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Article

Half-Symmetric Connections of Generalized Riemannian Spaces

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Abstract

In this article, we generalize Yano's concept of a half-symmetric affine connection. With respect to this generalization, we obtain five linearly independent curvature tensors. In the following, we examine which special kinds of affine connections may be the generalized half-symmetric affine connection. At the end of this work, we generalize the term of Killing's vector given by Yano to affine Killing, conformal Killing, projective Killing, harmonic, and covariant and contravariant analytic vectors.

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1. Introduction

The symmetric affine connection spaces [1–3], and Riemannian spaces as special cases of them, are subjects of research of different researchers and scientific groups. Those include J. Mikeš with his team [1,2], N. S. Sinyukov [3], and many others.

An n-dimensional manifold equipped with a symmetric affine connection ∇, whose coefficients are l^i_{jk}, $l^i_{jk} = l^i_{kj}$, is the n-dimensional symmetric affine connection space $\mathbb{A}_n$. A special subclass of the class of symmetric affine connection spaces is composed of the n-dimensional manifolds equipped with symmetric metric tensors g_{ij}. These spaces are n-dimensional Riemannian spaces $\mathbb{R}_n$.

The affine connection coefficients of Riemannian space $\mathbb{R}_n$ are Christoffel symbols:

$$\gamma^i_{jk} = \frac{1}{2} g^{ip}\left(g_{jp,k} - g_{jk,p} + g_{kp,j}\right), \tag{1}$$

where $\left[g^{ij}\right] = \left[g_{ij}\right]^{-1}$, a comma denotes partial derivation, $g_{ij,k} = \partial g_{ij}/\partial x^k$, and a summation by the mute index p, $p = 1, \ldots, n$, is implied.

From [1–3], it is well known that sums $l^i_{jk} + P^i_{jk}$ and $\gamma^i_{jk} + P^i_{jk}$ of the affine connection coefficients l^i_{jk} and γ^i_{jk} and a tensor P^i_{jk} of the type $(1,2)$ are coefficients of other affine connections.

The impact of special forms of this tensor on properties of affine connections on the $n = 2N$-dimensional manifold is studied in this research. The research starts with Section 2, where general definitions of n-dimensional and $2N$-dimensional affine connection spaces are reviewed. After that, we generalize the results presented in [1–3] by involving torsion in the affine connection, such as Yano's results [4] with different possibilities of defining special affine connections initially defined in Yano's work. The results of this research may be of special interest for applications in different scientific disciplines such as quantum physics, cosmology, astronomy, and many others.

2. Theoretical Background

In this research, we continue the research about half-symmetric affine connection started by K. Yano [4], T. Suguri and S. Nakayama [5], and S. Ishihara [6,7]. Yano's research is a special case of the study about non-symmetric affine connection space started by L. P. Eisenhart [8], and continued by S. Minčić [9–14], M. Stanković [15], Lj. S. Velimirović [14–16], M. Z. Petrović [16–18], and many others.

In this article, we recall basic definitions about symmetric and non-symmetric affine connection spaces. After that, the curvature tensors of these spaces are expressed. In Section 2, we present a definition of half-symmetric connection [4] and correlate it with the corresponding non-symmetric affine connection. In Section 3, we obtain a family of curvature tensors with respect to half-symmetric affine connection. The last result in this study is linearly independent curvature tensors obtained with respect to the half-symmetric affine connection.

2.1. 2N-Dimensional Riemannian Space

In this part, we adopt definitions of an N-dimensional Riemannian space from [1–3] for the corresponding-dimensional Riemannian spaces. A $2N$-dimensional differentiable manifold $\mathcal{M}_{2N}$ equipped with symmetric metric tensor $\underline{\hat{g}}$, whose components are $g_{\underline{ij}}$, $g_{\underline{ij}} = g_{\underline{ji}}$, is the Riemannian space $\mathbb{R}_{2N}$. We assume the regularity of matrix $\big[g_{\underline{ij}}\big]$, i.e., $\det\big[g_{\underline{ij}}\big] \neq 0$. The regularity of matrix $\big[g_{\underline{ij}}\big]$ allows the contravariant metric tensor $\underline{g}^{-1}$ to be defined by components such as $\big[g^{\underline{ij}}\big] = \big[g_{\underline{ij}}\big]^{-1}$.

The first quadratic form of space $\mathbb{R}_{2N}$ is

$$ds^2 = g_{\underline{pq}} dx^p dx^q. \tag{2}$$

The Christoffell symbols of space $\mathbb{R}_{2N}$ are

$$\Gamma_{i.\underline{jk}} = \frac{1}{2}\big(g_{\underline{ji},k} - g_{\underline{jk},i} + g_{\underline{ik},j}\big).$$

The Christoffel symbols of the second kind of space $\mathbb{R}_{2N}$, which are the affine connection coefficients of space $\mathbb{R}_{2N}$ are

$$\Gamma^i_{\underline{jk}} = g^{\underline{ip}}\Gamma_{p.\underline{jk}} = \frac{1}{2}g^{\underline{ip}}\big(g_{\underline{jp},k} - g_{\underline{jk},p} + g_{\underline{pk},j}\big) = \Gamma^i_{\underline{kj}}.$$

The covariant derivative of a tensor a^i_j of type $(1,1)$ with respect to the affine connection $\Gamma^i_{\underline{jk}}$ is

$$a^i_{j\underset{g}{|}k} = a^i_{j,k} + \Gamma^i_{\underline{pk}}a^p_j - \Gamma^p_{\underline{jk}}a^i_p.$$

2.2. Symmetric and Non-Symmetric Affine Connection Spaces

The generalization of the concept of Riemannian space consists of symmetric and non-symmetric affine connection spaces. An n-dimensional manifold $\mathcal{M}_n$ equipped with a symmetric metric affine connection $\overset{0}{\nabla}$, whose coefficients are $L^i_{\underline{jk}}$, $L^i_{\underline{jk}} = L^i_{\underline{kj}}$, is the symmetric affine connection space $\mathbb{A}_n$ $\big($see [1–3]$\big)$.

The covariant derivative of a tensor $\hat{a}$ of type $(1,1)$ with respect to the symmetric affine connection $\overset{0}{\nabla}$ is [1–3]

$$a^i_{j|k} = a^i_{j,k} + L^i_{\underline{pk}} a^p_j - L^p_{\underline{jk}} a^i_p,$$

where partial derivation is marked by a comma.

The corresponding Ricci identity is

$$a^i_{j|m|n} - a^i_{j|n|m} = a^p_j R^i_{pmn} - a^i_p R^p_{jmn},$$

for the curvature tensor of space $\mathbb{R}_n$ expressed as [3]

$$R^i_{jmn} = L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + L^p_{\underline{jm}} L^i_{\underline{pn}} - L^p_{\underline{jn}} L^p_{\underline{jm}}. \tag{3}$$

An n-dimensional manifold $\mathcal{M}_n$ equipped with a non-symmetric affine connection ∇, whose coefficients are L^i_{jk}, $L^i_{jk} \neq L^i_{kj}$ for at least one pair of indices (j,k), is the non-symmetric affine connection space $\mathbb{GA}_n$ (see [8–19]).

The symmetric and anti-symmetric parts of affine coefficients L^i_{jk} are

$$L^i_{\underline{jk}} = \frac{1}{2}\big(L^i_{jk} + L^i_{kj}\big), \quad T^i_{jk} = L^i_{\underset{\vee}{jk}} = \frac{1}{2}\big(L^i_{jk} - L^i_{kj}\big).$$

The components $L^i_{\underline{jk}}$ are components of coefficients of a symmetric affine connection. This symmetric affine connection is the affine connection of associated space $\mathbb{GA}_K$. The components $L^i_{\underset{\vee}{jk}}$ are components of a tensor of type $(1,2)$. The tensor $S^i_{jk} = 2L^i_{\underset{\vee}{jk}}$ is the torsion tensor of space $\mathbb{GA}_K$. It holds the equality $L^i_{jk} = L^i_{\underline{jk}} + L^i_{\underset{\vee}{jk}}$.

S. M. Minčić found four kinds of covariant derivatives of tensor $\hat{a}$ of type $(1,1)$ with respect to non-symmetric affine connection [9–14]

$$a^i_{j\underset{1}{|}k} = a^i_{j,k} + L^i_{pk} a^p_j - L^p_{jk} a^i_p, \tag{4}$$

$$a^i_{j\underset{2}{|}k} = a^i_{j,k} + L^i_{kp} a^p_j - L^p_{kj} a^i_p, \tag{5}$$

$$a^i_{j\underset{3}{|}k} = a^i_{j,k} + L^i_{pk} a^p_j - L^p_{kj} a^i_p, \tag{6}$$

$$a^i_{j\underset{4}{|}k} = a^i_{j,k} + L^i_{kp} a^p_j - L^p_{jk} a^i_p. \tag{7}$$

N. O. Vesić proved that three of these four kinds of covariant derivatives were linearly independent [20].

With respect to the four kinds of covariant derivatives (4)–(7), S. M. Minčić obtained four curvature tensors, eight derived curvature tensors, and fifteen curvature pseudotensors of space $\mathbb{GA}_n$.

Minčić's work has been continued by many scientists, including M. Stanković [15], M. Zlatanović [15,19], Lj. S. Velimirović [14–18], and many others.

