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ORIGINAL PAPER

Banach bimodule-valued positive maps: inequalities and representations

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Abstract

In this paper we construct representations of general positive sesquilinear maps with values in ordered Banach bimodules such as commutative and non-commutative L^1-spaces and the spaces of bounded linear operators from a C^*-algebra into the dual of another C^*-algebra are considered. As a starting point, a generalized Cauchy–Schwarz inequality is proved for these maps and a representation of bounded positive maps from a (quasi) *-algebra into such an ordered Banach bimodule is derived and some more inequalities for these maps are deduced. In particular, an extension of Paulsen's modified Kadison–Schwarz inequality for 2-positive maps to the case of general positive maps from a unital *-algebra into the space of trace-class operators on a separable Hilbert space and into the duals of von-Neumann algebras is obtained. Also, representations for completely positive maps with values in an ordered Banach bimodule and Cauchy–Schwarz inequality for infinite sums of such maps are provided. Concrete examples illustrate the results.

Keywords Representations · Modules · Positive sesquilinear maps · Completely positive sesquilinear maps · Normed spaces

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1 Introduction

Positive and completely positive maps play an important role in the theory of operator algebras and quantum information, see for instance [9]. This motivates a study of representations involving these maps. A classical result in representation theory is the famous Stinespring theorem [25] (see also [2, 20, 26] which gives a representation of completely positive maps from a C^*-algebra into the space of bounded operators in Hilbert space. There is an extensive literature on the theory of representations see, e.g., [24] and [23, Chapter 12]. Quite recently, in [6, Corollary 3.10] and [5, Corollary 3.3], a representation of general positive C^*-valued maps on arbitrary *-algebras has also been provided.

Now, C^*-algebras are just a special case of Banach quasi *-algebras and, moreover, every Banach quasi *-algebra is a Banach bimodule over a *-algebra.

Several beautiful and important mathematical structures (such as commutative and noncommutative L^p-spaces) are examples of Banach quasi *-algebras. Further, Hilbert C^*-modules [17, 18] and the spaces of bounded linear operators from a C^*-algebra into the dual of another C^*-algebra are other examples of (ordered) Banach bimodules over a *-algebra which are of interest for applications. All these facts motivated us to study positive and completely positive maps from a Banach quasi *-algebra into an (ordered) Banach bimodule over a *-algebra, and the main aim of this paper is therefore obtaining representations and related inequalities of such maps.

In a recent paper [6] we considered a GNS-like construction defined by positive sesquilinear maps on a quasi *-algebra taking their values in a C^*-algebra. In this paper we go some steps further and consider the possibility of replacing sesquilinear maps taking values in a C^*-algebra with sesquilinear maps with values in an ordered Banach bimodule $\mathfrak{Y}$ over some *-algebra $\mathfrak{Y}_0$. In this situation, an extension of the result of [6] is possible, provided that the sesquilinear form Φ, where we start from, satisfies a generalized Cauchy–Schwarz inequality. This discussion is developed in Sect. 3 where (Proposition 3.1) we characterize a class of ordered Banach bimodules $\mathfrak{Y}$ with the property that every $\mathfrak{Y}$-valued positive sesquilinear map satisfies a generalized Cauchy–Schwarz inequality. Concrete examples of such bimodules are, for instance, non-commutative L^1-spaces and the space of bounded linear operators from a von Neumann algebra into the dual of another von Neumann algebra. As a corollary, we obtain an extension of Paulsen's modified Kadison–Schwarz inequality for 2-positive maps between C^*-algebras to the case of general positive maps from a unital *-algebra into any of the above mentioned ordered Banach bimodules (Corollary 3.2) as well as Cauchy–Schwarz inequality for infinite sums of ordered Banach bimodule-valued positive sesquilinear maps (Proposition 3.3). Then we provide a representation of positive sesquilinear maps from a quasi *-algebra into such an ordered Banach bimodule. Moreover, we give examples of such maps. In the last part of Sect. 3, we obtain also a representation of positive ordered Banach bimodule-valued linear maps on a unital *-algebra (Corollary 3.2).

An important difference between representations of ordered Banach bimodule-valued positive maps given in Sect. 3 and the representations of C^*-valued positive maps given in [6] is that the ones from Sect. 3 are given in a proper Banach space and not just in a quasi-Banach space (as in [6]). This is due to the Cauchy–Schwarz inequality for positive Banach bimodule-valued sesquilinear maps given in Proposition 3.1. Positive maps from a unital C^*-algebra into the space of trace-class operators are a special case of the positive maps treated in Sect. 3. Since the space of trace-class operators is the subspace of the C^*-algebra of bounded linear operators on a Hilbert space, it follows that our results have an application in the theory of positive maps on C^*-algebras. This fact has also been a motivation for Sect. 3 in this paper. Thus, the purpose of Section 3 is not only generalizing the results from [6] from the case of positive C^*-valued maps to the case of positive Banach bimodule-valued maps, but also improving the results from [6] by providing representations of some classes of positive maps in Banach spaces and not just in quasi-Banach spaces as done in [6].

Section 4 is devoted to the *-representations induced by completely positive sesquilinear maps. This is actually a subject for which several approaches have been proposed in view of a possible generalization of the Stinespring theorem to different environments (see, for instance, [3, 11, 12, 14, 15, 19, 21]). Motivated by [21, Proposition 3.1], following an idea proposed in [22, Ch. 11] and adopted in [4] for partial *-algebras, we consider completely positive sesquilinear maps from a normed quasi *-algebra $\mathfrak{A}$ into a set of sesquilinear maps on a vector space which take values in an ordered Banach bimodule $\mathfrak{Y}$ and provide a generalization of the classical Stinespring theorem in this context, as well as applications of this result to Cauchy–Schwarz inequality for infinite sums of such maps (Corollary 4.6). Further, we provide a version of Radon–Nikodym theorem for these maps as well as for ordered Banach bimodule-valued bounded positive sesquilinear maps on a quasi *-algebra, thus generalizing, in this setting, the results from [26, Section 3.5] and give examples of such maps.

At the end of Sect. 4, we apply our results to obtain representations of bounded completely positive sesquilinear maps on a quasi *-algebra with values in the space of ordered Banach bimodule-valued, bounded, linear operators. As a consequence of these representations, in Proposition 4.12 we obtain Cauchy–Schwarz inequalities for such maps. This application has also been a motivation for constructing representations of completely positive maps. Also, we provide some examples as well as some other applications.

2 Preliminaries

A *quasi *-algebra* $(\mathfrak{A}, \mathfrak{A}_0)$ is a pair consisting of a vector space $\mathfrak{A}$ and a *-algebra $\mathfrak{A}_0$ contained in $\mathfrak{A}$ as a subspace and such that

- $\mathfrak{A}$ carries an involution $a \mapsto a^*$ extending the involution of $\mathfrak{A}_0$;
- $\mathfrak{A}$ is a bimodule over $\mathfrak{A}_0$ and the module multiplications extend the multiplication of $\mathfrak{A}_0$. In particular, the following associative laws hold:

$$(ca)d = c(ad); \quad a(cd) = (ac)d, \quad \forall\, a \in \mathfrak{A},\ c, d \in \mathfrak{A}_0;$$

- $(ac)^* = c^* a^*$, for every $a \in \mathfrak{A}$ and $c \in \mathfrak{A}_0$.

The *identity* or *unit element* of $(\mathfrak{A}, \mathfrak{A}_0)$, if any, is a necessarily unique element $\mathsf{e} \in \mathfrak{A}_0$, such that $a\mathsf{e} = a = \mathsf{e}a$, for all $a \in \mathfrak{A}$.

We will always suppose that

$$ac = 0,\ \forall c \in \mathfrak{A}_0 \Rightarrow a = 0$$
$$ac = 0,\ \forall a \in \mathfrak{A} \Rightarrow c = 0.$$

Clearly, both these conditions are automatically satisfied if $(\mathfrak{A}, \mathfrak{A}_0)$ has an identity e.

A quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is said to be *normed* if $\mathfrak{A}$ is a normed space, with a norm $\|\cdot\|$ enjoying the following properties

- there exists $\gamma > 0$ such that for every $a \in \mathfrak{A}$

$$\max\{\|ac\|, \|ca\|\} \le \gamma \|a\|, \quad \forall c \in \mathfrak{A}_0;$$

- $\|a^*\| = \|a\|,\ \forall a \in \mathfrak{A}$;
- $\mathfrak{A}_0$ is dense in $\mathfrak{A}[\|\cdot\|]$.

