B}(\mathcal{H})$ and $t \in [0,1]$, Conde et al. [15] defined the weighted numerical radius as

$$w_t(A) = \sup_{\|x\|=1} |\langle (A + (1-2t)A^*)x, x\rangle| = w(A + (1-2t)A^*).$$

In the same paper, the authors investigated various properties of the weighted numerical radius and obtained various inequalities such as

$$\int_0^{\frac{1}{2}} w_t(A)dt \leq \frac{3}{4}w(A) \ \text{ and } \ \frac{1}{4}w(A) \leq \int_{\frac{1}{2}}^1 w_t(A)dt.$$

This paper generalizes the concept of the weighted numerical radius by incorporating arbitrary functions. This approach significantly extends the findings of Conde et al. [15], whose work represents a specific instance of our more general framework. Our method of generalizing conditions through various functions is inspired by the work of Stojiljković and Dragomir [33], a trend subsequently adopted by other researchers. For example, Nayak [30] derived various numerical radius inequalities, and Zamani [38] applied generalized weighted functions to establish numerical radius inequalities within the $C^*$-algebra setting.

## 2 Preliminary results

We require the following result given by Kittaneh in [23].

**Lemma 2.1** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then*

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|. \tag{2.1}$$

The following lemma was given by Abu-Omar and Kittaneh in [2].

**Lemma 2.2** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then*

$$\sqrt{r(AB)} = \left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|, \tag{2.2}$$

*where $r(\cdot)$ is the spectral radius.*

**Lemma 2.3** *Let $x, y, e \in H$ with $\|e\| = 1$. Then we have*

$$|\langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{2}(\|x\| \left\|y\right\| + |\langle x, y\rangle|). \tag{2.3}$$

The last result is the Buzano extension of the Cauchy-Schwarz inequality, see [14].

**Lemma 2.4** (*Hölder-McCarthy inequality* [32, *p.* 20]). *Let $A \in \mathcal{B}(\mathcal{H})$, $A \geq 0$, and let $x \in \mathcal{H}$ be any unit vector. Then we have*

$$\langle Ax, x\rangle^l \leq \langle A^l x, x\rangle \ \textit{for} \ \ l \geq 1, \tag{2.4}$$

$$\langle A^l x, x\rangle \leq \langle Ax, x\rangle^l \ \textit{for} \ \ 0 < l \leq 1. \tag{2.5}$$

The direct sum of two copies of $\mathcal{H}$ is denoted by $\mathcal{H} \oplus \mathcal{H}$. If $A, B, C, D \in \mathcal{B}(\mathcal{H})$, then the operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be considered as an operator on $\mathcal{H} \oplus \mathcal{H}$, and is defined by $\begin{bmatrix} A & B \\ C & D \end{bmatrix} x = \begin{bmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{bmatrix}$ for all $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$.

The following lemma is well known. See, e.g., [21].

**Lemma 2.5** *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$w\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \max\{w(A), w(B)\}. \tag{2.6}$$

*In particular,*

$$w\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) = w(B).$$

*We also require the following result on the norm of the diagonal operator*:

$$\left\| \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right\| = \max\{\|A\|, \|D\|\}. \tag{2.7}$$

## 3 Main results

We define the following class of functions, which will be used throughout the paper. We say that $h, g$ belong to the class of $Q$ if $h, g : K \subset \mathbb{R} \to \mathbb{R}$.

**Definition 3.1** Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$. We define the generalized $h, g$ real and imaginary parts of an operator as

$$\mathrm{Re}_{h,g}(A) \stackrel{\text{def}}{=} h(t)A + g(t)A^*,$$
$$\mathrm{Im}_{h,g}(A) \stackrel{\text{def}}{=} \frac{g(t)A - h(t)A^*}{i}.$$

*Remark* 3.2 Setting $g(t) = h(1-t), K = [0,1]$, and letting $h(t) = t$, we recover the generalized real and imaginary parts of an operator defined by Sheikhhosseini et al. [31].

Setting $h(t) = g(t) = \frac{1}{2}$, we recover the original real and imaginary parts of an operator, which indeed shows that our definition encaptures the previously defined ones.

Considering the fact that a bounded linear operator can be decomposed into a real and imaginary parts of an operator, that is $A = B + iC$ for some self-adjoint operators $B, C \in \mathcal{B}(\mathcal{H})$, we propose the following decomposition using the generalized definitions of the real and imaginary parts of an operator $A$. By proceeding as usual, we consider $\mathrm{Re}_{h,g}(A) + i\mathrm{Im}_{h,g}(A) = (h(t) + g(t))A + (g(t) - h(t))A^*$.

**Definition 3.3** Let $A \in \mathcal{B}(\mathcal{H})$, and let $f, g \in Q$. We define the following generalized $h, g$ numerical radius:

$$w_{h,g}(A) \stackrel{\text{def}}{=} \sup_{\|x\|=1} |\langle (h(t) + g(t))A + (g(t) - h(t)A^*)x, x\rangle|. \tag{3.1}$$

*Remark* 3.4 Setting $h(t) = t, g(t) = 1 - t, K = [0,1]$ in (3.1), we obtain the $t$-weighted numerical radius given by Conde et al. ([15], Definition 1.1), that is, we have

$$w_t(A) = w((1-2t)A^* + A).$$

**Proposition 3.5** *Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$ be such that $h(t) + g(t) = 1$. Then*

$$(Re_{h,g}(A))^2 - Re_{h,g}(A^2) = 4h(t)g(t)(Im(A))^2. \tag{3.2}$$

*In particular, setting $h(t) = g(t) = \frac{1}{2}$, we obtain*

$$(Re_{\frac{1}{2},\frac{1}{2}}(A))^2 - Re_{\frac{1}{2},\frac{1}{2}}(A^2) = (Im(A))^2.$$

*Proof* We begin by calculating the difference on the left-hand side:

$$\begin{aligned}(Re_{h,g}(A))^2 - Re_{h,g}(A^2) &= h(t)(h(t)-1)A^2 \\ &\quad + g(t)(g(t)-1)(A^*)^2 + h(t)g(t)(AA^* + A^*A).\end{aligned}$$