N. O. Vesić [20,21] and D. J. Simjanović [21] completed the research realized in [20] where it was proved that just curvature tensors could be obtained from the differences $a^i_{j\underset{p}{|}m\underset{q}{|}n} - a^i_{j\underset{r}{|}n\underset{s}{|}m}$, $p,q,r,s \in \{1,2,3,4\}$.

The curvature tensors of space $\mathbb{GA}_n$ are elements of family

$$K^i_{jmn} = R^i_{jmn} + uT^i_{jm|n} + u'T^i_{jn|m} + vT^p_{jm}T^i_{pn} + v'T^p_{jn}T^i_{pm} + wT^p_{mn}T^i_{pj}. \tag{8}$$

The next theorem about Bianchi identities, as a generalization of Bianchi identities presented in [1–3], is proved in the following.

Theorem 1. *In a non-symmetric affine connection space $\mathbb{GA}_N$, with affine connection ∇ whose coefficients are $L^i_{jk} = L^i_{\underline{jk}} + L^i_{\underset{\vee}{jk}}$, the family of curvature tensors K^i_{jmn} satisfies the family of first generalized Bianchi identities*

$$\begin{aligned} K^i_{jmn} + K^i_{mnj} + K^i_{njm} &= (u - u')\big(T^i_{jm|n} - T^i_{jn|m} + T^i_{mn|j}\big) \\ &\quad + (v - v' + w)\big(T^p_{pn}T^i_{pn} - T^p_{jn}T^i_{pm} + T^p_{mn}T^i_{pj}\big) \end{aligned} \tag{9}$$

With respect to the covariant derivative with respect to the affine connection $L^i_{\underline{jk}}$ denoted by "$|$", the following equation holds:

$$\begin{aligned} K^i_{jmn|k} + K^i_{jnk|m} + K^i_{jkm|n} &= u\big(T^i_{jm|n|k} + T^i_{jn|k|m} + T^i_{jk|m|n}\big) \\ &\quad + u'\big(T^i_{jm|k|n} + T^i_{jn|m|k} + T^i_{jk|n|m}\big) \\ &\quad + v\big((T^p_{jn}T^i_{pk})_{|m} + (T^p_{jk}T^i_{pm})_{|n} + (T^p_{jm}T^i_{pn})_{|k}\big) \\ &\quad + v'\big((T^p_{jk}T^i_{pn})_{|m} + (T^p_{jm}T^i_{pk})_{|n} + (T^p_{jn}T^i_{pm})_{|k}\big) \\ &\quad + w\big((T^p_{nk}T^i_{pj})_{|m} + (T^p_{km}T^i_{pj})_{|n} + (T^p_{mn}T^i_{pj})_{|k}\big). \end{aligned} \tag{10}$$

Proof. To simplify calculations in this theorem, we involve the term

$$L^i_{\underline{jm}|n} = L^i_{\underline{jm},n} + L^i_{\underline{pn}}L^i_{\underline{pm}} - L^p_{\underline{jn}}L^i_{\underline{pm}} - L^p_{\underline{mn}}L^i_{\underline{jp}}. \tag{11}$$

The affine connection coefficients $L^i_{\underline{jk}}$ are not tensors. Thus, the equality (11) does not represent a covariant derivative of $L^i_{\underline{jk}}$ but just simplifies the writing of its right-hand side. This abbreviation is significant because if we define $L^i_{\underline{jm}|n|k} = L^i_{\underline{jm}|n,k} + L^i_{\underline{pk}}L^p_{\underline{jm}|n} - L^p_{\underline{jk}}L^i_{\underline{pm}|n} - L^p_{\underline{mk}}L^i_{\underline{jp}|n} - L^p_{\underline{nk}}L^i_{\underline{jm}|p}$ and substitute the definition (11) in this relation and antisymmetrize it by n and k, we get

$$L^i_{\underline{jm}|n|k} - L^i_{\underline{jm}|k|n} = R^i_{pnk}L^p_{\underline{jm}} - R^p_{jnk}L^i_{\underline{pm}} - R^p_{mnk}L^i_{\underline{pj}}. \tag{12}$$

We are now ready to start the proof of this theorem.

First, the curvature tensor of symmetric affine connection space $\mathbb{R}_N$ is

$$\begin{aligned} R^i_{jmn} &= L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + L^p_{\underline{jm}}L^i_{\underline{pn}} - L^p_{\underline{jn}}L^i_{\underline{pm}} \\ &= L^i_{\underline{jm}|n} - L^i_{\underline{jn}|m} - L^p_{\underline{jm}}L^i_{\underline{pn}} + L^p_{\underline{jn}}L^i_{\underline{pm}}, \end{aligned} \tag{13}$$

for $L^i_{\underline{jm}|n}$ defined by (11).

With respect to the definition of family K^i_{jmn} of curvature tensors, we obtain

$$\begin{aligned} K^i_{jmn} + K^i_{mnj} + K^i_{njm} &= R^i_{jmn} + R^i_{mnj} + R^i_{njm} + u\big(T^i_{jm|n} + T^i_{mn|j} + T^i_{nj|m}\big) \\ &\quad + u'\big(T^i_{jn|m} + T^i_{nm|j} + T^i_{mj|n}\big) + v\big(T^p_{jm}T^i_{pn} + T^p_{mn}T^i_{pj} + T^p_{nj}T^i_{pm}\big) \\ &\quad + v'\big(T^p_{jn}T^i_{pm} + T^p_{mj}T^i_{pn} + T^p_{nm}T^i_{pj}\big) + w\big(T^p_{mn}T^i_{pj} + T^p_{nj}T^i_{pm} + T^p_{jm}T^i_{pn}\big) \\ &= \big(L^i_{\underline{jm}|n} - L^i_{\underline{jn}|m} - L^p_{\underline{jm}}L^i_{\underline{pn}} + L^p_{\underline{jn}}L^i_{\underline{pm}}\big) \\ &\quad + \big(L^i_{\underline{mn}|j} - L^i_{\underline{mj}|n} - L^p_{\underline{mn}}L^i_{\underline{pj}} + L^p_{\underline{mj}}L^i_{\underline{pn}}\big) \\ &\quad + \big(L^i_{\underline{nj}|m} - L^i_{\underline{nm}|j} - L^p_{\underline{nj}}L^i_{\underline{pm}} + L^p_{\underline{nm}}L^i_{\underline{pj}}\big) \\ &\quad + (u - u')\big(T^i_{jm|n} - T^i_{jn|m} + T^i_{mn|j}\big) \\ &\quad + (v - v' + w)\big(T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm} + T^p_{mn}T^i_{pj}\big). \end{aligned}$$

along with relation (9), which proves the first part of this theorem.

After expressing the curvature tensor R^i_{jmn} in terms of $L^i_{\underline{jm}|n}$ and $L^i_{\underline{jn}|m}$ as in (13), we get

$$
\begin{aligned}
K^i_{jmn|k} = {} & L^i_{\underline{jm}|n|k} - L^i_{\underline{jn}|m|k} - L^p_{\underline{jm}|k} L^i_{\underline{pn}} - L^p_{\underline{jm}} L^i_{\underline{pn}|k} + L^p_{\underline{jn}|k} L^i_{\underline{pm}} + L^p_{\underline{jn}} L^i_{\underline{pm}|k} \\
& + uT^i_{jm|n|k} + u'T^i_{jn|m|k} + vT^p_{jm|k}T^i_{pn} + vT^p_{jm}T^i_{pn|k} \qquad (14) \\
& + v'T^p_{jn|k}T^i_{pm} + v'T^p_{jn}T^i_{pm|k} + wT^p_{mn|k}T^i_{pj} + wT^p_{mn}T^i_{pj|k}, \\
K^i_{jnk|m} = {} & L^i_{\underline{jn}|k|m} - L^i_{\underline{jk}|n|m} - L^p_{\underline{jn}|m} L^i_{\underline{pk}} - L^p_{\underline{jn}} L^i_{\underline{pk}|m} + L^p_{\underline{jk}|m} L^i_{\underline{pn}} + L^p_{\underline{jk}} L^i_{\underline{pn}|m} \\
& + uT^i_{jn|k|m} + u'T^i_{jk|n|m} + vT^p_{jn|m}T^i_{pk} + vT^p_{jn}T^i_{pk|m} \qquad (15) \\
& + v'T^p_{jk|m}T^i_{pn} + v'T^p_{jk}T^i_{pn|m} + wT^p_{nk|m}T^i_{pj} + wT^p_{nk}T^i_{pj|m}, \\
K^i_{jkm|n} = {} & L^i_{\underline{jk}|m|n} - L^i_{\underline{jm}|k|n} - L^p_{\underline{jk}|n} L^i_{\underline{pm}} - L^p_{\underline{jk}} L^i_{\underline{pm}|n} + L^p_{\underline{jm}|n} L^i_{\underline{pk}} + L^p_{\underline{jm}} L^i_{\underline{pk}|n} \\
& + uT^i_{jk|m|n} + u'T^i_{jm|k|n} + vT^p_{jk|n}T^i_{pm} + vT^p_{jk}T^i_{pm|n} \qquad (16) \\
& + v'T^p_{jm|n}T^i_{pk} + v'T^p_{jm}T^i_{pk|n} + wT^p_{km|n}T^i_{pj} + wT^p_{km}T^i_{pj|n}.
\end{aligned}
$$