If the normed vector space $\mathfrak{A}[\|\cdot\|]$ is complete, then $(\mathfrak{A}, \mathfrak{A}_0)$ is called a *Banach quasi *-algebra*. We refer to [10] for further details.

Throughout the paper we will denote by $\mathfrak{B}(X, Y)$ the space of bounded linear maps from the normed space X into the normed space Y. If $X = Y$ we will simply write $\mathfrak{B}(X) = \mathfrak{B}(X, X)$.

Definition 2.1 Let $\mathfrak{Y}$ be a Banach bimodule over the *-algebra $\mathfrak{Y}_0$. We say that $\mathfrak{Y}$ is an *ordered Banach bimodule* over $\mathfrak{Y}_0$ if

(i) $\mathfrak{Y}$ is ordered as a vector space; that is, $\mathfrak{Y}$ contains a (positive) closed cone $\mathfrak{K}$, i.e., $\mathfrak{K} \subset \mathfrak{Y}$ is such that $\mathfrak{K} + \mathfrak{K} \subset \mathfrak{K}$, $\lambda\mathfrak{K} \subset \mathfrak{K}$ for $\lambda \ge 0$ and $\mathfrak{K} \cap (-\mathfrak{K}) = \{0\}$;
(ii) $z^*\mathfrak{K}z \subset \mathfrak{K}, \quad \forall z \in \mathfrak{Y}_0$.

As usual, we will write $y_1 \le y_2$ whenever $y_2 - y_1 \in \mathfrak{K}$, with $y_1, y_2 \in \mathfrak{K}$ and we will sometimes suppose that $\mathfrak{Y}$ has an order-preserving norm in the sense that if $y_1 \le y_2$ with $y_1, y_2 \in \mathfrak{K}$, then also $\|y_1\|_{\mathfrak{Y}} \le \|y_2\|_{\mathfrak{Y}}$. Throughout the paper, Ω will denote a (locally) compact Hausdorff space and $C(\Omega)$ and $M(\Omega)$ will denote the space of continuous functions on Ω and the Banach space of all complex Radon measures on Ω equipped with the total variation norm, respectively.

Example 2.2 Examples of ordered Banach bimodules over a Banach *-algebra are provided by:

- if $\mathcal{H}$ is a separable Hilbert space, and $\mathfrak{B}_1(\mathcal{H})$ denotes the space of all trace-class operators, then $\mathfrak{Y} = (\mathfrak{B}_1(\mathcal{H}), \|\cdot\|_1)$ is a Banach bimodule over the von Neumann algebra $\mathfrak{Y}_0 = (\mathfrak{B}(\mathcal{H}), \|\cdot\|)$ of bounded operators on $\mathcal{H}$ equipped with the operator norm $\|\cdot\|$;
- the non commutative spaces $\mathfrak{Y} = L^p(\rho)$, over the Banach *-algebra $\mathfrak{Y}_0 = L^\infty(\rho)$, with ρ a faithful finite trace on a von Neumann algebra $\mathfrak{M}$, see [10, Section 5.6.1];
- $\mathfrak{Y} = \mathfrak{B}(\mathfrak{C}_1, \mathfrak{C}_2)$, the space of bounded linear operators from $\mathfrak{C}_1$ into $\mathfrak{C}_2$ equipped with the operator norm, where $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are unital C^*-algebras, $\mathfrak{Y}_0 = \mathfrak{C}_1$. If $A \in \mathfrak{B}(\mathfrak{C}_1, \mathfrak{C}_2)$ and $c \in \mathfrak{C}_1$, the module multiplication can be defined as

$$(A \cdot c)(d) = A(cd) \text{ and } (c \cdot A)(d) = A(dc), \quad d \in \mathfrak{C}_1.$$

Similar construction applies if we instead of $\mathfrak{C}_2$ consider the dual of $\mathfrak{C}_2$;
- $\mathfrak{Y} = \mathfrak{Y}_0 = \ell_2(C(\Omega))$, with $\ell_2(C(\Omega))$ the space of square summable sequences of functions in $C(\Omega)$ with Ω a compact Hausdorff space. It is a Banach bimodule over itself because it is a (non-unital) normed *-algebra.

The definitions of the positive cones are obvious in all these cases.

In the next results we will need the additional assumption that the norm on the ordered Banach bimodule $\mathfrak{Y}$ is order-preserving. Examples of Banach bimodules with an order-preserving norm are commutative L^p-spaces and $\ell_2(C(\Omega))$, with Ω a compact Hausdorff space. Furthermore, if $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are C^*-algebras with unit $e_{\mathfrak{C}_1}$ and $e_{\mathfrak{C}_2}$, respectively, the Banach ordered bimodule $\mathfrak{B}(\mathfrak{C}_1, \mathfrak{C}_2)$ is order-preserving (this follows from the fact that $\|\varphi\| = \|\varphi(e_{\mathfrak{C}_1})\|_{\mathfrak{C}_2} \leq \|\psi(e_{\mathfrak{C}_1})\|_{\mathfrak{C}_2} = \|\psi\|$, with $\varphi, \psi \in \mathfrak{B}(\mathfrak{C}_1, \mathfrak{C}_2)^+$ and $\varphi \leq \psi$, see e.g. [26]). Moreover, if $\mathcal{H}$ is a separable Hilbert space, the norm in $\mathfrak{B}_1(\mathcal{H})$ is clearly order-preserving.

Let $\mathfrak{X}$ be a vector space and $\mathfrak{Y}$ an ordered Banach bimodule over the *-algebra $\mathfrak{Y}_0$, with positive cone $\mathfrak{K}$. Let Φ be a $\mathfrak{Y}$-valued positive sesquilinear map on $\mathfrak{X} \times \mathfrak{X}$

$$\Phi : (x_1, x_2) \in \mathfrak{X} \times \mathfrak{X} \to \Phi(x_1, x_2) \in \mathfrak{Y}$$

i.e., a map with the properties

(i) $\Phi(x_1, x_1) \in \mathfrak{K}$,
(ii) $\Phi(\alpha x_1 + \beta x_2, \gamma x_3) = \overline{\gamma}[\alpha\Phi(x_1, x_3) + \beta\Phi(x_2, x_3)]$,

with $x_1, x_2, x_3 \in \mathfrak{X}$ and $\alpha, \beta, \gamma \in \mathbb{C}$.

The $\mathfrak{Y}$-valued positive sesquilinear map Φ is called *faithful* if

$$\Phi(x, x) = 0_{\mathfrak{Y}} \;\Rightarrow\; x = 0.$$

3 Representations induced by positive maps

It is quite apparent that the cornerstone of every extension of the GNS-construction is a Cauchy–Schwarz inequality. We begin with presenting some results in this direction.

Throughout this section we will assume that $\mathfrak{C}$ is a C^*-algebra with positive convex cone $\mathfrak{C}^+$ and norm $\|\cdot\|_{\mathfrak{C}}$, $\mathfrak{Y}$ is (an ordered) Banach bimodule over a *-algebra $\mathfrak{Y}_0$ (with $\mathfrak{Y}_0$ equipped with a not necessarily sub-multiplicative norm $\|\cdot\|_{\mathfrak{Y}_0}$) with the respective cones $\mathfrak{K}$ and $\mathfrak{K}_0$, with $\mathfrak{K}_0 = \{\sum_{i=1}^N z_i^* z_i,\ z_i \in \mathfrak{Y}_0, i = 1, \dots, N; N \in \mathbb{N}\}$.

A positive sesquilinear map Φ from a complex vector space $\mathfrak{X}$ into an ordered Banach bimodule $\mathfrak{Y}$ is said to satisfy the *Cauchy–Schwarz inequality* if, for every $x_1, x_2 \in \mathfrak{X}$ it holds that

$$\|\Phi(x_1, x_2)\|_{\mathfrak{Y}} \leq \|\Phi(x_1, x_1)\|_{\mathfrak{Y}}^{1/2} \|\Phi(x_2, x_2)\|_{\mathfrak{Y}}^{1/2}.$$

From now on, by ρ we will denote a faithful semifinite trace on a von Neumann algebra $\mathfrak{M}$.