Rewriting the right-hand side in terms of the real and imaginary parts of the operator, we obtain the following:

$$\begin{aligned}&A = \mathrm{Re}(A) + i\mathrm{Im}(A),\ A^* = \mathrm{Re}(A) - i\mathrm{Im}(A),\\ &A^2 = (\mathrm{Re}(A))^2 - (\mathrm{Im}(A))^2 + i\mathrm{Re}(A)\mathrm{Im}(A) + i\mathrm{Im}(A)\mathrm{Re}(A),\\ &(A^*)^2 = (\mathrm{Re}(A))^2 - (\mathrm{Im}(A))^2 - i\mathrm{Re}(A)\mathrm{Im}(A) - i\mathrm{Im}(A)\mathrm{Re}(A),\\ &AA^* = (\mathrm{Re}(A))^2 + (\mathrm{Im}(A))^2 + i(\mathrm{Im}(A)\mathrm{Re}(A) - \mathrm{Re}(A)\mathrm{Im}(A)),\\ &A^*A = (\mathrm{Re}(A))^2 + (\mathrm{Im}(A))^2 - i(\mathrm{Im}(A)\mathrm{Re}(A) - \mathrm{Re}(A)\mathrm{Im}(A)).\end{aligned}$$

Since $h(t) + g(t) = 1$, we have $h(t) - 1 = -g(t)$ and $g(t) - 1 = -h(t)$. Therefore,

$$\begin{aligned}(\mathrm{Re}_{h,g}(A))^2 - \mathrm{Re}_{h,g}(A^2) &= -h(t)g(t)A^2 - g(t)h(t)(A^*)^2 \\ &\quad + h(t)g(t)(AA^* + A^*A) \\ &= h(t)g(t)\left[(AA^* + A^*A) - (A^2 + (A^*)^2)\right].\end{aligned}$$

Next, we rewrite the terms inside the brackets using the Cartesian decomposition $A = \mathrm{Re}(A) + i\mathrm{Im}(A)$ and the provided expansions for $A^2, (A^*)^2, AA^*$, and $A^*A$:

$$\begin{aligned}&AA^* + A^*A = 2(\mathrm{Re}(A))^2 + 2(\mathrm{Im}(A))^2,\\ &A^2 + (A^*)^2 = 2(\mathrm{Re}(A))^2 - 2(\mathrm{Im}(A))^2.\end{aligned}$$

Substituting these sums back into the bracketed expression:

$$\begin{aligned}(\mathrm{Re}_{h,g}(A))^2 - \mathrm{Re}_{h,g}(A^2) &= h(t)g(t)\big[(2(\mathrm{Re}(A))^2 + 2(\mathrm{Im}(A))^2) \\ &\quad - (2(\mathrm{Re}(A))^2 - 2(\mathrm{Im}(A))^2)\big]\end{aligned}$$

$$= h(t)g(t)\big[2(\mathrm{Re}(A))^2 + 2(\mathrm{Im}(A))^2$$
$$- 2(\mathrm{Re}(A))^2 + 2(\mathrm{Im}(A))^2\big]$$
$$= h(t)g(t)\left[4(\mathrm{Im}(A))^2\right]$$
$$= 4h(t)g(t)(\mathrm{Im}(A))^2.$$

Carefully evaluating, we recover the intended identity. For the second one, simply set $h(t) = g(t) = \frac{1}{2}$. □

**Corollary 3.6** *Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$ be such that $h(t) + g(t) = 1$ and $h(t)g(t) \geq 0$. Then*

$$(Re_{h,g}(A))^2 - Re_{h,g}(A^2) \geq 0.$$

*In particular,*

$$(Re(A))^2 \geq Re(A^2).$$

*Proof* Considering (3.2), we have that for $h, g \in Q$,

$$(\mathrm{Re}_{h,g}(A))^2 - \mathrm{Re}_{h,g}(A^2) = 4h(t)g(t)(\mathrm{Im}(A))^2.$$

Here, we observe that since $(\mathrm{Im}(A))^2$ is positive and since $h(t)g(t)$ is positive, the right-hand side of the equality is positive, therefore the first inequality holds. The second one follows from setting $h(t) = \frac{1}{2}, g(t) = \frac{1}{2}$. □

*Remark* 3.7 Setting $h(t) = t, g(t) = 1 - t, K = [0,1]$, we recover the identity given by Conde et al. ([15], Corollary 1.1), that is

$$(\mathrm{Re}_t(A))^2 - \mathrm{Re}_t(A^2) = 4t(1-t)(\mathrm{Im}(A))^2.$$

*Remark* 3.8 Let $A \in \mathcal{B}(\mathcal{H})$, and let $K = [0,1]$. Then setting $h(t) = \sin^2\left(\frac{\pi t}{2}\right)$ and $g(t) = \cos^2\left(\frac{\pi t}{2}\right)$ in (3.2), we obtain the following equality:

$$\frac{\pi}{2}\int_0^1 \sqrt{((\mathrm{Re}_{\sin^2(\frac{\pi t}{2}),\cos^2(\frac{\pi t}{2})}(A))^2 - \mathrm{Re}_{\sin^2(\frac{\pi t}{2}),\cos^2(\frac{\pi t}{2})}(A^2))}dt = |\mathrm{Im}(A)|.$$

Setting $h(t) = t, g(t) = 1 - t$, we can also obtain

$$\frac{4}{\pi}\int_0^1 \sqrt{(\mathrm{Re}_{t,1-t}(A))^2 - \mathrm{Re}_{t,1-t}(A^2))}dt = |\mathrm{Im}(A)|.$$

The following properties are trivially checked and the proofs are therefore omitted.