With respect to the equalities (14)–(16), we obtain the following sum

$$
\begin{aligned}
K^i_{jmn|k} + K^i_{jnk|m} + K^i_{jkm|n} = {} & \big(L^i_{\underline{jm}|n|k} - L^i_{\underline{jm}|k|n}\big) - \big(L^i_{\underline{jn}|m|k} - L^i_{\underline{jn}|k|m}\big) \\
& + \big(L^i_{\underline{jk}|m|n} - L^i_{\underline{jk}|n|m}\big) + L^i_{\underline{pm}}\big(L^p_{\underline{jn}|k} - L^p_{\underline{jk}|n}\big) \\
& - L^i_{\underline{pn}}\big(L^p_{\underline{jm}|k} - L^p_{\underline{jk}|m}\big) + L^i_{\underline{pk}}\big(L^p_{\underline{jm}|n} - L^p_{\underline{jn}|m}\big) \\
& - L^p_{\underline{jm}}\big(L^i_{\underline{pn}|k} - L^i_{\underline{pk}|n}\big) + L^p_{\underline{jn}}\big(L^i_{\underline{pm}|k} - L^i_{\underline{pk}|m}\big) \\
& - L^p_{\underline{jk}}\big(L^i_{\underline{pm}|n} - L^i_{\underline{pn}|m}\big) + u\big(T^i_{jm|n|k} + T^i_{jn|k|m} + T^i_{jk|m|n}\big) \qquad (17) \\
& + u'\big(T^i_{jm|k|n} + T^i_{jn|m|k} + T^i_{jk|n|m}\big) \\
& + v\big((T^p_{jn}T^i_{pk})_{|m} + (T^p_{jk}T^i_{pm})_{|n} + (T^p_{jm}T^i_{pn})_{|k}\big) \\
& + v'\big((T^p_{jk}T^i_{pn})_{|m} + (T^p_{jm}T^i_{pk})_{|n} + (T^p_{jn}T^i_{pm})_{|k}\big) \\
& + w\big((T^p_{nk}T^i_{pj})_{|m} + (T^p_{km}T^i_{pj})_{|n} + (T^p_{mn}T^i_{pj})_{|k}\big).
\end{aligned}
$$

Based on Equations (12) and (13), we transform the previous relation to

$$
\begin{aligned}
K^i_{jmn|k} + K^i_{jnk|m} + K^i_{jkm|n} = {} & \big(R^i_{pnk}L^p_{\underline{jm}} - R^p_{jnk}L^i_{\underline{pm}} - R^p_{mnk}L^i_{\underline{jp}}\big) \\
& - \big(R^i_{pmk}L^p_{\underline{jn}} - R^p_{jmk}L^i_{\underline{pn}} - R^p_{nmk}L^i_{\underline{jp}}\big) \\
& + \big(R^i_{pmn}L^p_{\underline{jk}} - R^p_{jmn}L^i_{\underline{pk}} - R^p_{kmn}L^i_{\underline{jp}}\big) \\
& + L^i_{\underline{pm}}\big(R^p_{jnk} + L^q_{\underline{jn}}L^p_{\underline{qk}} - L^q_{\underline{jk}}L^p_{\underline{qn}}\big) \\
& - L^i_{\underline{pn}}\big(R^p_{jmk} + L^q_{\underline{jm}}L^p_{\underline{qk}} - L^q_{\underline{jk}}L^p_{\underline{qm}}\big) \\
& + L^i_{\underline{pk}}\big(R^p_{jmn} + L^q_{\underline{jm}}L^p_{\underline{qn}} - L^q_{\underline{jn}}L^p_{\underline{qm}}\big) \\
& - L^p_{\underline{jm}}\big(R^i_{pnk} + L^q_{\underline{pn}}L^i_{\underline{qk}} + L^q_{\underline{pk}}L^i_{\underline{qn}}\big) \\
& + L^p_{\underline{jn}}\big(R^i_{pmk} + L^q_{\underline{pm}}L^i_{\underline{qn}} - L^q_{\underline{pk}}L^i_{\underline{qm}}\big) \\
& - L^p_{\underline{jk}}\big(R^i_{pmn} + L^q_{\underline{pm}}L^i_{\underline{qn}} - L^q_{\underline{pn}}L^i_{\underline{qm}}\big) \\
& + u\big(T^i_{jm|n|k} + T^i_{jn|k|m} + T^i_{jk|m|n}\big) \\
& + u'\big(T^i_{jm|k|n} + T^i_{jn|m|k} + T^i_{jk|n|m}\big) \\
& + v\big((T^p_{jn}T^i_{pk})_{|m} + (T^p_{jk}T^i_{pm})_{|n} + (T^p_{jm}T^i_{pn})_{|k}\big)
\end{aligned}
$$

$$+ v'\big((T^p_{jk}T^i_{pn})_{|m} + (T^p_{jm}T^i_{pk})_{|n} + (T^p_{jn}T^i_{pm})_{|k}\big)$$
$$+ w\big((T^p_{nk}T^i_{pj})_{|m} + (T^p_{km}T^i_{pj})_{|n} + (T^p_{mn}T^i_{pj})_{|k}\big).$$

Since $R^i_{jmn} = -R^i_{jnm}$ and $R^p_{mnk} + R^p_{nkm} + R^p_{kmn} = 0$, we reduce the last equality to Equation (10), which completes the proof of this theorem. □

The equalities (9) and (10) are families of the first and second generalized Bianchi identities.

Linearly Independent Curvature Tensors

In an attempt to generalize initial research about curvature tensors of symmetric affine connection [1,3], S. M. Minčić [9–13] concluded that from the difference $a^i_{j\underset{p}{|}m\underset{q}{|}n} - a^i_{j\underset{r}{|}n\underset{s}{|}m}$, five linearly independent curvature tensors could be obtained:

$$\underset{1}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{pn} - L^p_{jn}L^i_{pm}, \tag{18}$$
$$\underset{2}{R}{}^i_{jmn} = L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{mp}, \tag{19}$$
$$\underset{3}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + 2L^p_{nm}T^i_{pj}, \tag{20}$$
$$\underset{4}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + 2L^p_{mn}T^i_{pj}, \tag{21}$$
$$\underset{5}{R}{}^i_{jmn} = L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + \frac{1}{2}\big(L^p_{jm}L^i_{pn} + L^p_{mj}L^i_{np} - L^p_{jn}L^i_{mp} - L^p_{nj}L^i_{pm}\big). \tag{22}$$

These five linearly independent curvature tensors are expressed as functions of the curvature tensor R^i_{jmn} of associated space $\mathbb{A}_n$ and torsion tensor as

$$\underset{1}{R}{}^i_{jmn} = R^i_{jmn} + T^i_{jm|n} - T^i_{jn|m} + T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}, \tag{23}$$
$$\underset{2}{R}{}^i_{jmn} = R^i_{jmn} - T^i_{jm|n} + T^i_{jn|m} + T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}, \tag{24}$$
$$\underset{3}{R}{}^i_{jmn} = R^i_{jmn} + T^i_{jm|n} + T^i_{jn|m} - T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm} - 2T^p_{mn}T^i_{pj}, \tag{25}$$
$$\underset{4}{R}{}^i_{jmn} = R^i_{jmn} + T^i_{jm|n} + T^i_{jn|m} - T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm} + 2T^p_{mn}T^i_{pj}, \tag{26}$$
$$\underset{5}{R}{}^i_{jmn} = R^i_{jmn} + T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm}. \tag{27}$$

In the research of N. O. Vesić [20,21] and D. J. Simjanović [21], the six linearly independent curvature tensors of space $\mathbb{GA}_n$ were obtained. Six linearly independent curvature tensors of this space are

$$\begin{aligned}
\underset{1}{\tilde{R}}{}^i_{jmn} &= R^i_{jmn} + T^i_{jm|n} - T^i_{jn|m} + T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm} + 2T^p_{mn}T^i_{pj},\\
\underset{2}{\tilde{R}}{}^i_{jmn} &= R^i_{jmn} + T^i_{jm|n} - T^i_{jn|m} + T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm},\\
\underset{3}{\tilde{R}}{}^i_{jmn} &= R^i_{jmn} + T^i_{jm|n} - T^i_{jn|m} - T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm},\\
\underset{4}{\tilde{R}}{}^i_{jmn} &= R^i_{jmn} + T^i_{jm|n} - T^i_{jn|m} - T^p_{jm}T^i_{pn} - 3T^p_{jn}T^i_{pm},\\
\underset{5}{\tilde{R}}{}^i_{jmn} &= R^i_{jmn} + T^i_{jm|n} + T^i_{jn|m} + T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm},\\
\underset{6}{\tilde{R}}{}^i_{jmn} &= R^i_{jmn} - T^p_{jm|n} - T^p_{jn|m} - T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}.
\end{aligned} \tag{28}$$

The curvature tensors $\underset{1}{R}{}^i_{jmn}, \dots, \underset{6}{R}{}^i_{jmn}$ are the elements of family (8) for the following six $c = (u, u', v, v', w)$: $\underset{1}{c} = (1, -1, 1, -1, 2)$, $\underset{2}{c} = (1, -1, 1, 1, 0)$, $\underset{3}{c} = (1, -1, -1, -1, 0)$, $\underset{4}{c} = (1, -1, -1, -3, 0)$, $\underset{5}{c} = (1, 1, 1, 1, 0)$, and $\underset{6}{c} = (-1, -1, -1, -1, 0)$. If we substitute these values of u, u', v, v', and w into Equations (9) and (10), we obtain generalized Bjanchi identities which correspond to these six curvature tensors.