Proposition 3.1 *Let $\mathfrak{X}$ be a complex vector space. Assume that $\mathfrak{Y}$, $\mathfrak{Y}_0$, $\mathfrak{K}$ and $\mathfrak{C}$ are as above. Let $\Phi : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{Y}$ be a positive sesquilinear map. Let Ω, Υ be locally compact Hausdorff spaces, $\mathcal{H}$ a separable Hilbert space, $\mathfrak{M}$ a von Neumann algebra and ρ a faithful trace on $\mathfrak{M}$. Then, the Cauchy–Schwarz inequality holds in each one of the following cases*

1. $\mathfrak{Y} = L^2(\Omega)$,
2. $\mathfrak{Y} = \mathfrak{B}(\mathfrak{M}, \mathfrak{B}_1(\mathcal{H}))$, $\mathfrak{Y}_0 = \mathfrak{M}$,
3. $\mathfrak{Y} = \mathfrak{M}^*$, *the dual of* $\mathfrak{M}$,
4. $\mathfrak{Y} = M(\Omega)$ *and* $\mathfrak{Y}_0 = C(\Omega)$,
5. $\mathfrak{Y} = \mathfrak{B}(\mathfrak{M}, M(\Omega))$, $\mathfrak{Y}_0 = \mathfrak{M}$,
6. $\mathfrak{Y} = \mathfrak{B}(C(\Omega), C(\Upsilon))$, *or* $\mathfrak{Y} = \mathfrak{B}(\mathfrak{M}, C(\Omega))$,
7. $\mathfrak{Y} = L^1(\rho)$ *or* $\mathfrak{Y} = L^1(\Omega)$,
8. $\mathfrak{Y} = \mathfrak{B}(\mathfrak{M}, \widetilde{\mathfrak{M}}^*)$ *and* $\mathfrak{Y}_0 = \mathfrak{M}$, *with* $\widetilde{\mathfrak{M}}$ *another von Neumann algebra,*
9. $\mathfrak{Y} = \mathfrak{Y}_0 = \ell_2(C(\Omega))$.

Proof We just give a sketch of the proof of each case.

(1) Let S be a simple function on Ω, then $S = \sum_{j=1}^N c_j \chi_{A_j}$. Then, for each j, the map $\varphi_j(x_1, x_2) = \int_\Omega \Phi(x_1, x_2)\chi_{A_j} d\mu$, $x_1, x_2 \in \mathfrak{X}$, is a positive sesquilinear form, so we can apply the Cauchy–Schwarz inequality and get:

$$\left| \int_\Omega \Phi(x_1, x_2) S d\mu \right| \leq \|\Phi(x_1, x_1)\|_2^{1/2} \|\Phi(x_2, x_2)\|_2^{1/2} \|S\|_2, \quad \forall x_1, x_2 \in \mathfrak{X};$$

Then consider a sequence $\{S_n\}_n$ of simple functions such that $S_n \to \overline{\Phi(x_1, x_2)}$ in $L^2(\Omega, \mu)$ as $n \to \infty$. By taking limits in the above inequality, we get the Cauchy–Schwarz inequality.

(2) Let $P \in \mathfrak{M}$ and $Q \in \mathfrak{B}(\mathcal{H})$ be orthogonal projections. Define $\varphi_{P,Q} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ by

$$\varphi_{P,Q}(x_1, x_2) = tr(Q(\Phi(x_1, x_2)(P))), \quad \forall x_1, x_2 \in \mathfrak{X}. \tag{3.1}$$

Then $\varphi_{P,Q}$ is a positive sesquilinear form; hence, using the Cauchy–Schwarz inequality, we get

$$|tr(Q(\Phi(x_1, x_2)(P)))| \leq (tr(Q(\Phi(x_1, x_1)(P))))^{1/2} (tr(Q(\Phi(x_2, x_2)(P))))^{1/2} \tag{3.2}$$

for all $x_1, x_2 \in \mathfrak{X}$. Let now $S \in \mathfrak{M}$ and $T \in \mathfrak{B}(\mathcal{H})$ be two simple operator functions, so $S = \sum_{i=1}^N \alpha_i P_i$ and $T = \sum_{j=1}^M \beta_j Q_j$ with $P_{i_1} P_{i_2} = Q_{j_1} Q_{j_2} = 0$ whenever $i_1 \neq i_2$ and $j_1 \neq j_2$, $\sum_{i=1}^N P_i = \mathbb{I}_{\mathfrak{M}}$ and $\sum_{j=1}^M Q_j = \mathbb{I}_{\mathcal{H}}$. By (3.2) and some computations we get that

$$|tr(T(\Phi(x_1,x_2)(S)))| \le \sum_{i,j} |\alpha_i||\beta_j||tr(Q_j(\Phi(x_1,x_2)(P_i)))|$$
$$\le \|T\|\|S\|(tr(\Phi(x_1,x_1)(\mathbb{I}_{\mathfrak{M}})))^{1/2}(tr(\Phi(x_2,x_2)(\mathbb{I}_{\mathfrak{M}})))^{1/2}$$
$$\le \|T\|\|S\|\|\Phi(x_1,x_1)\|^{1/2}\|\Phi(x_2,x_2)\|^{1/2}, \quad \forall x_1,x_2 \in \mathfrak{X}. \tag{3.3}$$

Let now $U \in \mathfrak{M}$ and $W \in \mathfrak{B}(\mathcal{H})$ be unitary. Then, by the spectral theorem, we can find sequences $\{S_n\}_n \subset \mathfrak{M}$ and $\{T_n\}_n \subset \mathfrak{B}(\mathcal{H})$ that converge in the operator norm to U and W, respectively. Then, since

$$|tr(B(\Phi(x_1,x_2)(A)))| \le tr(|B(\Phi(x_1,x_2)(A))|)$$
$$\le \|B\|\|\Phi(x_1,x_2)(A)\|_1 \le \|B\|\|A\|\|\Phi(x_1,x_2)\|, \tag{3.4}$$

for all $A \in \mathfrak{M}$, $B \in \mathfrak{B}(\mathcal{H})$, $x_1, x_2 \in \mathfrak{X}$, it follows that

$$tr(T_n(\Phi(x_1,x_2)(S_n))) \to tr(W(\Phi(x_1,x_2)(U))), \text{ as } n \to \infty.$$

Also, $\|T_n\|\|S_n\| \to \|W\|\|U\| = 1$ as $n \to \infty$, hence, by using inequality (3.3), passing to the limits, we deduce that

$$|tr(W(\Phi(x_1,x_2)(U)))| \le \|\Phi(x_1,x_1)\|^{1/2}\|\Phi(x_2,x_2)\|^{1/2}, \tag{3.5}$$

for all $x_1, x_2 \in \mathfrak{X}$. This inequality extends easily to convex combinations $\sum_{i=1}^N \lambda_i U_i, \sum_{j=1}^M \gamma_j W_j$.
The Russo–Dye theorem then allows to get the following inequality for $A \in \mathfrak{M}$ and $B \in \mathfrak{B}(\mathcal{H})$ with $\|A\|, \|B\| < 1$,

$$|tr(B(\Phi(x_1,x_2)(A)))| \le \|\Phi(x_1,x_1)\|^{1/2}\|\Phi(x_2,x_2)\|^{1/2}, \quad \forall x_1,x_2 \in \mathfrak{X}. \tag{3.6}$$

Finally, let $A \in \mathfrak{M}$ with $\|A\| = 1$, and $V \in \mathfrak{B}(\mathcal{H})$ be the partial isometry with $V(\Phi(x_1,x_2)(A)) = |\Phi(x_1,x_2)(A)|$ and choose sequences $\{A_n\}_n \subset \mathfrak{M}$ and $\{V_n\}_n \subset \mathfrak{B}(\mathcal{H})$, with $\|A_n\|, \|V_n\| < 1$ for all n, such that $A_n \to A$ and $V_n \to V$ in the operator norm as $n \to \infty$. Once again, by using (3.4) and a limit procedure we get

$$\|\Phi(x_1,x_2)(A)\|_1 = tr(V(\Phi(x_1,x_2)(A))) \le \|\Phi(x_1,x_1)\|^{1/2}\|\Phi(x_2,x_2)\|^{1/2},$$

for all $x_1, x_2 \in \mathfrak{X}$. Finally, by taking the supremum over the unit ball in $\mathfrak{M}$, we conclude that

$$\|\Phi(x_1,x_2)\| \le \|\Phi(x_1,x_1)\|^{1/2}\|\Phi(x_2,x_2)\|^{1/2}, \quad \forall x_1,x_2 \in \mathfrak{X}.$$

(3) It is a special case of (2): if $\mathcal{H}$ is one dimensional, then $\mathfrak{B}_1(\mathcal{H}) = \mathbb{C}$.