**Proposition 3.9** *Let $A, B \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$. Then*

(1) $w_{\frac{1}{2},\frac{1}{2}}(A) = w(A)$,

(2) $w_{h(t),g(t)}(A) \leq w(A)(|h(t) + g(t)| + |g(t) - h(t)|)$,

(3) $w_{h(t),g(t)}(A+B) \le w_{h(t),g(t)}(A) + w_{h(t),g(t)}(B)$,
(4) $f(t) = w_{h(t),g(t)}(A)$ *is convex when* $h, g$ *are linear.*

*In particular, we recover the bound given by Conde et al.* [15], *that is* $w_t(A) \le 2w(A)$ *from the second property and by using the fact that our generalized numerical radius reduces to the generalized numerical radius, that is by setting* $h(t) = t, g(t) = 1-t, K = [0,1]$.

**Proposition 3.10** *Let* $A \in \mathcal{B}(\mathcal{H})$, *and let* $h, g \in Q$. *Then* $w_{h,g}(A)$ *is self-adjoint, that is*

$$w_{h,g}(A) = w_{h,g}(A^*). \tag{3.3}$$

*Proof* We start with the definition

$$\begin{aligned} w_{h,g}(A) &= w((h(t)+g(t))A + (g(t)-h(t))A^*) \\ &= w((h(t)+g(t))(\mathrm{Re}(A) + i\mathrm{Im}(A)) + (g(t)-h(t))(\mathrm{Re}(A) - i\mathrm{Im}(A))) \\ &= w(2g(t)\mathrm{Re}(A) + 2h(t)i\mathrm{Im}(A)). \end{aligned}$$

Similarly, we get

$$w_{h,g}(A^*) = w(2g(t)\mathrm{Re}(A) - 2h(t)i\mathrm{Im}(A)).$$

Since $w(T) = w(T^*)$ for every operator $T$, these two identities are the same. □

**Theorem 3.11** *Let* $A \in \mathcal{B}(\mathcal{H})$, *and let* $h, g \in Q$. *Then*

$$\frac{w_{h,g}(A)}{2l} \le w(A) \le \frac{w_{h,g}(A)}{l_*} \tag{3.4}$$

*where* $l_* = \max\{|h(t)+g(t)| - |g(t)-h(t)|, |g(t)-h(t)| - |h(t)+g(t)|\}$ *and* $l = \max\{|h(t)+g(t)|, |g(t)-h(t)|\}$.

*Proof* Consider

$$w_{h,g}(A) = w\left((h(t)+g(t))A + (g(t)-h(t))A^*\right).$$

Applying the well-known inequality $w(X-Y) \ge w(X) - w(Y)$ for bounded linear operators $X$ and $Y$, we take $X = (h(t)+g(t))A$ and $Y = -(g(t)-h(t))A^*$. Then

$$\begin{aligned} w_{h,g}(A) = w(X-Y) &\ge w(X) - w(Y) \\ &= |h(t)+g(t)|w(A) - |g(t)-h(t)|w(A^*) \\ &= (|h(t)+g(t)| - |g(t)-h(t)|)w(A). \end{aligned}$$

By symmetry, if instead we take $X' = (g(t)-h(t))A^*$ and $Y' = -(h(t)+g(t))A$, we obtain

$$\begin{aligned} w_{h,g}(A) = w(X'-Y') &\ge w(X') - w(Y') \\ &= |g(t)-h(t)|w(A^*) - |h(t)+g(t)|w(A) \end{aligned}$$

$$= (|g(t) - h(t)| - |h(t) + g(t)|)w(A).$$

Therefore, we have

$$\begin{aligned} w_{h,g}(A) &\geq \max\{|h(t) + g(t)| - |g(t) - h(t)|, |g(t) - h(t)| - |h(t) + g(t)|\}w(A) \\ &= l_* w(A). \end{aligned}$$

□

*Remark* 3.12 Setting $h(t) = e^{2t}, g(t) = e^t, K = [0, 1]$ and integrating the previous inequalities (3.4), we obtain the following inequalities:

$$\begin{aligned} &\int_0^{\frac{1}{2}} w_{e^{2t},e^t}(A)dt \leq (e + 2\sqrt{e} - 3)w(A), \\ &2(e - \sqrt{e})w(A) \leq \int_{\frac{1}{2}}^1 w_{e^{2t},e^t}(A)dt. \end{aligned}$$

**Corollary 3.13** *Let* $A \in \mathcal{B}(\mathcal{H})$, *and let* $h, g \in Q, l = \max\{|h(t) + g(t)|, |g(t) - h(t)|\}$. *Then*

$$|g(t)| \, \|Re(A)\| \leq \frac{w_{h,g}(A)}{2} \leq lw(A).$$

*In particular,*

$$(\sqrt{e} - 1) \, \|Re(A)\| \leq \frac{1}{2} \int_0^{\frac{1}{2}} w_{e^{2t},e^t}(A)dt \leq \frac{1}{2}(e + 2\sqrt{e} - 3)w(A). \tag{3.5}$$

*Proof* Consider the following:

$$\begin{aligned} &4|g(t)| \, \|\mathrm{Re}(A)\| = 2|g(t)|w(A + A^*) = w_{h,g}(A + A^*) \\ &\leq 2w_{h,g}(A) \\ &\leq 4lw(A). \end{aligned}$$

The first equality is obvious, the first inequality is due to (3.3) and Proposition 3.9 (3), and the second inequality is due to (3.4). □

*Remark* 3.14 Replacing $A$ by $iA^*$ in the previous corollary, we obtain the following result:

$$|g(t)| \, \|\mathrm{Im}(A)\| \leq \frac{w((h(t) + g(t))A^* - (g(t) - h(t))A)}{2} \leq lw(A).$$

Similarly, we have

$$\begin{aligned} (\sqrt{e} - 1) \, \|\mathrm{Im}(A)\| &\leq \frac{1}{2} \int_0^{\frac{1}{2}} w((e^t + e^{2t})A^* - A(e^t - e^{2t}))dt \\ &\leq \frac{1}{2}(e + 2\sqrt{e} - 3)w(A). \end{aligned}$$

Combining Remark 3.14 and the inequality (3.5), we obtain the following corollary.