The linearly independent curvature tensors $\underset{1}{\tilde{R}}{}^i_{jmn}, \ldots, \underset{5}{\tilde{R}}{}^i_{jmn}$ given by (23)–(27) and $\underset{3}{\tilde{R}}{}^i_{jmn}$ given by (28) are linearly independent. Hence, our study in this research is based on these six linearly independent curvature tensors.

2.3. Almost Complex Manifolds

The $n = 2N$-dimensional affine connection spaces were studied in Yano's work [4]. A $2N$-dimensional manifold $\mathcal{M}_{2N} = \mathcal{M}_{2N}(x^1, \ldots, x^{2N})$ equipped with a structural affinor F^h_i which satisfies the equality

$$F^s_i F^i_t = -\delta^s_t, \tag{29}$$

is an almost complex manifold [4].

The operators O^{hs}_{ri} and ${}^*O^{hs}_{ri}$ are defined as

$$O^{hs}_{ri} = \frac{1}{2}\left(\delta^h_r \delta^s_i - F^h_r F^s_i\right),$$
$${}^*O^{hs}_{ri} = \frac{1}{2}\left(\delta^h_r \delta^s_i + F^h_r F^s_i\right).$$

The affine connection of an almost complex space, whose coefficients are L^i_{jk}, is the F-connection if the affinor F^h_i is covariantly constant with respect to that connection, i.e., $F^j_{i\underset{1}{|}k} = 0$. Since

$$F^j_{i|k} = F^j_{i\underset{1}{|}k} - T^j_{lk} F^l_i + T^l_{ik} F^j_l,$$

the F-connection satisfies the equality

$$F^j_{i|k} = -T^j_{lk} F^l_i + T^l_{ik} F^j_l.$$

The F-connection L^i_{jk} is the half-symmetric connection if its torsion tensor satisfies the equality

$$O^{hs}_{rk} O^{kt}_{ij} S^r_{st} = 0.$$

For the half-symmetric affine connection, the next equivalences are satisfied:

$$\begin{aligned}
S^h_{ij} &= F^s_j F^h_r S^r_{is} + F^t_i F^s_j S^h_{ts} + F^h_r F^t_i S^r_{tj},\\
T^h_{ij} &= F^s_j F^h_r T^r_{is} + F^t_i F^s_j T^h_{ts} + F^h_r F^t_i T^r_{tj},\\
F^s_i F^i_t &= -\delta^s_t,\\
F^j_{i|k} &= -T^j_{lk} F_i{}^l + T^l_{ik} F_l{}^j.
\end{aligned} \tag{30}$$

2.4. Motivation

The structure $F^i_j = g^{ip} F_{pj}$ is used in mathematics (F-plannar mappings [22–24]). Invariants for F-planar mappings and transformation rules caused by F^i_j are the main subjects of that research.

This structure is also widely applied in physics. The anti-symmetric tensor F_{ij} corresponds to electromagnetism [25]. In [26], the importance of the structure $F_{ij}F^{ij}$ for the Einstein–Maxwell cosmological model is demonstrated.

In this article, our attention is focused on theoretical aspects of transformations of affine connection caused by F^i_j, but these results will be directly applied in future generalizations of gravity theory. Because our research generalizes Yano's results, the tensor F_{ij} is anti-symmetric by i and j.

2.5. Research Purposes

In [27], we studied the affine connection spaces whose affine connection coefficients were of the form

$$\overset{1}{L}{}^i_{jk} = L^i_{\underline{jk}} - \frac{1}{2} F^i_p F^p_{j;k}, \tag{31}$$

for a covariant derivative with respect to the affine connection $L^i_{\underline{jk}}$. This is a special type of half-symmetric affine connection. In this article, we reduce our subject of research to the affine connection spaces whose affine connection coefficients are

$$\overset{1}{\Gamma}{}^i_{jk} = \Gamma^i_{\underline{jk}} - \frac{1}{2} F^i_p F^p_{j\underset{g}{|}k}, \tag{32}$$

This research start with a special case of half-symmetric affine connection. Following the methodology used in Yano's work [4], we restrict that concept for general affine connections expressed as in (32). After that, the F-connections (affine connections which cause affinors to vanish by covariant differentiation) are studied. The term of the Killing vector [4] is generalized with respect to the definition of a half-symmetric affine connection (32).

3. Results in Almost Hermitian Spaces

The almost complex manifold $\mathcal{M}_{2N}$ generated with the positive definite quadratic form (2) such that the following equations are satisfied

$$g_{\underline{ij}} = F^t_i F^s_j g_{\underline{ts}}, \tag{33}$$
$$F_{ij} = F^t_j g_{\underline{ti}}, \tag{34}$$

is the almost Hermitian space. An almost Hermitian space in which the following equation is satisfied

$$F^i_{j\underset{g}{|}k} + F^i_{k\underset{g}{|}j} = 0$$

is an almost Tachibana space (nearly Kählerian space) [4].

Based on Equations (30) and (33), the following equality holds $F_{ij} = -F_{ji}$. The Riemannian metric $g_{\underline{ij}}$ which satisfies (33) is the Hermitian metric. The almost complex space $\mathcal{M}_{2N}$ equipped with the Hermitian metric is the almost Hermitian space.

In the almost Hermitian space, we may analyze the special case of half-symmetric affine connection (31), given by (32). The affine connection (32) is the first connection (canonic connection). For the affine connection (32), the following equalities are satisfied

$$g_{\underline{ij}\underset{1}{|}k} = 0, \tag{35}$$
$$F_{ij\underset{1}{|}k} = 0,$$

for the covariant derivative $\underset{1}{|}$ with respect to the affine connection whose coefficients are $\overset{1}{\Gamma}{}^i_{jk}$.

Let us prove the next theorem.

Theorem 2. *Let $\Gamma^i_{\underline{jk}}$ be the second-kind Christoffell symbols in an almost Hermitian space. For the affine connection coefficients $\overset{1}{\Gamma}{}^i_{jk}$, the next equalities are satisfied*

$$\overset{1}{S}{}^i_{jk} = -\overset{2}{S}{}^i_{jk} = 0 \iff F^i_{j\underset{g}{|}k} = F^i_{k\underset{g}{|}j}, \quad \overset{1}{T}{}^i_{jk} = -\overset{2}{T}{}^i_{jk} = 0 \iff F^i_{j\underset{g}{|}k} = F^i_{k\underset{g}{|}j},$$

i.e., $\overset{1}{\Gamma}{}^i_{jk} = \overset{1}{\Gamma}{}^i_{\underline{jk}}$ *if and only if the tensor* $F^i_{\underset{g}{j|k}}$ *is symmetric by* j *and* k.

Proof. Based on Equation (32), the torsion tensor and its half are

$$\overset{1}{S}{}^i_{jk} = 2\overset{1}{\Gamma}{}^i_{\underset{\vee}{jk}} = \frac{1}{2} F^i_p \left(F^p_{\underset{g}{k|j}} - F^p_{\underset{g}{j|k}} \right), \tag{36}$$

$$\overset{1}{T}{}^i_{jk} = \overset{1}{\Gamma}{}^i_{\underset{\vee}{jk}} = \frac{1}{4} F^i_p \left(F^p_{\underset{g}{k|j}} - F^p_{\underset{g}{j|k}} \right).$$

The symmetric and dual affine connection of the connection (32) are, respectively,

$$\overset{1}{\Gamma}{}^i_{\underline{jk}} = \frac{1}{2}\left(\overset{1}{\Gamma}{}^i_{jk} + \overset{1}{\Gamma}{}^i_{kj} \right) = \Gamma^i_{\underline{jk}} - \frac{1}{4} F^i_p \left(F^p_{\underset{g}{j|k}} + F^p_{\underset{g}{k|j}} \right), \tag{37}$$

$$\overset{2}{\Gamma}{}^i_{jk} = \overset{1}{\Gamma}{}^i_{kj} = \overset{1}{\Gamma}{}^i_{jk} - \overset{1}{S}{}^i_{jk} = \Gamma^i_{\underline{jk}} - \frac{1}{2} F^i_p F^p_{\underset{g}{k|j}}. \tag{38}$$

The affine connection (32) can be expressed as

$$\overset{1}{\Gamma}{}^i_{jk} = \overset{1}{\Gamma}{}^i_{\underline{jk}} + \overset{1}{\Gamma}{}^i_{\underset{\vee}{jk}} = \overset{1}{\Gamma}{}^i_{\underline{jk}} + \overset{1}{T}{}^i_{jk} = \overset{1}{\Gamma}{}^i_{\underline{jk}} + \frac{1}{4} F^i_p \left(F^p_{\underset{g}{k|j}} - F^p_{\underset{g}{j|k}} \right).$$

The torsion tensor $\overset{2}{S}{}^i_{jk} = \overset{1}{S}{}^i_{kj} = 2\overset{1}{T}{}^i_{kj}$, such as its half $\overset{2}{T}{}^i_{jk} = \overset{1}{T}{}^i_{kj} = \frac{1}{4} F^i_p \left(F^p_{\underset{g}{j|k}} - F^p_{\underset{g}{k|j}} \right)$, because of their anti-symmetries by j and k, satisfy the next equalities

$$\overset{2}{S}{}^i_{jk} = -\overset{1}{S}{}^i_{jk}, \quad \overset{2}{T}{}^i_{jk} = -\overset{1}{T}{}^i_{jk},$$

which completes proof of this theorem. □

The next theorem gives the necessary and sufficient condition for a dual connection to be an F-connection.