(4) We can apply the Cauchy–Schwarz inequality to every positive sesquilinear form $\varphi_E(x_1, x_2) = \Phi(x_1, x_2)(E)$, with E a given Borel subset of Ω, and therefore by considering a partition $\{E_n\}_{n=1}^N$ of Ω we obtain

$$\begin{aligned}\sum_{n=1}^{N} |\Phi(x_1, x_2)(E_n)| &\leq \sum_{n=1}^{N} (\Phi(x_1, x_1)(E_n))^{1/2} (\Phi(x_2, x_2)(E_n))^{1/2} \\ &\leq \left(\sum_{n=1}^{N} \Phi(x_1, x_1)(E_n)\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} \Phi(x_2, x_2)(E_n)\right)^{\frac{1}{2}}\end{aligned}$$

hence, by taking the supremum of both sides, we get the Cauchy–Schwarz inequality.

(5) Let $P \in \mathfrak{M}$ be an orthogonal projection and E be a Borel subset of Ω. The map $\varphi_{P,E} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ given by

$$\varphi_{P,E}(x_1, x_2) = \Phi(x_1, x_2)(P)(E), \quad \forall x_1, x_2 \in \mathfrak{X}$$

is a positive sesquilinear form, hence, by similar arguments as in the proof of (4) we can deduce that

$$\|\Phi(x_1, x_2)(P)\| \leq (\Phi(x_1, x_1)(P))^{1/2} (\Phi(x_2, x_2)(P))^{1/2}, \quad \forall x_1, x_2 \in \mathfrak{X}.$$

Since this holds for an arbitrary orthogonal projection $P \in \mathfrak{M}$, we can then proceed further in a similar way as in the proof of (2) and deduce Cauchy–Schwarz inequality for Φ. We leave the details to the reader.

(6) The map $\widetilde{\varphi}_t : \mathfrak{X} \times \mathfrak{X} \to (C(\Omega))^* \cong M(\Omega)$ given by

$$\widetilde{\varphi}_t(x_1, x_2)(f) = (\Phi(x_1, x_2)(f))(t), \quad \forall f \in C(\Omega), x_1, x_2 \in \mathfrak{X}$$

is sesquilinear and positive, hence, the Cauchy–Schwarz inequality holds for $\widetilde{\varphi}_t$ by case (4). Therefore, for every $f \in C(\Omega)$, with $\|f\| \leq 1$ and every $t \in \Upsilon$, we get

$$\begin{aligned}|(\Phi(x_1, x_2)(f))(t)| &\leq \|\widetilde{\varphi}_t(x_1, x_2)\| \leq \|\widetilde{\varphi}_t(x_1, x_1)\|^{1/2} \|\widetilde{\varphi}_t(x_2, x_2)\|^{1/2} \\ &\leq \left(\sup_{\substack{g \in C(\Omega), \\ \|g\|_\infty \leq 1}} \|\Phi(x_1, x_1)(g)\|_\infty \right)^{1/2} \\ &\quad \left(\sup_{\substack{h \in C(\Omega), \\ \|h\|_\infty \leq 1}} \|\Phi(x_2, x_2)(g)\|_\infty \right)^{1/2} \\ &\leq \|\Phi(x_1, x_1)\|^{1/2} \|\Phi(x_2, x_2)\|^{1/2}.\end{aligned}$$

The statement follows by taking supremums.

The case $\mathfrak{Y} = \mathfrak{B}(\mathfrak{M}, C(\Omega))$ can be treated similarly by using (3).

(7) Consider the map $\iota : L^1(\rho) \to \widetilde{\mathfrak{M}}^*$ given by

$$\iota(T)(S) = \rho(ST), \quad \forall T \in L^1(\rho), S \in \mathfrak{M}.$$

It is not hard to check that ι is an isometry. Moreover, since $\iota(T)(S) = \rho(S^{1/2}TS^{1/2}) \geq 0$ whenever S and T are positive, it follows that ι preserves positivity. Hence, if $\Phi : \mathfrak{X} \times \mathfrak{X} \to L^1(\rho)$ is a positive sesquilinear map, then $\iota \circ \Phi : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{M}^*$ is also a positive sesquilinear map. By (3) and since ι is an isometry, we get that

$$\begin{aligned} \|\Phi(x_1, x_2)\|_1 &= \|\iota \circ \Phi(x_1, x_2)\| \leq \|\iota \circ \Phi(x_1, x_1)\|^{1/2} \|\iota \circ \Phi(x_2, x_2)\|^{1/2} \\ &= \|\Phi(x_1, x_2)\|_1^{1/2} \|\Phi(x_1, x_2)\|_1^{1/2}, \quad \forall x_1, x_2 \in \mathfrak{X}. \end{aligned}$$

Of course, an analogous result holds in the commutative case (i.e. $\mathfrak{Y} = L^1(\Omega)$).

(8) In the proof of (2), given $P \in \mathfrak{M}$ and $Q \in \mathfrak{B}(\mathcal{H})$, we have considered the sesquilinear form $\varphi_{P,Q}$ in (3.1). Now, given $S \in \mathfrak{M}$ and $T \in \widetilde{\mathfrak{M}}$, we can consider, instead, the sesquilinear form $\varphi_{S,T} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ given by

$$\varphi_{S,T}(x_1, x_2) = \Phi(x_1, x_2)(S)(T), \quad \forall x_1, x_2 \in \mathfrak{X}$$

and proceed analogously to the proof of (2).

(9) We can apply the Cauchy–Schwarz inequality to all the positive sesquilinear forms $\varphi_{n,t}(x_1, x_2) = (\Phi(x_1, x_2))_n(t)$ with $(\Phi(x_1, x_2))_n$ the n-th component of the sequence $\Phi(x_1, x_2)$ and then applying the Cauchy–Schwarz inequality for the inner product in $\ell_2(C(\Omega))$.

□

From now on $\mathfrak{Y}$ will denote any Banach bimodule considered in Proposition 3.1. The next corollary is motivated by modified Schwarz inequality for 2-positive maps (see [20, Ch. 3, Ex. 3.4, p.40] and [8, Lemma 2.6])

Corollary 3.2 *If $\mathfrak{A}$ is a unital *-algebra and $\omega : \mathfrak{A} \to \mathfrak{Y}$ is a positive linear map, then*

$$\|\omega(b^*a)\|_{\mathfrak{Y}} \leq \|\omega(a^*a)\|_{\mathfrak{Y}}^{1/2} \|\omega(b^*b)\|_{\mathfrak{Y}}^{1/2}, \quad \forall a, b \in \mathfrak{A}.$$

Proof This follows from Proposition 3.1 by considering the sesquilinear map $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{Y}$ given by $\Phi(a, b) = \omega(b^*a)$, for all $a, b \in \mathfrak{A}$. □

Now we give an extension of Proposition 3.1.

Proposition 3.3 *Let $\mathfrak{X}$ be a vector space and assume that the norm on $\mathfrak{Y}$ is order-preserving. If $\{\Phi_n\}_n$ is a sequence of positive sesquilinear maps $\Phi_n : \mathfrak{X} \times \mathfrak{X} \to$*

$\mathfrak{Y}$ *and* $\{x_n\}_n$, $\{\widetilde{x}_n\}_n$ *are sequences in* $\mathfrak{X}$ *such that the series* $\sum_{n=1}^{\infty} \Phi_n(x_n, x_n)$ *and* $\sum_{n=1}^{\infty} \Phi_n(\widetilde{x}_n, \widetilde{x}_n)$ *are convergent in* $\mathfrak{Y}$ *then*

$$\left\| \sum_{n=1}^{\infty} \Phi_n(x_n, \widetilde{x}_n) \right\|_{\mathfrak{Y}} \leq \left\| \sum_{n=1}^{\infty} \Phi_n(x_n, x_n) \right\|_{\mathfrak{Y}}^{1/2} \left\| \sum_{n=1}^{\infty} \Phi_n(\widetilde{x}_n, \widetilde{x}_n) \right\|_{\mathfrak{Y}}^{1/2}.$$

Proof The statement can be proved by arguments similar to the first part of [18, Example 1.3.5] due to Proposition 3.1 and the fact that the norm of $\mathfrak{Y}$ is order-preserving. □

For a detailed overview of some other operator-inequalities for positive maps we refer to [7, 16].