**Corollary 3.15** *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}(\sqrt{e}-1)\,\|A\| &\le (\sqrt{e}-1)(\|Re(A)\|+\|Im(A)\|)\\ &\le \frac{1}{2}\int_0^{\frac{1}{2}}(w((e^t+e^{2t})A^*-A(e^t-e^{2t}))+w_{e^{2t},e^t}(A))dt\\ &\le (e+2\sqrt{e}-3)w(A).\end{aligned}$$

**Proposition 3.16** *Let $A \in \mathcal{B}(\mathcal{H})$. Then the following inequality holds for $t \in [0,1]$:*

$$w(A)=\sup_{\theta\in\mathbb{R}}\{\frac{w_{t,t+1}(e^{i\theta}A)}{2+2t}\}.$$

*Proof* Consider Corollary 3.13. In the proof of the upper bound, we will use a different estimate, a weaker one, namely

$$4|g(t)|\,\|\mathrm{Re(A)}\| \le 2w_{h,g}(A) \le 2(|h+g|+|h-g|)w(A),$$

which is trivially obtained by invoking Proposition 3.9 (2). From here, one has

$$\|\mathrm{Re(A)}\| \le \frac{1}{2|g(t)|}w_{h,g}(A) \le \frac{(|h+g|+|h-g|)}{2|g(t)|}w(A).$$

Now, setting $e^{i\theta}A$ instead of $A$ and taking the supremum over $\theta \in \mathbb{R}$ while setting $g(t) = t+1, h(t) = t, K = [0,1]$, we obtain

$$\sup_{\theta\in\mathbb{R}}\left\|\mathrm{Re}(e^{i\theta}\mathrm{A})\right\| \le \sup_{\theta\in\mathbb{R}}\frac{1}{2(t+1)}w_{t,t+1}(e^{i\theta}A) \le w(A).$$

Now, using the well-known fact that $w(A) = \sup_{\theta\in\mathbb{R}}\left\|\mathrm{Re}(e^{i\theta}A)\right\|$, we obtain the desired equality. □

One can show in a similar manner that the following formula holds.

*Remark* 3.17 Let $A \in \mathcal{B}(\mathcal{H})$, and let $K = [-2,2]$. Then the following formula for the numerical radius holds:

$$w(A)=\sup_{\theta\in\mathbb{R}}\left\{\frac{w_{\sin(t),t^2+1}(e^{i\theta}A)}{2(t^2+1)}\right\}.$$

In particular, one can find arbitrary many alternative formulas for the numerical radius, one only has to choose functions $h, g$ such that

$$\frac{|h(t)+g(t)|+|h(t)-g(t)|}{2|g(t)|}=1.$$

We utilize a refinement of the Hermite-Hadamard inequality, which goes as follows:

$$f\left(\frac{1}{2}\right) \le \int_0^1 f\left(st+\frac{1-t}{2}\right)dt$$

$$\leq \int_0^1 f(t)dt \leq \frac{1}{2}\int_0^1 \left[f\left(\frac{1-s}{2}t\right) + f\left(\frac{1+s}{2} + \frac{1-s}{2}t\right)\right]dt$$
$$\leq \frac{f(0)+f(1)}{2},$$

for a convex function $f : [0,1] \to \mathbb{R}$, and $s \in [0,1]$, which is obtained by combining the refinement of the lower bound given by Dragomir in [16], and of the upper bound given by Yang and Hong in [37]. This enables us to obtain the following result.

*Remark* 3.18 Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$ be linear. Then utilizing the fact that $f(t) = w_{h(t),g(t)}(A)$ is convex from Proposition 3.9 (4), the following chain of inequalities holds:

$$\begin{aligned} w_{h\left(\frac{1}{2}\right),g\left(\frac{1}{2}\right)}(A) &\leq \int_0^1 w_{h\left(st+\frac{1-s}{2}\right),g\left(st+\frac{1-s}{2}\right)}(A)dt \\ &\leq \int_0^1 w_{h(t),g(t)}(A)dt \\ &\leq \frac{1}{2}\int_0^1 \Bigg[ w_{h\left(\frac{1-s}{2}t\right),g\left(\frac{1-s}{2}t\right)}(A) \\ &+ w_{h\left(\frac{1+s}{2}+\frac{1-s}{2}t\right),g\left(\frac{1+s}{2}+\frac{1-s}{2}t\right)}(A)\Bigg]dt \\ &\leq \frac{w_{h(0),g(0)}(A) + w_{h(1),g(1)}(A)}{2}, \end{aligned}$$

for all $s \in [0,1]$.

Setting $h(t) = 1-t, g(t) = t$, we obtain the following chain of inequalities, which in particular provides a refinement of the inequality $w(A) \leq \|\mathrm{Re}(A)\| + \|\mathrm{Im}(A)\|$ as

$$\begin{aligned} w(A) &\leq \int_0^1 w_{\frac{1}{2}-st+\frac{s}{2},st+\frac{1-s}{2}}(A)dt \\ &\leq \int_0^1 w_{1-t,t}(A)dt \\ &\leq \frac{1}{2}\int_0^1 \left[ w_{1-\frac{1-s}{2}t,\frac{1-s}{2}t}(A) + w_{\frac{1}{2}(1-s-t+st),\frac{1+s}{2}+\frac{1-s}{2}t}(A)\right]dt \\ &\leq \|\mathrm{Re}(A)\| + \|\mathrm{Im}(A)\| \end{aligned}$$

for all $s \in [0,1]$.