Theorem 3. *The dual connection* $\overset{2}{\Gamma}{}^i_{jk}$ *of the affine connection* $\overset{1}{\Gamma}{}^i_{jk}$ *is an F-connection if and only if the equality* $F^i_{\underset{g}{j|k}} = \frac{1}{2} F^i_{\underset{g}{k|j}} - \frac{1}{2} F^i_{\underset{g}{p|q}} F^q_j F^p_k$ *is satisfied.*

Proof. Because $\overset{1}{\Gamma}{}^i_{jk}$ is the F-connection, the next equality holds $F^i_{\underset{1}{j|k}} = 0$, where $\underset{1}{|}$ is the covariant derivative with respect to the affine connection $\overset{1}{\Gamma}{}^i_{jk}$. Based on the equality $F^i_{\underset{1}{j|k}} = 0$ and Equation (38), we obtain

$$F^i_{\underset{2}{j|k}} = F^i_{\underset{g}{j|k}} - \frac{1}{2} F^i_{\underset{g}{k|j}} + \frac{1}{2} F^i_{\underset{g}{p|q}} F^q_j F^p_k,$$

where $\underset{2}{|}$ is the covariant derivative with respect to the affine connection $\overset{2}{\Gamma}{}^i_{jk}$. □

The next theorem is a logical extension of the previous one.

Theorem 4. *The symmetric affine connection* $\overset{1}{\Gamma}{}^i_{\underline{jk}}$ *of the connection* $\overset{1}{\Gamma}{}^i_{jk}$ *is an F-connection if and only if the equality* $F^i_{j\underset{g}{|}k} = \frac{1}{2}F^i_{k\underset{g}{|}j} - \frac{1}{2}F^i_{p\underset{g}{|}q}F^q_j F^p_k$ *holds.*

Proof. From the equality $F^i_{j\underset{1}{|}k} = 0$ and Equation (37), we obtain

$$F^i_{j\underset{0}{|}k} = \frac{1}{2}\left(F^i_{j\underset{g}{|}k} - \frac{1}{2}F^i_{k\underset{g}{|}j} + \frac{1}{2}F^i_{p\underset{g}{|}q}F^q_j F^p_k\right),$$

where $\underset{0}{|}$ is the covariant derivative with respect to the symmetric affine connection $\overset{1}{\Gamma}{}^i_{\underline{jk}}$. □

Theorem 5. *The symmetric part* $\overset{1}{\Gamma}{}^i_{\underline{jk}}$ *of the affine connection* $\overset{1}{\Gamma}{}^i_{jk}$ *is a metric connection if and only if the next equality is satisfied* $F^i_{j\underset{g}{|}k} + F^i_{k\underset{g}{|}j} = 0$*, i.e., if and only if an almost Hermitian space is an almost Tachibana space.*

Proof. Based on Equation (30) and the covariant derivative $\underset{g}{|}$, we get

$$F^p_{i\underset{g}{|}k}F_{jp} + F^p_{j\underset{g}{|}k}F_{ip} = 0. \tag{39}$$

With respect to Equation (39), we obtain the following equality

$$g_{\underline{ij}\underset{0}{|}k} = \frac{1}{4}F_{pk}\left(F^p_{j\underset{g}{|}i} + F^p_{i\underset{g}{|}j}\right),$$

which completes the proof of this theorem. □

Theorem 6. *The tensor* F_{ij} *is covariantly constant with respect to the symmetric part* $\overset{1}{\Gamma}{}^i_{\underline{jk}}$ *of the affine connection* $\overset{1}{\Gamma}{}^i_{jk}$ *if and only if the equality* $F_{ij\underset{g}{|}k} = \frac{1}{2}F_{kj\underset{g}{|}i} + \frac{1}{2}F_{ik\underset{g}{|}j}$ *is satisfied.*

Proof. Using Equations (29) and (34), we get

$$F_{ip}F^p_j = -g_{\underline{ij}}, \tag{40}$$

$$F_{pi}F^p_j = g_{\underline{ij}}. \tag{41}$$

From Equations (34), (40), and (41), one obtains

$$F_{ij\underset{0}{|}k} = \frac{1}{2}\left(F_{ij\underset{g}{|}k} + \frac{1}{2}F_{jk\underset{g}{|}i} + \frac{1}{2}F_{ki\underset{g}{|}j}\right),$$

such as

$$F_{ij\underset{0}{|}k} = F_{ij,k} - L^p_{\underline{ik}}F_{pj} - L^p_{\underline{jk}}F_{ip} = F_{ij,k} - \overset{1}{\Gamma}{}^p_{\underline{ik}}F_{pj} - \overset{1}{\Gamma}{}^p_{\underline{jk}}F_{ip},$$

These relations, together with the anti-symmetry of F_{ij} by i and j (see [4], p. 126), complete the proof of this theorem. □

Theorem 7. *The dual connection* $\overset{2}{\Gamma}{}^i_{jk}$ *of the affine connection* $\overset{1}{\Gamma}{}^i_{jk}$ *is a metric connection if and only if the equality* $F^i_{\underset{g}{j|k}} + F^i_{\underset{g}{k|j}} = 0$ *is satisfied, i.e., if and only if the almost Hermitian space is the Tachibana space.*

Proof. With respect to $g_{\underset{g}{\underline{ij}|k}} = 0$, we obtain

$$g_{\underset{2}{\underline{ij}|k}} = \frac{1}{2} F_{pk} \left(F^p_{\underset{g}{j|i}} + F^p_{\underset{g}{i|j}} \right),$$

which proves the theorem. □

Theorem 8. *The tensor* F_{ij} *is covariantly constant with respect to the dual affine connection* $\overset{2}{\Gamma}{}^i_{jk}$ *of the affine connection* $\overset{1}{\Gamma}{}^i_{jk}$ *if and only if the next equality* $F_{\underset{g}{ij|k}} = \frac{1}{2} F_{\underset{g}{kj|i}} + \frac{1}{2} F_{\underset{g}{ik|j}}$ *holds.*

Proof. Based on Equations (34), (40) and (41), we obtain

$$F_{\underset{2}{ij|k}} = F_{\underset{g}{ij|k}} + \frac{1}{2} F_{\underset{g}{jk|i}} + \frac{1}{2} F_{\underset{g}{ki|j}}. \tag{42}$$

The equality (42), together with the relation $F_{ij} = -F_{ji}$ (see [4], p. 126), confirms the validity of this theorem. □

The properties of torsion tensor (36) with respect to the affinor F^i_j are examined below. For the torsion tensor of first connection (36), we obtain

$$F^i_p \overset{1}{S}{}^p_{jk} = \frac{1}{2} \left(F^i_{\underset{g}{j|k}} - F^i_{\underset{g}{k|j}} \right),$$

$$F^q_p \overset{1}{S}{}^p_{qi} = -\frac{1}{2} F^r_{\underset{g}{i|r}} = \frac{1}{2} F_i,$$

$$F^p_j \overset{1}{S}{}^i_{pk} = -\frac{1}{2} \left(F^i_{\underset{g}{j|k}} - F^i_t F^p_j F^t_{\underset{g}{k|p}} \right),$$

$$F^p_j F^q_k \overset{1}{S}{}^i_{pq} = -\frac{1}{2} \left(F^i_{\underset{g}{j|p}} F^p_k - F^i_{\underset{g}{k|p}} F^p_j \right).$$

Hence, the following relations are satisfied:

$$\overset{1}{S}{}^i_{jk} = 0 \iff F^i_{\underset{g}{j|k}} = F^i_{\underset{g}{k|j}}, \tag{43}$$

$$F^i_p \overset{1}{S}{}^p_{jk} = 0 \iff F^i_{\underset{g}{j|k}} = F^i_{\underset{g}{k|j}}, \tag{44}$$

$$F^q_p \overset{1}{S}{}^p_{qi} = 0 \iff F_i = 0,$$

$$F^p_j \overset{1}{S}{}^i_{pk} = 0 \iff F^i_{\underset{g}{j|k}} = F^i_t F^p_j F^t_{\underset{g}{k|p}},$$

$$F^p_j \overset{1}{S}{}^i_{pk} = 0 \iff F^i_{\underset{g}{j|k}} = F^i_t F^p_j F^t_{\underset{g}{k|p}},$$

$$F^p_j F^q_k \overset{1}{S}{}^i_{pq} = 0 \iff F^i_{\underset{g}{j|p}} F^p_k = F^i_{\underset{g}{k|p}} F^p_j.$$

The Nijenhuis tensor in an almost Hermitian space is

$$N^i_{jk} = F^p_k\left(F^i_{\underset{g}{j|p}} - F^i_{\underset{g}{p|j}}\right) - F^p_j\left(F^i_{\underset{g}{k|p}} - F^i_{\underset{g}{p|k}}\right).$$

When this tensor vanishes, the almost Hermitian space is a Hermitian one.

These expressions complete the proof of the two following theorems.