Definition 3.4 Let $\mathfrak{X}$ be a vector space. A $\mathfrak{Y}$-valued faithful positive sesquilinear map Φ on $\mathfrak{X} \times \mathfrak{X}$ is said to be a $\mathfrak{Y}$*-valued inner product* on $\mathfrak{X}$ and we often write $\langle x_1 | x_2 \rangle_\Phi := \Phi(x_1, x_2)$, $x_1, x_2 \in \mathfrak{X}$.

A $\mathfrak{Y}$-valued inner product on $\mathfrak{X}$

$$\Phi : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{Y}$$

induces a norm $\|\cdot\|_\Phi$ on $\mathfrak{X}$:

$$\|x\|_\Phi := \sqrt{\|\langle x|x\rangle_\Phi\|_{\mathfrak{Y}}} = \sqrt{\|\Phi(x, x)\|_{\mathfrak{Y}}}, \quad x \in \mathfrak{X},$$

since

$$\|x_1 + x_2\|_\Phi \leq \|x_1\|_\Phi + \|x_2\|_\Phi, \qquad \forall x_1, x_2 \in \mathfrak{X};$$

this can be shown similarly to what done in [6].

The space $\mathfrak{X}$ is then a normed space w.r.to the norm $\|\cdot\|_\Phi$.

Definition 3.5 Let $\mathfrak{X}$ be a complex vector space and Φ be a $\mathfrak{Y}$-valued inner product on $\mathfrak{X}$. If $\mathfrak{X}$ is complete w.r. to the norm $\|\cdot\|_\Phi$, then $\mathfrak{X}$ is called a *Banach space with* $\mathfrak{Y}$*-valued inner product* or for short a $B_{\mathfrak{Y}}$*-space*.

If Φ is not faithful, we can consider the subspace of $\mathfrak{X}$

$$\mathfrak{N}_\Phi = \{x_1 \in \mathfrak{X} : \ \Phi(x_1, x_2) = 0_{\mathfrak{Y}}, \forall x_2 \in \mathfrak{X}\}.$$

By Proposition 3.1, then

$$\mathfrak{N}_\Phi = \{x \in \mathfrak{X} : \ \Phi(x, x) = 0_{\mathfrak{Y}}\}.$$

We denote by $\Lambda_\Phi(x)$ the coset of $\mathfrak{X}/\mathfrak{N}_\Phi$ containing $x \in \mathfrak{X}$; i.e., $\Lambda_\Phi(x) = x + \mathfrak{N}_\Phi$ and define a $\mathfrak{Y}$-valued inner product on $\mathfrak{X}/\mathfrak{N}_\Phi$ as follows:

$$\langle \Lambda_\Phi(x_1)|\Lambda_\Phi(x_2)\rangle_\Phi := \Phi(x_1, x_2), \quad x_1, x_2 \in \mathfrak{X}.$$

The associated norm is:

$$\|\Lambda_\Phi(x)\|_\Phi := \sqrt{\|\Phi(x, x)\|_{\mathfrak{Y}}}, \quad x \in \mathfrak{X}.$$

The quotient space $\mathfrak{X}/\mathfrak{N}_\Phi = \Lambda_\Phi(\mathfrak{X})$ is a normed space (see [6]).

Let $\mathcal{K}$ be a $B_{\mathfrak{Y}}$-space and $D(T)$ be a dense subspace of $\mathcal{K}$. A linear map $T : D(T) \to \mathcal{K}$ is said Φ-*adjointable* if there exists a linear map T^* defined on a subspace $D(T^*) \subset \mathcal{K}$ such that

$$\Phi(T\xi, \eta) = \Phi(\xi, T^*\eta), \quad \forall \xi \in D(T), \eta \in D(T^*).$$

Let $\mathcal{D}$ be a dense subspace of $\mathcal{K}$ and let us consider the following families of linear operators acting on $\mathcal{D}$:

$$\mathcal{L}^\dagger(\mathcal{D}, \mathcal{K}) = \{T\,\Phi\text{-adjointable}, D(T) = \mathcal{D};\; D(T^*) \supset \mathcal{D}\}$$
$$\mathcal{L}^\dagger(\mathcal{D}) = \{T \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{K}) : T\mathcal{D} \subset \mathcal{D};\; T^*\mathcal{D} \subset \mathcal{D}\}.$$

The involution in $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{K})$ is defined by $T^\dagger := T^* \upharpoonright \mathcal{D}$, the restriction of T^*, the Φ-adjoint of T, to $\mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra.

Remark 3.6 If $T \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{K})$ then T is closable and its Φ-adjoint T^* is closed (it can be shown similarly as in [6, Remark 2.8] due to Proposition 3.1). Moreover, the space $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{K})$ is a *partial *-algebra* [1] with respect to the following operations: the usual sum $T_1 + T_2$, the scalar multiplication λT, the involution $T \mapsto T^\dagger := T^* \upharpoonright \mathcal{D}$ and the *(weak)* partial multiplication $\square$ of two operators $T_1, T_2 \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{K})$ defined whenever there exists $W \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{K})$ such that

$$\Phi(T_2\xi, T_1^\dagger\eta) = \Phi(W\xi, \eta), \quad \forall \xi, \eta \in \mathcal{D}.$$

Due to the density of $\mathcal{D}$ in $\mathcal{K}$, the element W, if it exists, is unique. We put $W = T_1 \square T_2$.

Definition 3.7 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit e. Let $\mathcal{D}$ be a dense subspace of a certain $B_{\mathfrak{Y}}$-space $\mathcal{K}$ with $\mathfrak{Y}$-valued inner product $\langle\cdot|\cdot\rangle_{\mathcal{K}}$. A linear map π from $\mathfrak{A}$ into $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{K})$ is called a **-representation* of $(\mathfrak{A}, \mathfrak{A}_0)$, if the following properties are fulfilled:

(i) $\pi(a^*) = \pi(a)^\dagger := \pi(a)^* \upharpoonright \mathcal{D}, \quad \forall\, a \in \mathfrak{A}$;
(ii) for $a \in \mathfrak{A}$ and $c \in \mathfrak{A}_0$, $\pi(a) \square \pi(c)$ is well-defined and $\pi(a) \square \pi(c) = \pi(ac)$.

We assume that for every *-representation π of $(\mathfrak{A}, \mathfrak{A}_0)$, $\pi(\mathsf{e}) = \mathbb{I}_{\mathcal{D}}$, the identity operator on the space $\mathcal{D}$.

The *-representation π is said to be

- *closable* if there exists $\widetilde{\pi}$ the closure of π, defined as $\widetilde{\pi}(a) = \overline{\pi(a)} \upharpoonright \widetilde{\mathcal{D}}$ where $\widetilde{\mathcal{D}}$ is the completion under the graph topology t_π defined by the seminorms $\xi \in \mathcal{D} \to \|\xi\|_{\mathcal{K}} + \|\pi(a)\xi\|_{\mathcal{K}}$, $a \in \mathfrak{A}$, with $\|\cdot\|_{\mathcal{K}}$ the norm induced by the inner product on $\mathcal{K}$;
- *closed* if $\mathcal{D}[t_\pi]$ is complete;
- *cyclic* if there exists $\xi \in \mathcal{D}$ such that $\pi(\mathfrak{A}_0)\xi$ is dense in $\mathcal{K}$ in its norm topology. In this case ξ is called *cyclic vector*.