Consequently, we obtain the following:

$$\begin{aligned} w(A) &\leq \sup_{s\in[0,1]} \int_0^1 w_{\frac{1}{2}-st+\frac{t}{2},st+\frac{1-t}{2}}(A)dt \\ &\leq \int_0^1 w_{1-t,t}(A)dt \\ &\leq \inf_{s\in[0,1]}\left[\frac{1}{2}\int_0^1 (w_{1-\frac{1-s}{2}t,\frac{1-s}{2}t}(A) + w_{\frac{1}{2}(1-s-t+st),\frac{1+s}{2}+\frac{1-s}{2}t}(A))dt\right] \\ &\leq \|\mathrm{Re}(A)\| + \|\mathrm{Im}(A)\|\,. \end{aligned}$$

**Proposition 3.19** *Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$. Then*

$$\sqrt{2}\,\left\|g(t)Re(A) + h(t)Im(A)\right\| \leq w_{h,g}(A).$$

*Proof* The proof is similar to the one given by Conde et al. ([15], Proposition 2.4), and is therefore omitted. □

*Remark* 3.20 Setting $g(t) = h(t) = \frac{1}{2}$, we recover the inequality given by Conde et al. ([15], Proposition 2.4), that is,

$$\frac{1}{\sqrt{2}}\,\|\mathrm{Re}(A) + \mathrm{Im}(A)\| \leq w(A).$$

*Remark* 3.21 Setting $g(t) = 1 - t, h(t) = t, K = [0,1]$, we recover the inequality given by Conde et al. ([15], Proposition 2.4)

$$\sqrt{2}\,\|(1-t)\mathrm{Re}(A) + t\mathrm{Im}(A)\| \leq w_{t,1-t}(A) = w_t(A).$$

**Theorem 3.22** *Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$. Then*

$$\left\||g(t)|Re(A) \pm |h(t)|Im(A)\right\| \leq \frac{w_{h,g}(A)}{\sqrt{2}} \leq \sqrt{2}\,\left\|g^2(t)Re^2(A) + h^2(t)Im^2(A)\right\|^{\frac{1}{2}}.$$

*Proof* Consider the following for any unit vector $x \in \mathcal{H}$:

$$\begin{aligned}
&|\langle (A(h(t)+g(t)) + (g(t)-h(t))A^*)x, x\rangle|^2 \\
&= |\langle (2g(t)\mathrm{Re}(A) + 2h(t)i\mathrm{Im}(A))x, x\rangle|^2 \\
&= 4g^2(t)\langle \mathrm{Re}(A)x, x\rangle^2 + 4h^2(t)\langle \mathrm{Im}(A)x, x\rangle^2 \\
&\geq \frac{\Big(2|g(t)||\langle \mathrm{Re}(A)x, x\rangle| + 2|h(t)||\langle \mathrm{Im}(A)x, x\rangle|\Big)^2}{2} \\
&\geq \frac{\Big(|\langle 2g(t)\mathrm{Re}(A)x, x\rangle \pm \langle 2h(t)\mathrm{Im}(A)x, x\rangle|\Big)^2}{2} \\
&= 2\Big(|\langle g(t)\mathrm{Re}(A)x, x\rangle \pm \langle h(t)\mathrm{Im}(A)x, x\rangle|\Big)^2.
\end{aligned}$$

Taking the supremum over all the unit vectors, we obtain the desired inequality on the left-hand side. To obtain the right-hand side, simply use the Cauchy-Schwarz inequality as

$$\begin{aligned}
&|\langle (A(h(t)+g(t)) + (g(t)-h(t))A^*)x, x\rangle|^2 \\
&= 4g^2(t)\langle \mathrm{Re}(A)x, x\rangle^2 + 4h^2(t)\langle \mathrm{Im}(A)x, x\rangle^2 \\
&\leq 4g^2(t)\,\|\mathrm{Re}(A)x\|^2 + 4h^2(t)\,\|\mathrm{Im}(A)x\|^2 \\
&= \langle (4g^2(t)\mathrm{Re}^2(A) + 4h^2(t)\mathrm{Im}^2(A))x, x\rangle.
\end{aligned}$$

Taking the supremum over all the unit vectors, we obtain the desired inequality. □

*Remark* 3.23 In particular, setting $h(t) = \sin\left(\frac{\pi}{2}t\right), g(t) = 1 - \sin\left(\frac{\pi}{2}t\right)$, we obtain

$$\begin{aligned}&\left\| \left(1 - \sin\left(\frac{\pi}{2}t\right)\right) \operatorname{Re}(A) \pm \sin\left(\frac{\pi}{2}t\right) \operatorname{Im}(A) \right\| \\ &\leq \frac{w_{\sin(\frac{\pi}{2}t),1-\sin(\frac{\pi}{2}t)}(A)}{\sqrt{2}} \\ &\leq \sqrt{2} \left\| \left(1 - \sin\left(\frac{\pi}{2}t\right)\right)^2 \operatorname{Re}(A) + \sin^2\left(\frac{\pi}{2}t\right) \operatorname{Im}(A) \right\|^{\frac{1}{2}}.\end{aligned}$$

**Theorem 3.24** *Let $A \in \mathcal{B}(\mathcal{H})$, let $R := \max\{g^2(t), h^2(t)\}$, and let $r := \min\{g^2(t), h^2(t)\}$. Then*

$$r \left\| |A|^2 + |A^*|^2 \right\| \leq w_{h,g}^2(A) \leq 2R \left\| |A|^2 + |A^*|^2 \right\|.$$

*Proof* Consider the expression we obtained by defining the $w_{h,g}(A)$ numerical radius, and consider the decomposition into the real and imaginary parts. Then for every unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned}&|\langle((h(t) + g(t))A + (g(t) - h(t))A^*)x, x\rangle|^2 \\ &= (2g(t)\langle\operatorname{Re}(A)x, x\rangle)^2 + (2h(t)\langle\operatorname{Im}(A)x, x\rangle)^2 \\ &= 4g^2(t)\langle\operatorname{Re}(A)x, x\rangle^2 + 4h^2(t)\langle\operatorname{Im}(A)x, x\rangle^2 \\ &\leq 4g^2(t)\langle\operatorname{Re}^2(A)x, x\rangle + 4h^2(t)\langle\operatorname{Im}^2(A)x, x\rangle \\ &\leq 4R\langle(\operatorname{Re}^2(A) + \operatorname{Im}^2(A))x, x\rangle = 2R\langle(AA^* + A^*A)x, x\rangle,\end{aligned}$$

where in the last equality we used the fact that $\operatorname{Re}^2(A) + \operatorname{Im}^2(A) = \frac{AA^*+A^*A}{2}$. Taking the supremum over all the unit vectors, we obtain the desired inequality. Regarding the lower bound, consider the following:

$$\begin{aligned}&|\langle((h(t) + g(t))A + (g(t) - h(t))A^*)x, x\rangle|^2 \\ &= (2g\langle\operatorname{Re}(A)x, x\rangle)^2 + (2h\langle\operatorname{Im}(A)x, x\rangle)^2 \\ &\geq \frac{1}{2}(2|g(t)||\langle\operatorname{Re}(A)x, x\rangle| + 2|h(t)||\langle\operatorname{Im}(A)x, x\rangle|)^2 \\ &\geq 2r|\langle(\operatorname{Re}(A) \pm \operatorname{Im}(A))x, x\rangle|^2.\end{aligned}$$