Theorem 9. *In an almost Hermitian space, if Equation (43) or (44) is satisfied, then the almost Hermitian space is a Hermitian one.*

Theorem 10. *An almost Hermitian space is an almost semi-Kählerian space ($F_i = 0$) if and only if the equality $F^q_p \overset{1}{S}{}^p_{qi} = 0$ holds.*

A Killing vector v^i of Riemannian space $\mathbb{R}_{2N}$ (almost Hermitian space) is the vector which satisfies Killing's equations:

$$\mathcal{L}_v g_{\underline{ij}} = v^p g_{\underset{g}{\underline{ij}|p}} + g_{\underline{pj}} v^p_{\underset{g}{|i}} + g_{\underline{ip}} v^p_{\underset{g}{|j}} = 0. \tag{45}$$

Because $g_{\underset{g}{\underline{ij}|p}} = 0$, the previous Killing equations reduce to

$$\mathcal{L}_v g_{\underline{ij}} = v_{\underset{g}{i|j}} + v_{\underset{g}{j|i}} = 0.$$

The vector v^i is Killing's vector with respect to the first connection (32) if

$$\overset{1}{\mathcal{L}}_v g_{\underline{ij}} = 0,$$

for $\overset{1}{\mathcal{L}}$ defined as

$$\overset{1}{\mathcal{L}}_v g_{\underline{ij}} = v^p g_{\underset{1}{\underline{ij}|p}} + g_{\underline{pj}} v^p_{\underset{1}{|i}} + g_{\underline{ip}} v^p_{\underset{1}{|j}} = 0.$$

Theorem 11. *In an almost Hermitian space, the next equation holds:*

$$\overset{1}{\mathcal{L}}_v g_{\underline{ij}} = \mathcal{L}_v g_{\underline{ij}} + \frac{1}{2} v_t F^t_p \left(F^p_{\underset{g}{i|j}} + F^p_{\underset{g}{j|i}}\right).$$

Proof. Since the equality $g_{\underset{1}{\underline{ij}|p}} = 0$ is satisfied with respect to the Equation (35), the previous Killing equations for the first connection reduce to

$$\overset{1}{\mathcal{L}}_v g_{\underline{ij}} = v_{\underset{1}{i|j}} + v_{\underset{1}{j|i}} = 0.$$

After some computing, we get

$$v_{\underset{1}{i|j}} + v_{\underset{1}{j|i}} = v_{\underset{g}{i|j}} + v_{\underset{g}{j|i}} + \frac{1}{2} v_t F^t_p \left(F^p_{\underset{g}{i|j}} + F^p_{\underset{g}{j|i}}\right),$$

which completes this proof. □

Corollary 1. *If an almost Hermitian space is an almost Tachibana space, then the next equality holds*

$$\overset{1}{\mathcal{L}}_v g_{\underline{ij}} = \mathcal{L}_v g_{\underline{ij}}.$$

Corollary 2. *If v^i is a Killing vector in an almost Hermitian space, then the following equality is satisfied*

$$\overset{1}{\mathcal{L}}_v g_{\underline{ij}} = \frac{1}{2} v_t F^t_p \left(F^p_{i\underset{g}{|}j} + F^p_{j\underset{g}{|}i} \right).$$

Corollary 3. *In an almost Hermitian space, let a vector v^i be the Killing vector. Then, the equality $\overset{1}{\mathcal{L}}_v g_{\underline{ij}} = 0$ holds if and only if the almost Hermitian space is the almost Tachibana space.*

Based on Equation (3), we obtain the curvature tensor $\overset{1}{R}{}^i_{jmn} = \overset{1}{\Gamma}{}^i_{\underline{jm},n} - \overset{1}{\Gamma}{}^i_{\underline{jn},m} + \overset{1}{\Gamma}{}^p_{\underline{jm}} \overset{1}{\Gamma}{}^i_{\underline{pn}} - \overset{1}{\Gamma}{}^p_{\underline{jn}} \overset{1}{\Gamma}{}^i_{\underline{pm}}$ with respect to the symmetric affine connection $\overset{1}{\Gamma}{}^i_{\underline{jk}}$:

$$\begin{aligned}
\overset{1}{R}{}^i_{jmn} = {} & \frac{3}{4} R^i_{jmn} - \frac{1}{4} F^i_p F^s_j R^p_{smn} - \frac{1}{4} F^i_p \left(F^p_{m\underset{g}{|}jn} - F^p_{n\underset{g}{|}jm} \right) \\
& - \frac{1}{16} \left(3F^i_{p\underset{g}{|}n} + F^s_p F^q_n F^i_{q\underset{g}{|}s} \right) \left(F^p_{j\underset{g}{|}m} + F^p_{m\underset{g}{|}j} \right) \\
& + \frac{1}{16} \left(3F^i_{p\underset{g}{|}m} + F^s_p F^q_m F^i_{q\underset{g}{|}s} \right) \left(F^p_{j\underset{g}{|}n} + F^p_{n\underset{g}{|}j} \right),
\end{aligned} \tag{46}$$

where R^i_{jmn} and $\underset{g}{|}$ are the curvature tensor and covariant derivative with respect to the Christoffell symbols $\Gamma^i_{\underline{jk}}$. From Equation (46), we get:

$$\begin{aligned}
\underset{1}{\overset{1}{R}}{}^i_{jmn} = {} & \frac{1}{2} R^i_{jmn} - \frac{1}{2} F^i_p F^s_j R^p_{smn} - \frac{1}{4} F^i_{p\underset{g}{|}n} F^p_{j\underset{g}{|}m} + \frac{1}{4} F^i_{p\underset{g}{|}m} F^p_{j\underset{g}{|}n}, \\
\underset{2}{\overset{1}{R}}{}^i_{jmn} = {} & R^i_{jmn} - \frac{1}{2} F^i_p \left(F^p_{m\underset{g}{|}jn} - F^p_{n\underset{g}{|}jm} \right) - \frac{1}{2} F^i_{p\underset{g}{|}n} F^p_{m\underset{g}{|}j} + \frac{1}{2} F^i_{p\underset{g}{|}m} F^p_{n\underset{g}{|}j} \\
& - \frac{1}{4} F^i_{p\underset{g}{|}q} F^q_s \left(F^s_{m\underset{g}{|}j} F^p_n - F^s_{n\underset{g}{|}j} F^p_m \right), \\
\underset{3}{\overset{1}{R}}{}^i_{jmn} = {} & R^i_{jmn} - \frac{1}{2} F^i_p \left(F^p_{j\underset{g}{|}mn} - F^p_{n\underset{g}{|}jm} \right) - \frac{1}{2} F^i_{p\underset{g}{|}n} F^p_{j\underset{g}{|}m} + \frac{1}{4} F^i_{p\underset{g}{|}m} F^p_{n\underset{g}{|}j} + \frac{1}{4} F^p_{n\underset{g}{|}m} F^i_{p\underset{g}{|}j} \\
& - \frac{1}{4} F^i_{p\underset{g}{|}q} F^q_s \left(F^s_{j\underset{g}{|}m} F^p_n - F^s_{n\underset{g}{|}m} F^p_j \right), \\
\underset{4}{\overset{1}{R}}{}^i_{jmn} = {} & R^i_{jmn} - \frac{1}{2} F^i_p \left(F^p_{j\underset{g}{|}mn} - F^p_{n\underset{g}{|}jm} \right) - \frac{1}{2} F^i_{p\underset{g}{|}n} F^p_{j\underset{g}{|}m} + \frac{1}{4} F^i_{p\underset{g}{|}m} F^p_{n\underset{g}{|}j} + \frac{1}{4} F^p_{m\underset{g}{|}n} F^i_{p\underset{g}{|}j} \\
& - \frac{1}{4} F^i_{p\underset{g}{|}q} F^q_s \left(F^s_{j\underset{g}{|}m} F^p_n - F^s_{m\underset{g}{|}n} F^p_j \right), \\
\underset{5}{\overset{1}{R}}{}^i_{jmn} = {} & \frac{3}{4} R^i_{jmn} - \frac{1}{4} F^i_p F^s_j R^p_{smn} - \frac{1}{4} F^i_p \left(F^p_{m\underset{g}{|}jn} - F^p_{n\underset{g}{|}jm} \right) - \frac{1}{8} F^i_{p\underset{g}{|}n} F^p_{j\underset{g}{|}m} - \frac{1}{4} F^i_{p\underset{g}{|}n} F^p_{m\underset{g}{|}j} \\
& + \frac{1}{4} F^i_{p\underset{g}{|}m} F^p_{j\underset{g}{|}n} + \frac{1}{8} F^i_{p\underset{g}{|}m} F^p_{n\underset{g}{|}j} - \frac{1}{8} F^i_{p\underset{g}{|}q} F^q_s \left(F^s_{m\underset{g}{|}j} F^p_n - F^s_{j\underset{g}{|}n} F^p_m \right).
\end{aligned}$$

A vector v^i of Riemannian (almost Hermitian) space $\mathbb{R}_{2N}$ which satisfies the affine Killing equations

$$\mathcal{L}_v\Gamma^i_{\underline{jk}} = v^i_{\underset{g}{|}jk} + R^i_{jkp}v^p = 0,$$

is the affine Killing vector.

A vector v^i is the affine Killing vector with respect to the first connection (32) if the following equalities are satisfied

$$\overset{1}{\mathcal{L}}_v\Gamma^i_{\underline{jk}} = v^i_{\underset{1}{|}jk} + R^i_{jkp}v^p = 0. \tag{47}$$

Let us prove the next theorem.