Definition 3.8 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra. We denote by $\mathcal{I}^{\mathfrak{Y}}_{\mathfrak{A}_0}(\mathfrak{A})$ the set of those positive sesquilinear maps $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{Y}$ such that:

(i) $\Lambda_\Phi(\mathfrak{A}_0) = \mathfrak{A}_0/\mathfrak{N}_\Phi$ is dense in the completion $\widetilde{\mathfrak{A}}$ of $\mathfrak{A}/\mathfrak{N}_\Phi$ w.r. to the norm $\|\cdot\|_\Phi$;
(ii) are left-invariant, i.e., $\Phi(ac, d) = \Phi(c, a^*d)$, $\forall\, a \in \mathfrak{A},\ c, d \in \mathfrak{A}_0$.

If $\mathfrak{A} = \mathfrak{A}_0$ and $\omega : \mathfrak{A} \to \mathfrak{Y}$ is a positive linear map, then $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{Y}$ given by

$$\Phi(a, b) = \omega(b^*a), \quad \forall a, b \in \mathfrak{A}$$

satisfies the conditions of Definition 3.8, i.e. $\Phi \in \mathcal{I}^{\mathfrak{Y}}_{\mathfrak{A}_0}(\mathfrak{A})$.

Proposition 3.9 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit e and Φ be a $\mathfrak{Y}$-valued left-invariant positive sesquilinear map on $\mathfrak{A} \times \mathfrak{A}$. The following statements are equivalent:*

(i) $\Phi \in \mathcal{I}^{\mathfrak{Y}}_{\mathfrak{A}_0}(\mathfrak{A})$;
(ii) *there exist a $B_{\mathfrak{Y}}$-space $\mathcal{K}_\Phi$ with $\mathfrak{Y}$-valued inner product $\langle\cdot|\cdot\rangle_{\mathcal{K}_\Phi}$, a dense subspace $\mathcal{D}_\Phi \subseteq \mathcal{K}_\Phi$ and a closed cyclic *-representation $\pi : \mathfrak{A} \to \mathcal{L}^\dagger(\mathcal{D}_\Phi, \mathcal{K}_\Phi)$ with cyclic vector ξ_Φ such that*

$$\langle \pi(a)\xi|\eta\rangle_{\mathcal{K}_\Phi} = \langle \xi|\pi(a^*)\eta\rangle_{\mathcal{K}_\Phi}, \quad \forall \xi, \eta \in \mathcal{D}_\Phi, a \in \mathfrak{A}$$

and such that

$$\Phi(a, b) = \langle \pi(a)\xi_\Phi|\pi(b)\xi_\Phi\rangle_{\mathcal{K}_\Phi}, \quad \forall a, b \in \mathfrak{A}.$$

Proof The proof proceeds along the lines of that one of [6, Theorem 3.2], due to the Cauchy–Schwarz inequality for positive $\mathfrak{Y}$-valued sesquilinear maps (Proposition 3.1) and due to Remark 3.6. □

Remark 3.10 By the same arguments as in [6, Corollary 3.5] one can show that every $\mathfrak{Y}$-valued, *bounded*, left-invariant, positive sesquilinear map on a unital normed quasi *-algebra belongs to $\mathcal{I}^{\mathfrak{Y}}_{\mathfrak{A}_0}(\mathfrak{A})$.

In the following example we construct some $\mathfrak{Y}$-valued, bounded, left-invariant, positive sesquilinear maps.

Example 3.11 Let ρ be a finite trace on a von Neumann algebra $\mathfrak{M}$. Let $W \in L^\infty(\rho)$, with $W \geq 0$. Let $k \in C([0, \|W\|] \times [0, \|W\|])$ be such that $k \geq 0$. Then, for each $x \in [0, \|W\|]$, the function $\eta_x : [0, \|W\|] \to \mathbb{C}$ defined by $\eta_x(t) = k(x, t)$ is a continuous, positive function on $[0, \|W\|]$. Therefore, by the functional calculus,

$\eta_x(W)$ defines a positive operator in $L^\infty(\rho)$. Let us define, for every $x \in [0, \|W\|]$ and $X, Y \in L^2(\rho)$,

$$\varphi(X, Y)(x) = \rho(X\eta_x(W)Y^*).$$

Then, $\varphi(X, Y) \in C([0, \|W\|])$ for all $X, Y \in L^2(\rho)$. Moreover, $\varphi : L^2(\rho) \times L^2(\rho) \to C([0, \|W\|])$ is a bounded, left-invariant, positive sesquilinear map (see [6]).

Let $f : [0, 1] \to [0, \|W\|]$ be measurable. For every $X, Y \in L^2(\rho)$, let $\psi : L^2(\rho) \times L^2(\rho) \to L^2([0, 1])$ defined by $\psi(X, Y) = \phi(X, Y) \circ f$. Then one can check that ψ is a bounded, left-invariant, positive sesquilinear map.

Finally, if $(\mathfrak{A}, \mathfrak{A}_0)$ is a unital quasi *-algebra and $\Psi : \mathfrak{A} \times \mathfrak{A} \to L^1(\Omega)$ is a bounded left-invariant positive sesquilinear map, then the induced map $\widetilde{\Psi} : \mathfrak{A} \times \mathfrak{A} \to M(\Omega)$ given by $d\widetilde{\Psi}(a, b) = \Psi(a, b)d\mu$, for all $a, b \in \mathfrak{A}$ and some fixed positive measure $\mu \in M(\Omega)$, is also a bounded left-invariant positive sesquilinear map.

Corollary 3.12 *Let $\mathfrak{A}$ be a *–algebra with unit e and let ω be a positive linear $\mathfrak{Y}$-valued map on $\mathfrak{A}$. Then, there exists a $B_{\mathfrak{Y}}$-space $\mathcal{K}_\Phi$ whose norm is induced by a $\mathfrak{Y}$-valued inner product $\langle\cdot|\cdot\rangle_{\mathcal{K}_\Phi}$, a dense subspace $\mathcal{D}_\omega \subseteq \mathcal{K}_\Phi$ and a closed cyclic *–representation Π_ω of $\mathfrak{A}$ with domain $\mathcal{D}_\omega$, such that*

$$\omega(b^*ac) = \langle \Pi_\omega(a)\Lambda_\omega(c)|\Lambda_\omega(b)\rangle_{\mathcal{K}_\Phi}, \quad \forall a, b, c \in \mathfrak{A}.$$

Moreover, there exists a cyclic vector η_ω, such that

$$\omega(a) = \langle \Pi_\omega(a)\eta_\omega|\eta_\omega\rangle_{\mathcal{K}_\Phi}, \quad \forall\, a \in \mathfrak{A}.$$

The representation is unique up to unitary equivalence.

Proof The statement can be proved analogously to [6, Corollary 3.10], due to Proposition 3.9. □

4 Representations induced by completely positive sesquilinear maps

In this section, we extend the results of Sect. 3 to the case of $\mathfrak{Y}$-valued completely positive sesquilinear maps.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be vector spaces. We will denote by $\mathcal{S}_{\mathfrak{Y}}(\mathfrak{X})$ the space of all $\mathfrak{Y}$-valued sesquilinear maps on $\mathfrak{X}$.

Definition 4.1 Let $\mathfrak{X}$ be a normed vector space, $\mathfrak{Y}$ an ordered Banach module over the *-algebra $\mathfrak{Y}_0$, with positive cone $\mathfrak{K}$ and $(\mathfrak{A}, \mathfrak{A}_0)$ a normed quasi *-algebra with unit. The sesquilinear map $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathcal{S}_{\mathfrak{Y}}(\mathfrak{X})$ is called

- *bounded* if there exists a constant $M > 0$ such that

$$\|\Phi(a, b)(x_1, x_2)\|_{\mathfrak{Y}} \le M\|a\|\|b\|\|x_1\|\|x_2\|, \quad \forall a, b \in \mathfrak{A}, x_1, x_2 \in \mathfrak{X};$$

- *left-invariant* if $\Phi(ac, d) = \Phi(c, a^*d), \quad \forall a \in \mathfrak{A}, c, d \in \mathfrak{A}_0$;

- *completely positive* if for every $N \in \mathbb{N}$, $a_1, \dots, a_N \in \mathfrak{A}$, $x_1, \dots, x_N \in \mathfrak{X}$,

$$\sum_{i,j=1}^{N} \Phi(a_i, a_j)(x_i, x_j) \in \mathfrak{K}.$$

The Stinespring Theorem is the main result on completely positive linear maps. In the following we consider certain *completely positive sesquilinear maps* taking values on the space of all $\mathfrak{Y}$-valued sesquilinear maps on a vector space, see Theorem 4.2 below.

Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a normed quasi *-algebra with unit e, $\mathfrak{X}$ be a normed complex vector space. Let the cone $\mathfrak{K}$ in $\mathfrak{Y}$ be closed and $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathcal{S}_{\mathfrak{Y}}(\mathfrak{X})$ be a left-invariant positive sesquilinear map. Consider the algebraic tensor product $\mathfrak{A} \otimes \mathfrak{X}$ and its subset

$$\mathcal{N}_\Phi = \left\{ \sum_{i=1}^{n} a_i \otimes x_i \in \mathfrak{A} \otimes \mathfrak{X} | \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi(a_i, a_j)(x_i, x_j) = 0_{\mathfrak{Y}} \right\}.$$

Hence, it is easy to check that the quotient space $(\mathfrak{A} \otimes \mathfrak{X})/\mathcal{N}_\Phi$ is a normed space.

Theorem 4.2 *Let $\mathfrak{X}$ be a normed complex vector space. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a normed quasi *-algebra with unit e and $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathcal{S}_{\mathfrak{Y}}(\mathfrak{X})$ a left-invariant sesquilinear map. Let Φ be bounded with bound $M > 0$. Then, Φ is completely positive if and only if there exist a $B_{\mathfrak{Y}}$-space $\mathcal{K}_\Phi$, a dense subspace $\mathcal{D}_\Phi$ of $\mathcal{K}_\Phi$, a closed *-representation π of $\mathfrak{A}$ in $\mathcal{L}^\dagger(\mathcal{D}_\Phi, \mathcal{K}_\Phi)$ and a bounded linear operator $V : \mathfrak{X} \to \mathcal{D}_\Phi$ such that $\pi(\mathfrak{A})V\mathfrak{X} = (\mathfrak{A} \otimes \mathfrak{X})/\mathcal{N}_\Phi$ and, for all $a, b \in \mathfrak{A}$ and $x_1, x_2 \in \mathfrak{X}$, it holds that*

$$\Phi(a, b)(x_1, x_2) = \langle \pi(a)Vx_1 | \pi(b)Vx_2 \rangle_{\mathcal{K}_\Phi}.$$

In this case $\|V\|^2 \le M\|\mathsf{e}\|^2$. Moreover, the triple $(\pi, V, \mathcal{D}_\Phi)$ is such that $\overline{\pi(\mathfrak{A})V\mathfrak{X}} = \mathcal{K}_\Phi$.

Proof The proof goes along the lines of that of [4] where more traditional sesquilinear forms were considered thanks to Proposition 3.1 and Remark 3.6. We omit the details. □

In the following we will refer to a triple $(\pi, V, \mathcal{D}_\Phi)$ as in Theorem 4.2 as a *Stinespring triple decomposing* Φ.

Definition 4.3 Let $\mathfrak{X}_1, \mathfrak{X}_2$ be two $B_{\mathfrak{Y}}$-spaces with respect to the norms $\|\cdot\|_\Phi$ and $\|\cdot\|_\Psi$, respectively, induced by two $\mathfrak{Y}$-valued inner products $\Phi : \mathfrak{X}_1 \times \mathfrak{X}_1 \to \mathfrak{Y}$ and $\Psi : \mathfrak{X}_2 \times \mathfrak{X}_2 \to \mathfrak{Y}$. A surjective operator $U : \mathfrak{X}_1 \to \mathfrak{X}_2$ is said *unitary* if

$$\langle U\xi | U\eta \rangle_\Psi = \langle \xi | \eta \rangle_\Phi, \quad \forall \xi, \eta \in \mathfrak{X}_1.$$

In analogy to Corollary 3.10 in [4], the following result can be proved.

Proposition 4.4 *Let $\mathfrak{X}$ be a normed complex vector space. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a normed quasi *-algebra with unit* e *and $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathcal{S}_{\mathfrak{Y}}(\mathfrak{X})$ be a bounded left-invariant sesquilinear map. If Φ is completely positive then π and V in the triple $(\pi, V, \mathcal{D}_\Phi)$ are uniquely determined by Φ up to unitary equivalence, i.e., if $(\pi, V, \mathcal{D}_\Phi)$ and $(\pi_1, V_1, \mathcal{E}_\Phi)$ are two Stinespring triples decomposing Φ, there exists a unitary operator U such that $UV = V_1$, $U\mathcal{D}_\Phi = \mathcal{E}_\Phi$ and $\pi(a) = U^{-1}\pi_1(a)U$, for all $a \in \mathfrak{A}$.*

Remark 4.5 By Proposition 4.4 all the Stinespring triples $(\pi, V, \mathcal{D}_\Phi)$ decomposing Φ are such that $\overline{\pi(\mathfrak{A})V\mathfrak{X}} = \mathcal{K}_\Phi$.

By Theorem 4.2, and similar arguments to those in the proof of Proposition 3.3, we deduce the following

Corollary 4.6 *Let $\mathfrak{X}$ be a normed complex vector space. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a normed quasi *-algebra with unit* e *and $\{\Phi_n\}_n$ be a sequence of bounded, left-invariant, completely positive sesquilinear maps $\Phi_n : \mathfrak{A} \times \mathfrak{A} \to \mathcal{S}_{\mathfrak{Y}}(\mathfrak{X})$. Let $\{a_n\}_n$, $\{\widetilde{a}_n\}_n$ be sequences in $\mathfrak{A}$ and $\{x_n\}_n$, $\{\widetilde{x}_n\}_n$ be sequences in $\mathfrak{X}$ such that the series $\sum_{n=1}^{\infty} \Phi_n(a_n, a_n)(x_n, x_n)$ and $\sum_{n=1}^{\infty} \Phi_n(\widetilde{a}_n, \widetilde{a}_n)(\widetilde{x}_n, \widetilde{x}_n)$ are convergent in $\mathfrak{Y}$. Then the series $\sum_{n=1}^{\infty} \Phi_n(a_n, \widetilde{a}_n)(x_n, \widetilde{x}_n)$ is convergent and*

$$\left\| \sum_{n=1}^{\infty} \Phi_n(a_n, \widetilde{a}_n)(x_n, \widetilde{x}_n) \right\|_{\mathfrak{Y}} \leq \left\| \sum_{n=1}^{\infty} \Phi_n(a_n, a_n)(x_n, x_n) \right\|_{\mathfrak{Y}}^{1/2} \left\| \sum_{n=1}^{\infty} \Phi_n(\widetilde{a}_n, \widetilde{a}_n)(\widetilde{x}_n, \widetilde{x}_n) \right\|_{\mathfrak{Y}}^{1/2}. \tag{4.1}$$

Following [26, Section 3.5] we now provide a generalization of the Radon-Nikodym theorem for completely positive sesquilinear maps with values in the space $\mathcal{S}_{\mathfrak{Y}}(\mathfrak{X})$ of $\mathfrak{Y}$-valued sesquilinear maps on $\mathfrak{X} \times \mathfrak{X}$.