Taking the supremum over all the unit vectors, we obtain

$$w_{h,g}^2(A) \geq 2r \|\operatorname{Re}(A) \pm \operatorname{Im}(A)\|^2 = 2r \left\|(\operatorname{Re}(A) \pm \operatorname{Im}(A))^2\right\|.$$

Further, we have

$$\begin{aligned}&2w_{h,g}^2(A) \geq 2r(\left\|(\operatorname{Re}(A) + \operatorname{Im}(A))^2\right\| + \left\|(\operatorname{Re}(A) - \operatorname{Im}(A))^2\right\|) \\ &\geq 2r(\left\|(\operatorname{Re}(A) + \operatorname{Im}(A))^2 + (\operatorname{Re}(A) - \operatorname{Im}(A))^2\right\|) \\ &= 4r \left\|\operatorname{Re}^2(A) + \operatorname{Im}^2(A)\right\| = 2r \left\|AA^* + A^*A\right\|.\end{aligned}$$

Therefore, we obtain $w_{h,g}^2(A) \geq r \|AA^* + A^*A\|$. □

**Theorem 3.25** *Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q, \alpha \in [0,1]$. Then*

$$w_{h,g}(A) \le \min\{w_1, w_2\}, \tag{3.6}$$

*where*

$$w_1 = \frac{1}{2}\Bigg[|g(t) - h(t)|\Bigg( \max\{\|A\|^{2\alpha}, \|A\|^{2(1-\alpha)}\} + \sqrt{r(|A|^{2\alpha}|A^*|^{2(1-\alpha)})}\Bigg) + |g(t) + h(t)|\Bigg( \max\{\|A\|^{2\alpha}, \|A\|^{2(1-\alpha)}\} + \sqrt{r(|A^*|^{2\alpha}|A|^{2(1-\alpha)})}\Bigg),$$

*and*

$$w_2 = \frac{1}{2}\Bigg[|g(t) + h(t)|\Bigg( \max\{\|A\|^{2\alpha}, \|A\|^{2(1-\alpha)}\} + \sqrt{r(|A|^{2\alpha}|A^*|^{2(1-\alpha)})}\Bigg) + |g(t) - h(t)|\Bigg( \max\{\|A\|^{2\alpha}, \|A\|^{2(1-\alpha)}\} + \sqrt{r(|A^*|^{2\alpha}|A|^{2(1-\alpha)})}\Bigg).$$

*Proof* Let $x \in \mathcal{H}$ be an arbitrary unit vector. Then

$$\begin{aligned}
&|\langle((h(t) + g(t))A + (g(t) - h(t))A^*)x, x\rangle| \\
&\le |h(t) + g(t)||\langle Ax, x\rangle| + |g(t) - h(t)||\langle A^*x, x\rangle| \\
&\le |(h(t) + g(t)|\sqrt{\langle |A|^{2\alpha}x, x\rangle\langle |A^*|^{2(1-\alpha)}x, x\rangle} \\
&+ |g(t) - h(t)|\sqrt{\langle |A^*|^{2\alpha}x, x\rangle\langle |A|^{2(1-\alpha)}x, x\rangle} \quad \text{(by (1.1))} \\
&\le \frac{1}{2}(|h(t) + g(t)|\langle(|A|^{2\alpha} + |A^*|^{2(1-\alpha)})x, x\rangle \\
&+ |g(t) - h(t)|\langle(|A^*|^{2\alpha} + |A|^{2(1-\alpha)})x, x\rangle) \\
&\text{(by the arithmetic-geometric mean inequality)} \\
&\le \frac{1}{2}(|h(t) + g(t)|\, \big\||A|^{2\alpha} + |A^*|^{2(1-\alpha)}\big\| + |g(t) - h(t)|\, \big\||A^*|^{2\alpha} + |A|^{2(1-\alpha)}\big\|) \\
&\le \frac{1}{2}\Bigg[|h(t) + g(t)|\Bigg( \max\{\big\||A|^{2\alpha}\big\|, \big\||A^*|^{2(1-\alpha)}\big\|\} + \big\||A|^{\alpha}|A^*|^{(1-\alpha)}\big\|\Bigg) \\
&+ |g(t) - h(t)|\Bigg( \max\{\big\||A^*|^{2\alpha}\big\|, \big\||A|^{2(1-\alpha)}\big\|\} + \big\||A^*|^{\alpha}|A|^{(1-\alpha)}\big\|\Bigg)\Bigg] \\
&\text{(by (2.1))} \\
&= \frac{1}{2}\Bigg[|h(t) + g(t)|\Bigg( \max\{\|A\|^{2\alpha}, \|A\|^{2(1-\alpha)}\} + \sqrt{r(|A|^{2\alpha}|A^*|^{2(1-\alpha)})}\Bigg) \\
&+ |h(t) - g(t)|\Bigg( \max\{\|A\|^{2\alpha}, \|A\|^{2(1-\alpha)}\} + \sqrt{r(|A^*|^{2\alpha}|A|^{2(1-\alpha)})}\Bigg)\Bigg] \quad \text{(by (2.2))}.
\end{aligned}$$

Taking the supremum over all unit vectors, we obtain the first inequality. Using (3.3) and applying the same procedure, we now deduce the desired inequality. □

*Remark* 3.26 Setting $h(t) = g(t) = \frac{1}{2}, \alpha = \frac{1}{2}$ in (3.6), we obtain a refinement of the inequality given by Bhunia and Paul ([13], Theorem 2.1), that is

$$w(A) \le \min\left\{\frac{1}{2}(\|A\| + \sqrt{r(|A||A^*|)}), \frac{1}{2}(\|A\| + \sqrt{r(|A^*||A|)}\right\}.$$