Theorem 12. *In an almost Hermitian space, the next equation is satisfied:*

$$\overset{1}{\mathcal{L}}_v\Gamma^i_{\underline{jk}} = \mathcal{L}_v\Gamma^i_{\underline{jk}} - \frac{1}{2}\left(F^i_tF^t_{p\underset{g}{|}k}v^p_{\underset{g}{|}j} + F^t_jF^p_{t\underset{g}{|}k}v^i_{\underset{g}{|}p} + F^i_tF^t_{p\underset{g}{|}j}v^p_{\underset{g}{|}k}\right) - \frac{1}{2}v^p\left(F^i_tF^t_{p\underset{g}{|}jk} + \frac{1}{2}F^i_{t\underset{g}{|}k}F^t_{p\underset{g}{|}j} - \frac{1}{2}F^i_tF^q_jF^s_{q\underset{g}{|}k}F^t_{p\underset{g}{|}s}\right).$$

Proof. Equation (47) reduces to

$$\overset{1}{\mathcal{L}}_v\Gamma^i_{\underline{jk}} = v^i_{\underset{g}{|}jk} + R^i_{jkp}v^p - \frac{1}{2}\left(F^i_tF^t_{p\underset{g}{|}k}v^p_{\underset{g}{|}j} + F^t_jF^p_{t\underset{g}{|}k}v^i_{\underset{g}{|}p} + F^i_tF^t_{p\underset{g}{|}j}v^p_{\underset{g}{|}k}\right) - \frac{1}{2}v^p\left(F^i_tF^t_{p\underset{g}{|}jk} + \frac{1}{2}F^i_{t\underset{g}{|}k}F^t_{p\underset{g}{|}j} - \frac{1}{2}F^i_tF^q_jF^s_{q\underset{g}{|}k}F^t_{p\underset{g}{|}s}\right) = 0.$$

After some computation, we get

$$v^i_{\underset{1}{|}jk} + R^i_{jkp}v^p = v^i_{\underset{g}{|}jk} + R^i_{jkp}v^p - \frac{1}{2}\left(F^i_tF^t_{p\underset{g}{|}k}v^p_{\underset{g}{|}j} + F^t_jF^p_{t\underset{g}{|}k}v^i_{\underset{g}{|}p} + F^i_tF^t_{p\underset{g}{|}j}v^p_{\underset{g}{|}k}\right) - \frac{1}{2}v^p\left(F^i_tF^t_{p\underset{g}{|}jk} + \frac{1}{2}F^i_{t\underset{g}{|}k}F^t_{p\underset{g}{|}j} - \frac{1}{2}F^i_tF^q_jF^s_{q\underset{g}{|}k}F^t_{p\underset{g}{|}s}\right),$$

which completes the proof of this theorem. □

Corollary 4. *If v^i is an affine Killing vector in an almost Hermitian space, then the next relation holds*

$$\overset{1}{\mathcal{L}}_v\Gamma^i_{\underline{jk}} = -\frac{1}{2}\left(F^i_tF^t_{p\underset{g}{|}k}v^p_{\underset{g}{|}j} + F^t_jF^p_{t\underset{g}{|}k}v^i_{\underset{g}{|}p} + F^i_tF^t_{p\underset{g}{|}j}v^p_{\underset{g}{|}k}\right) - \frac{1}{2}v^p\left(F^i_tF^t_{p\underset{g}{|}jk} + \frac{1}{2}F^i_{t\underset{g}{|}k}F^t_{p\underset{g}{|}j} - \frac{1}{2}F^i_tF^q_jF^s_{q\underset{g}{|}k}F^t_{p\underset{g}{|}s}\right).$$

A vector v^i of Riemannian (almost Hermitian) space $\mathbb{R}_{2N}$ which satisfies the conformal Killing equations

$$\mathcal{L}_vg_{\underline{ij}} = v_{i\underset{g}{|}j} + v_{j\underset{g}{|}i} = 2\Phi g_{\underline{ij}},$$

for a scalar function Φ, is the conformal Killing vector.

A vector v^i is the conformal Killing vector with respect to the first connection (32) if the next equation is satisfied

$$\overset{1}{\mathcal{L}}_v g_{\underline{ij}} = v_{i\underset{1}{|}j} + v_{j\underset{1}{|}i} = 2\Phi g_{\underline{ij}}.$$

The previous conformal Killing equations reduce to

$$\overset{1}{\mathcal{L}}_v g_{\underline{ij}} = v_{i\underset{g}{|}j} + v_{j\underset{g}{|}i} + \frac{1}{2} v_t F^t_p \left(F^p_{i\underset{g}{|}j} + F^p_{j\underset{g}{|}i} \right) = 2\Phi g_{\underline{ij}}.$$

Corollary 5. *If a vector v^i is a conformal Killing vector in an almost Hermitian space, then the following holds*

$$\overset{1}{\mathcal{L}}_v g_{\underline{ij}} = 2\Phi g_{\underline{ij}} + \frac{1}{2} v_t F^t_p \left(F^p_{i\underset{g}{|}j} + F^p_{j\underset{g}{|}i} \right).$$

Corollary 6. *Let v^i be a conformal Killing vector in an almost Hermitian space. In this case, the equality $\overset{1}{\mathcal{L}}_v g_{\underline{ij}} = 2\Phi g_{\underline{ij}}$ holds if and only if the almost Hermitian space is an almost Tachibana one.*

A vector v^i of Riemannian (almost Hermitian) space $\mathbb{R}_{2N}$ is the projective Killing vector with respect to the symmetric affine connection if the next relation holds (the projective Killing equations):

$$\mathcal{L}_v \Gamma^i_{\underline{jk}} = v^i_{\underset{g}{|}jk} + R^i_{jkp} v^p = \psi_j A^i_k + \psi_k A^i_j,$$

for the gradient vector ψ_i.

A vector v^i is the projective Killing vector with respect to the first connection (32) if the projective Killing equations hold

$$\overset{1}{\mathcal{L}}_v \Gamma^i_{\underline{jk}} = v^i_{\underset{1}{|}jk} + R^i_{jkp} v^p = \psi_j A^i_k + \psi_k A^i_j.$$

The previous projective Killing equation reduce to

$$\begin{aligned} \overset{1}{\mathcal{L}}_v \Gamma^i_{\underline{jk}} = {} & v^i_{\underset{g}{|}jk} + R^i_{jkp} v^p - \frac{1}{2} \left(F^i_t F^t_{p\underset{g}{|}k} v^p_{\underset{g}{|}j} + F^t_j F^p_{t\underset{g}{|}k} v^i_{\underset{g}{|}p} + F^i_t F^t_{p\underset{g}{|}j} v^p_{\underset{g}{|}k} \right) \\ & - \frac{1}{2} v^p \left(F^i_t F^t_{p\underset{g}{|}jk} + \frac{1}{2} F^i_{t\underset{g}{|}k} F^t_{p\underset{g}{|}j} - \frac{1}{2} F^i_t F^q_j F^s_{q\underset{g}{|}k} F^t_{p\underset{g}{|}s} \right) \\ = {} & \psi_j A^i_k + \psi_k A^i_j. \end{aligned}$$

Corollary 7. *If a vector v^i is a projective Killing vector, then the next equality holds*

$$\begin{aligned} \overset{1}{\mathcal{L}}_v \Gamma^i_{\underline{jk}} = {} & \psi_j A^i_k + \psi_k A^i_j - \frac{1}{2} \left(F^i_t F^t_{p\underset{g}{|}k} v^p_{\underset{g}{|}j} + F^t_j F^p_{t\underset{g}{|}k} v^i_{\underset{g}{|}p} + F^i_t F^t_{p\underset{g}{|}j} v^p_{\underset{g}{|}k} \right) \\ & - \frac{1}{2} v^p \left(F^i_t F^t_{p\underset{g}{|}jk} + \frac{1}{2} F^i_{t\underset{g}{|}k} F^t_{p\underset{g}{|}j} - \frac{1}{2} F^i_t F^q_j F^s_{q\underset{g}{|}k} F^t_{p\underset{g}{|}s} \right). \end{aligned}$$

A vector v^i of Riemannian (almost Hermitian) space $\mathbb{R}_{2N}$ is the harmonic vector if it satisfies the following equations

$$\begin{aligned} & v_{i\underset{g}{|}j} - v_{j\underset{g}{|}i} = 0, \\ & v^i_{\underset{g}{|}i} = 0. \end{aligned}$$

A vector v^i is harmonic with respect to the first connection (32) if the following equations hold

$$v_{i\underset{1}{|}j} - v_{j\underset{1}{|}i} = 0,$$
$$v^i_{\underset{1}{|}i} = 0.$$

The last two equations reduce to

$$v_{i\underset{1}{|}j} - v_{j\underset{1}{|}i} = v_{i\underset{g}{|}j} - v_{j\underset{g}{|}i} + \frac{1}{2}F^p_t v_p \left(F^t_{i\underset{g}{|}j} - F^t_{j\underset{g}{|}i} \right),$$
$$v^i_{\underset{1}{|}i} = v^i_{\underset{g}{|}i} + \frac{1}{2}F^t_p F^i_{t\underset{g}{|}i} v^p,$$

i.e.,

$$v_{i\underset{1}{|}j} - v_{j\underset{1}{|}i} = v_{i\underset{g}{|}j} - v_{j\underset{g}{|}i} + \frac{1}{2}F^p_t v_p \left(F^t_{i\underset{g}{|}j} - F^t_{j\underset{g}{|}i} \right),$$
$$v^i_{\underset{1}{|}i} = v^i_{\underset{g}{|}i} - \frac{1}{2}F^t_p F_t v^p.$$

Corollary 8. *If v^i is a harmonic vector of almost Hermitian space, then the next equalities hold:*

$$v_{i\underset{1}{|}j} - v_{j\underset{1}{|}i} = \frac{1}{2}F^p_t v_p \left(F^t_{i\underset{g}{|}j} - F^t_{j\underset{g}{|}i} \right),$$
$$v^i_{\underset{1}{|}i} = -\frac{1}{2}F^t_p F_t v^p.$$

Corollary 9. *Let v^i be a harmonic vector in an almost Hermitian space. This vector is harmonic with respect to the first affine connection if and only if the almost Hermitian space is the almost Kähler one ($F_i = 0$) and the tensor $F^i_{j\underset{g}{|}k}$ is symmetric by j and k.*

Corollary 10. *An almost Hermitian space is the almost Kähler one ($F_i = 0$) if and only if the equality $v^i_{\underset{1}{|}i} = v^i_{\underset{g}{|}i}$ holds.*

A vector v^i is a contravariant almost analytic vector of an almost Hermitian space if it satisfies the following equalities:

$$\mathcal{L}_v F^i_j = v^p F^i_{j\underset{g}{|}p} - F^p_j v^i_{\underset{g}{|}p} + F^i_p v^p_{\underset{g}{|}j} = 0.$$

A vector v^i is the contravariant almost analytic vector with respect to the first connection (32) if the next equalities hold

$$\overset{1}{\mathcal{L}}_v F^i_j = v^p F^i_{j\underset{1}{|}p} - F^p_j v^i_{\underset{1}{|}p} + F^i_p v^p_{\underset{1}{|}j} = 0.$$

Because the first connection is an F-connection ($F^i_{j\underset{1}{|}k} = 0$), the last equation becomes

$$\overset{1}{\mathcal{L}}_v F^i_j = -F^p_j v^i_{\underset{1}{|}p} + F^i_p v^p_{\underset{1}{|}j} = 0.$$

We prove the next theorem.

Theorem 13. *In an almost Hermitian space, the following equation holds*

$$\overset{1}{\mathcal{L}}_v F^i_j = \mathcal{L}_v F^i_j - v^t \left(F^i_{\underset{g}{j|t}} - \frac{1}{2} F^i_{\underset{g}{t|j}} - \frac{1}{2} F^i_p F^q_j F^p_{\underset{g}{t|q}} \right).$$

Proof. After some calculations, one gets

$$\overset{1}{\mathcal{L}}_v F^i_j = \mathcal{L}_v F^i_j - v^t \left(F^i_{\underset{g}{j|t}} - \frac{1}{2} F^i_{\underset{g}{t|j}} - \frac{1}{2} F^i_p F^q_j F^p_{\underset{g}{t|q}} \right),$$

which completes the proof of this theorem. □

Corollary 11. *If a vector v^i is a contravariant almost analytic vector in an almost Hermitian space, then the next equation holds:*

$$\overset{1}{\mathcal{L}}_v F^i_j = -v^t \left(F^i_{\underset{g}{j|t}} - \frac{1}{2} F^i_{\underset{g}{t|j}} - \frac{1}{2} F^i_p F^q_j F^p_{\underset{g}{t|q}} \right).$$

Corollary 12. *In an almost Hermitian space, the equality $\overset{1}{\mathcal{L}}_v F^i_j = \mathcal{L}_v F^i_j$ holds if and only if the equality $F^i_{\underset{g}{j|k}} = \frac{1}{2} F^i_{\underset{g}{k|j}} + \frac{1}{2} F^i_p F^q_j F^p_{\underset{g}{k|q}}$ holds.*

Corollary 13. *In an almost Hermitian space, the equality $\overset{1}{\mathcal{L}}_v F^i_j = \mathcal{L}_v F^i_j$ holds if and only if the dual and symmetric connections, $\overset{2}{\Gamma}{}^i_{jk}$ and $\overset{1}{\Gamma}{}^i_{\underline{jk}}$, are F-connections.*

Proof. Because the next equalities hold

$$F^i_{\underset{g}{j|k}} = \frac{1}{2} F^i_{\underset{g}{k|j}} + \frac{1}{2} F^i_p F^q_j F^p_{\underset{g}{k|q}} = \frac{1}{2} F^i_{\underset{g}{k|j}} - \frac{1}{2} F^i_{\underset{g}{p|q}} F^q_j F^p_k,$$

and considering Theorems 3 and 4, we complete the proof of this corollary. □

Corollary 14. *Let a vector v^i be covariant almost analytic ($\mathcal{L}_v F^i_j = 0$) in an almost Hermitian space. The vector v^i is contravariant almost analytic with respect to the first connection (32) ($\overset{1}{\mathcal{L}}_v F^i_j = 0$) if and only if the dual and symmetric connections $\overset{2}{\Gamma}{}^i_{jk}$ and $\overset{1}{\Gamma}{}^i_{\underline{jk}}$ of connection $\overset{1}{\Gamma}{}^i_{jk}$ are F-connections.*

A vector v_i is a covariant almost analytic vector of an almost Hermitian space if it satisfies the equation

$$\left(F^p_{\underset{g}{i|j}} - F^p_{\underset{g}{j|i}} \right) v_p - F^p_j v_{\underset{g}{i|p}} + F^p_i v_{\underset{g}{p|j}} = 0.$$

A vector v_i is a covariant analytic vector with respect to the first connection (32) if the next equation holds

$$\left(F^p_{\underset{1}{i|j}} - F^p_{\underset{1}{j|i}} \right) v_p - F^p_j v_{\underset{1}{i|p}} + F^p_i v_{\underset{1}{p|j}} = 0.$$

Because the first connection is an F-connection ($F^i_{j|k} = 0$ with the covariant derivative taken with respect to connection 1), the last equation transforms into:

$$-F^p_j v_{i\underset{1}{|}p} + F^p_i v_{p\underset{1}{|}j} = 0.$$

Finally, we present the following theorem.

Theorem 14. *In an almost Hermitian space, the following equality holds*

$$-F^p_j v_{i\underset{1}{|}p} + F^p_i v_{p\underset{1}{|}j} = -F^p_j v_{i\underset{g}{|}p} + F^p_i v_{p\underset{g}{|}j} + \frac{1}{2}F^p_{i\underset{g}{|}j} v_p + \frac{1}{2}F^t_i F^q_j F^p_{t\underset{g}{|}q} v_p.$$

Proof. After some computing, we get

$$-F^p_j v_{i\underset{1}{|}p} + F^p_i v_{p\underset{1}{|}j} = -F^p_j v_{i\underset{g}{|}p} + F^p_i v_{p\underset{g}{|}j} + \frac{1}{2}F^p_{i\underset{g}{|}j} v_p + \frac{1}{2}F^t_i F^q_j F^p_{t\underset{g}{|}q} v_p,$$

which confirms the validity of this theorem. □

Corollary 15. *If a vector v_i is a covariant almost analytic one in an almost Hermitian space, then*

$$-F^p_j v_{i\underset{1}{|}p} + F^p_i v_{p\underset{1}{|}j} = v_p\left(F^p_{j\underset{g}{|}i} - \frac{1}{2}F^p_{i\underset{g}{|}j} + \frac{1}{2}F^t_i F^q_j F^p_{t\underset{g}{|}q}\right).$$

Corollary 16. *Let a vector v_i be a covariant almost analytic one in an almost Hermitian space. The vector v_i is a covariant almost analytic vector with respect to the first connection* (32) *if and only if the following equation holds*

$$F^i_{k\underset{g}{|}j} = \frac{1}{2}F^i_{j\underset{g}{|}k} - \frac{1}{2}F^t_j F^q_k F^i_{t\underset{g}{|}q} = \frac{1}{2}F^i_{j\underset{g}{|}k} + \frac{1}{2}F^i_t F^q_k F^t_{j\underset{g}{|}q}.$$

4. Conclusions

In this study, we analyzed the special half-symmetric affine connection initiated by Christoffell symbols (32).

We analyzed the dual connection and obtained the necessary and sufficient condition for it to be an F-connection. In particular, the necessary and sufficient condition for the symmetric part of a half-symmetric affine connection to be an F-connection was presented.

Five linearly independent curvature tensors were obtained with respect to this affine connection.

Next, we reviewed the definition of a Tachibana space [4] and generalized it to the definition of an almost Tachibana space. After that, we obtained the necessary and sufficient condition for a dual half-symmetric connection to be a metric connection. It was proved that a dual connection of a half-symmetric connection (32) and its symmetric part were metric connections if and only if the almost Hermitian space equipped with the half-symmetric connection was an almost Tachibana space.

The necessary condition for an almost Hermitian space to be a Hermitian space was presented. The necessary and sufficient condition for an almost Kählerian space was presented as well.

In the last part of this research, motivated by Yano's research [4], we generalized the concept of Killing vector by defining the affine Killing vector, conformal Killing vector, projective Killing vector, harmonic vector, and covariant and contravariant analytic vectors.

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