**Theorem 3.27** *Let $A \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q, \alpha \in [0,1]$. Then*

$$w_{h,g}(A) \le \left[(h(t)+g(t))^2(\max\{\left\||A|^{4\alpha}\right\|, \left\||A^*|^{4(1-\alpha)}\right\|\} + \sqrt{r(|A|^{4\alpha}|A^*|^{4(1-\alpha)})}) + (g(t)-h(t))^2\left(\frac{\left\||A|^4+|A^*|^4\right\|}{2} + w(A^2)\right)\right]^{\frac{1}{2}}.$$

*Proof* The proof is similar to that of Theorem 3.25. Let $x \in \mathcal{H}$ be an arbitrary unit vector. Then

$$\begin{aligned}
&|\langle((h(t)+g(t))A + (g(t)-h(t))A^*)x, x\rangle|^2 \\
&\le (|h(t)+g(t)||\langle Ax,x\rangle| + |g(t)-h(t)||\langle A^*x,x\rangle|)^2 \\
&\le 2((h(t)+g(t))^2|\langle Ax,x\rangle|^2 + (g(t)-h(t))^2|\langle A^*x,x\rangle|^2) \\
&\le 2(h(t)+g(t))^2\langle|A|^{2\alpha}x,x\rangle\langle|A^*|^{2(1-\alpha)}x,x\rangle \\
&+ (g(t)-h(t))^2(\left\|A^*x\right\|\|Ax\| + |\langle A^*x, Ax\rangle|) \\
&\le (h(t)+g(t))^2(\langle|A|^{2\alpha}x,x\rangle^2 + \langle|A^*|^{2(1-\alpha)}x,x\rangle^2) \\
&+ (g(t)-h(t))^2\left(\frac{\langle|A|^2x,x\rangle^2 + \langle|A^*|^2x,x\rangle^2}{2} + |\langle A^*x,Ax\rangle|\right) \\
&\le (h(t)+g(t))^2\langle(|A|^{4\alpha} + A^*|^{4(1-\alpha)})x,x\rangle \\
&+ (g(t)-h(t))^2\left(\frac{\langle(|A|^4+|A^*|^4)x,x\rangle}{2} + |\langle A^*x,Ax\rangle|\right) \\
&\le (h(t)+g(t))^2\left\||A|^{4\alpha} + |A^*|^{4(1-\alpha)}\right\| \\
&+ (g(t)-h(t))^2\left(\frac{\left\||A|^4+|A^*|^4\right\|}{2} + |\langle A^*x,Ax\rangle|\right) \\
&\le (h(t)+g(t))^2(\max\{\left\||A|^{4\alpha}\right\|, \left\||A^*|^{4(1-\alpha)}\right\|\} + \left\||A|^{2\alpha}|A^*|^{2(1-\alpha)}\right\|) \\
&+ (g(t)-h(t))^2\left(\frac{\left\||A|^4+|A^*|^4\right\|}{2} + w((A^*)^2)\right) \\
&= (h(t)+g(t))^2(\max\{\left\||A|^{4\alpha}\right\|, \left\||A^*|^{4(1-\alpha)}\right\|\} + \sqrt{r(|A|^{4\alpha}|A^*|^{4(1-\alpha)})}) \\
&+ (g(t)-h(t))^2\left(\frac{\left\||A|^4+|A^*|^4\right\|}{2} + w(A^2)\right).
\end{aligned}$$

First we utilize the fact that $f(t) = t^2$ is convex, then we proceed similarly on the first term as in the previous theorem with the exception of using (2.4). We use (2.3) for the second term, as well as the arithmetic-geometric mean inequality, and then use (2.4). The proof is finished by taking the supremum and root on both sides. □

## 4 Inequalities for operator matrices

**Theorem 4.1** *Let $A,B,C,D \in \mathcal{B}(\mathcal{H})$, and let $h,g \in Q, \alpha \in [0,1], k \geq 1$, where $R = \max\{g^2(t), h^2(t)\}$. Then*

$$w_{h,g}^{2k}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq 2^{4k-2}R^k \max\{\left\| |C|^{4k\alpha} + |B^*|^{4(1-\alpha)} + |A|^{4k\alpha} + |A^*|^{4k(1-\alpha)} \right\|,$$

$$\left\| |B|^{4k\alpha} + |C^*|^{4(1-\alpha)} + |D|^{4k\alpha} + |D^*|^{4k(1-\alpha)} \right\|\}. \tag{4.1}$$

*Proof* Let $x \in \mathcal{H} \oplus \mathcal{H}$ be an arbitrary unit vector. Also let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $K_1 = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ and $K_2 = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$. Now, consider

$$\begin{aligned}
&\left|\left\langle(((g(t)+h(t))T + (g(t)-h(t))T^*))x,x\right\rangle\right|^{2k} \\
&= \left(4g^2(t)\left|\langle Re(T)x,x\rangle\right|^2 + 4h^2(t)\left|\langle Im(T)x,x\rangle\right|^2\right)^k \\
&\leq \left(4R|\langle Tx,x\rangle|^2\right)^k \\
&\leq 2^{2k}R^k\left(|\langle K_1x,x\rangle| + |\langle K_2x,x\rangle|\right)^{2k} \\
&\leq 2^{4k-1}R^k\left(|\langle K_1x,x\rangle|^{2k} + |\langle K_2x,x\rangle|^{2k}\right) \text{ (by the convexity of } f(t) = t^{2k} \text{ for } k \geq 1) \\
&\leq 2^{4k-1}R^k\left(\langle |K_1|^{2\alpha}x,x\rangle^k \langle |K_1^*|^{2(1-\alpha)}x,x\rangle^k + \langle |K_2|^{2\alpha}x,x\rangle^k \langle |K_2^*|^{2(1-\alpha)}x,x\rangle^k\right) \\
&\quad \text{(by (1.1))} \\
&\leq 2^{4k-1}R^k\left(\langle |K_1|^{2k\alpha}x,x\rangle \langle |K_1^*|^{2k(1-\alpha)}x,x\rangle + \langle |K_2|^{2k\alpha}x,x\rangle \langle |K_2^*|^{2k(1-\alpha)}x,x\rangle\right) \\
&\quad \text{(by (2.4))} \\
&\leq 2^{4k-2}R^k\langle(|K_1|^{4k\alpha} + |K_1^*|^{4k(1-\alpha)} + |K_2|^{4k\alpha} + |K_2^*|^{4k(1-\alpha)})x,x\rangle \\
&\quad \text{(by the arithmetic-geometric mean inequality and (2.4))}.
\end{aligned}$$

Taking the supremum over all the unit vectors and calculating the terms while utilizing equality (2.7), we obtain the desired inequality. □

*Example* 4.2 Setting $h(t) = t, g(t) = 1-t, \alpha = \frac{1}{2}$, and letting $K = [0,1]$, we reduce to the generalized $t$-weighted numerical radius defined by Conde et al. [15]. Now, setting $T = \begin{bmatrix} 1 & 10 \\ 1 & 1 \end{bmatrix}$, $k = 1$ and evaluating the inequality (4.1) and the inequality of Ammar et al. ([6], Theorem 2.13), we obtain the following numerical values for some of the inequalities:

$$169 \leq 412 \text{ (using the inequality (4.1))},$$
$$169 \leq 808 \text{ (the inequality given by Ammar et al. [6])},$$

which shows that for a particular operator matrix, our inequality is way sharper.

**Theorem 4.3** *Let $B, C \in \mathcal{B}(\mathcal{H})$, and let $h, g \in Q$, where $R = \max\{g^2(t), h^2(t)\}$. Then*

$$w_{h,g}^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq 2R\max\{w(BC), w(CB)\} + R\max\{\left\| |C|^2 + |B^*|^2 \right\|, \left\| |B|^2 + |C^*|^2 \right\|\}.$$

*Proof* Let $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, and let $x$ be an arbitrary vector in $H \oplus H$. Then we proceed starting from the first inequality of the previous theorem as follows:

$$\begin{aligned}
&|\langle((g(t)+h(t))T + (g(t)-h(t))T^*)x, x\rangle|^2 \\
&\leq 4R|\langle Tx, x\rangle|^2 \leq 2R(\|Tx\| \, \|T^*x\| + |\langle T^2x, x\rangle|) \text{ (by (2.3))} \\
&\leq 2R\sqrt{(\|Tx\|^2 + |\langle T^2x, x\rangle|)(\|T^*x\|^2 + |\langle T^2x, x\rangle|)} \\
&\text{(by the Cauchy-Schwarz inequality)} \\
&\leq 2R|\langle T^2x, x\rangle| + R(\langle |T|^2x, x\rangle + \langle |T^*|^2x, x\rangle) \\
&\text{(by the arithmetic-geometric mean inequality).}
\end{aligned}$$

Taking the supremum over all the unit vectors and utilizing (2.7), we obtain the desired inequality. □

*Remark* 4.4 Setting $h(t) = g(t) = \frac{1}{2}$, we recover the inequality given by Bani-Domi and Kittaneh ([10], Remark 2.5), that is

$$w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2}\max\{w(BC), w(CB)\} + \frac{1}{4}\max\{\left\| |C|^2 + |B^*|^2 \right\|, \left\| |B|^2 + |C^*|^2 \right\|\}. \tag{4.2}$$

*Remark* 4.5 Setting $B = C$ in (4.2), we recover the inequality

$$w^2(B) \leq \frac{1}{2}w(B^2) + \frac{1}{4}\left\| |B|^2 + |B^*|^2 \right\|,$$

which, in view of the power inequality for the numerical radius, yields the sharp inequality obtained by Kittaneh [25].

**Theorem 4.6** *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$, and let $R := \max\{g^2(t), h^2(t)\}$. Then*

$$w_{h,g}^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \le 2\max\{w_{h,g}^2(A), w_{h,g}^2(D)\}$$
$$+ 4R\max\{w(BC), w(CB)\}$$
$$+ 2R\max\{\left\| |C|^2 + |B^*|^2 \right\|, \left\| |B|^2 + |C^*|^2 \right\|\}.$$

*Proof* Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, and let $x$ be an arbitrary unit vector in $H \oplus H$. Then the following inequality holds:

$$\begin{aligned} w_{h,g}^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &= w_{h,g}^2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \\ &\le \left(w_{h,g}\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + w_{h,g}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right)^2 \\ &\le 2w_{h,g}^2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + 2w_{h,g}^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right). \end{aligned}$$

Regarding the first term, we use the modified version of Lemma 2.5, which follows from the proof of the standard result regarding the numerical radius of a diagonal matrix, and for the second term, we use the previously obtained result. □

## 5 Conclusion

This paper introduces a powerful generalized framework for numerical radius inequalities in Hilbert spaces. We have defined novel generalized real and imaginary parts of an operator, $Re_{h,g}(A)$ and $Im_{h,g}(A)$, leading to a new generalized numerical radius, $w_{h,g}(A)$. In particular, we have proved that $w_{h,g}(A)$ satisfies the usual numerical radius properties, such as self-adjointness and subadditivity, and we also have noted that our generalized definition of the numerical radius reduces to the well-known definition of the numerical radius, namely the $t$ numerical radius given by [15]. It is important to note that since all of our results generalize the results given by Conde et al. [15]. Beyond establishing this foundation of standard properties of numerical radius, the research provides key inequalities, including a new identity for $(\mathrm{Re}_{h,g}(A))^2 - \mathrm{Re}_{h,g}(A^2)$, along with various bounds for $w_{h,g}(A)$. The exploration of convexity further refines existing numerical radius bounds. It is also important to note that we have given two new ways to calculate $w(A)$ via $w_{h,g}(A)$ and the functions $h, g$. This generalized approach, particularly the adaptability of $w_{h,g}(A)$ through the functions $h$ and $g$, offers a significantly more versatile methodology for analyzing operator inequalities in functional analysis, with promising implications for dynamical systems, where operator behavior might evolve over time.

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