Proposition 4.7 *Let $\mathfrak{X}$ be a normed complex vector space, $(\mathfrak{A}, \mathfrak{A}_0)$ be a normed quasi *-algebra with unit* e *and $\Psi, \Phi : \mathfrak{A} \times \mathfrak{A} \to \mathcal{S}_{\mathfrak{Y}}(\mathfrak{X})$ be bounded, left-invariant, completely positive sesquilinear maps. Suppose that there exists $\gamma > 0$ such that, for every $N \in \mathbb{N}$, for all $a_1, \dots, a_N \in \mathfrak{A}$, $x_1, \dots, x_N \in \mathfrak{X}$*

$$\sum_{i,j=1}^{N} \Psi(a_i, a_j)(x_i, x_j) \leq \gamma \sum_{i,j=1}^{N} \Phi(a_i, a_j)(x_i, x_j).$$

Let $(\pi_\Psi, V_\Psi, \mathcal{D}_\Psi)$ and $(\pi_\Phi, V_\Phi, \mathcal{D}_\Phi)$ be two Stinespring triples decomposing Ψ and Φ, respectively. Then, there exists a linear operator $T : \pi_\Phi(\mathfrak{A})V_\Phi\mathfrak{X} \to \mathcal{K}_\Psi$ such that:

(i) *$TV_\Phi = V_\Psi$;*
(ii) *$T\pi_\Phi(a) = \pi_\Psi(a) \,\square\, T$ on $\pi_\Phi(\mathfrak{A}_0)V_\Phi\mathfrak{X}$, for every $a \in \mathfrak{A}$,*

and for all $a, b \in \mathfrak{A}$, $x_1, x_2 \in \mathfrak{X}$,

$$\Psi(a, b)(x_1, x_2) = \langle T\pi_\Phi(a)V_\Phi x_1 | T\pi_\Phi(b)V_\Phi x_2 \rangle_{\mathcal{K}_\Psi}.$$

Moreover, if the norm $\|\cdot\|_{\mathfrak{Y}}$ *in* $\mathfrak{Y}$ *is order-preserving, then* T *is bounded, defined on* $\mathcal{K}_\Phi = \overline{\pi_\Phi(\mathfrak{A})V_\Phi\mathfrak{X}}$ *and* $\|T\| \leq \sqrt{\gamma}$*. In this case, for every* $a \in \mathfrak{A}$*,* $T\pi_\Phi(a) = \pi_\Psi(a) \,\Box\, T$ *on the whole* $\mathcal{K}_\Phi$*.*

Proof Let $N \in \mathbb{N}$ and $a_1, \dots, a_N \in \mathfrak{A}$ and $x_1, \dots, x_N \in \mathfrak{X}$, then

$$\begin{aligned}\left\langle \sum_{i=1}^N \pi_\Psi(a_i)V_\Psi x_i \middle| \sum_{j=1}^N \pi_\Psi(a_j)V_\Psi x_j \right\rangle_{\mathcal{K}_\Psi} &= \sum_{i,j=1}^N \Psi(a_i, a_j)(x_i, x_j) \\ &\leq \gamma \sum_{i,j=1}^N \Phi(a_i, a_j)(x_i, x_j) \\ &= \gamma \left\langle \sum_{i=1}^N \pi_\Phi(a_i)V_\Phi x_i \middle| \sum_{j=1}^N \pi_\Phi(a_j)V_\Phi x_j \right\rangle_{\mathcal{K}_\Phi}.\end{aligned} \tag{4.2}$$

If $\sum_{i=1}^N \pi_\Phi(a_i)V_\Phi x_i = 0$, then also $\sum_{i=1}^N \pi_\Psi(a_i)V_\Psi x_i = 0$, hence, we can define a linear operator T by

$$T \sum_{i=1}^N \pi_\Phi(a_i)V_\Phi x_i = \sum_{i=1}^N \pi_\Psi(a_i)V_\Psi x_i, \tag{4.3}$$

for all $a_i \in \mathfrak{A}$, $x_i \in \mathfrak{X}$, $i \in \{1, \dots, N\}$.

Moreover, by taking $a = \mathsf{e} \in \mathfrak{A}_0$ then $TV_\Phi = V_\Psi$, since $\pi_\Phi(\mathsf{e}) = \mathbb{I}_{\mathcal{D}_\Phi}$ and $\pi_\Psi(\mathsf{e}) = \mathbb{I}_{\mathcal{D}_\Psi}$.

Then $T \,\Box\, \pi_\Phi(a) = \pi_\Psi(a) \,\Box\, T$, for every $a \in \mathfrak{A}$. Indeed, if now $a \in \mathfrak{A}$, $c \in \mathfrak{A}_0$, $x \in \mathfrak{X}$, by (4.3)

$$T(\pi_\Phi(a) \,\Box\, \pi_\Phi(c))V_\Phi x = T(\pi_\Phi(a)^{\dagger *}\pi_\Phi(c))V_\Phi x = (T\pi_\Phi(a)^{\dagger *})\pi_\Phi(c)V_\Phi x,$$

on the other hand,

$$\begin{aligned}T(\pi_\Phi(a) \,\Box\, \pi_\Phi(c))V_\Phi x &= T\pi_\Phi(ac)V_\Phi x = \pi_\Psi(ac)V_\Psi x \\ &= \pi_\Psi(a)^{\dagger *}\pi_\Psi(c)V_\Psi x = \pi_\Psi(a)^{\dagger *}(T\pi_\Phi(c)V_\Phi x) \\ &= (\pi_\Psi(a)^{\dagger *}T)\pi_\Phi(c)V_\Phi x = (\pi_\Psi(a) \,\Box\, T)\pi_\Phi(c)V_\Phi x.\end{aligned}$$

Finally, if $y_1 \leq y_2$, with $y_1, y_2 \in \mathfrak{K}$ implies that also $\|y_1\|_{\mathfrak{Y}} \leq \|y_2\|_{\mathfrak{Y}}$ then, by (4.2), we have that T is bounded, hence it extends to the closure $\mathcal{K}_\Phi = \overline{\pi_\Phi(\mathfrak{A})V_\Phi\mathfrak{X}}$ and $\|T\| \leq \sqrt{\gamma}$. Since the set $\pi_\Phi(\mathfrak{A}_0)V_\Phi\mathfrak{X} = (\mathfrak{A}_0 \otimes \mathfrak{X})/\mathcal{N}_\Phi$ is dense in $\mathcal{K}_\Phi$, we conclude that $T\pi_\Phi(a) = \pi_\Psi(a) \,\Box\, T$ on $\mathcal{K}_\Phi$, for every $a \in \mathfrak{A}$. The last statement follows by the very construction of T. □

A Radon Nikodym-like theorem holds for positive sesquilinear maps Φ, Ψ and not just for completely positive sesquilinear maps.

Proposition 4.8 *Let $\mathfrak{X}$ be a normed vector space and let the norm $\|\cdot\|_{\mathfrak{Y}}$ preserve the order in $\mathfrak{Y}$. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a normed quasi *-algebra with unit* e *and $\Phi, \Psi : \mathfrak{A} \times \mathfrak{A} \to \mathcal{S}_{\mathfrak{Y}}(\mathfrak{X})$ be bounded left-invariant positive sesquilinear maps such that*

$$\Psi(a, a)(x, x) \leq \gamma\, \Phi(a, a)(x, x), \quad \forall a \in \mathfrak{A}, x \in \mathfrak{X}$$

*for some $\gamma > 0$. Let $\mathcal{K}_\Phi$ and $\mathcal{K}_\Psi$ be $B_{\mathfrak{Y}}$-spaces, $\mathcal{D}_\Phi$ and $\mathcal{D}_\Psi$ dense subspaces of $\mathcal{K}_\Phi$ and $\mathcal{K}_\Psi$, respectively, $\pi_\Phi : \mathfrak{A} \to \mathcal{L}^\dagger(\mathcal{D}_\Phi, \mathcal{K}_\Phi)$ and $\pi_\Psi : \mathfrak{A} \to \mathcal{L}^\dagger(\mathcal{D}_\Psi, \mathcal{K}_\Psi)$ closed cyclic *-representations of Φ and Ψ, respectively, with cyclic vectors ξ_Φ and ξ_Ψ, respectively, as in Proposition 3.9. Then there exists a unique operator $T : \mathcal{K}_\Phi \to \mathcal{K}_\Psi$, with $\|T\| \leq \sqrt{\gamma}$ such that for all $a, b \in \mathfrak{A}$,*

$$\Psi(a, b) = \langle T\pi_\Phi(a)\xi_\Phi | T\pi_\Phi(b)\xi_\Phi\rangle_{\mathcal{K}_\Psi}.$$

Moreover, $T\pi_\Phi(a) = \pi_\Psi(a) \,\Box\, T$ on $\pi_\Phi(\mathfrak{A}_0)\xi_\Phi$, for all $a \in \mathfrak{A}$.

Proof The proof goes along the lines of that of Proposition 4.7, by applying the representations of Φ and Ψ of Proposition 3.9 and by letting $T\pi_\Phi(a)\xi_\Phi = \pi_\Psi(a)\xi_\Psi$ for all $a \in \mathfrak{A}$, see also Remark 3.10. □

We give now some examples.

Example 4.9 Let $\mathfrak{Y}$ be either a C^*-algebra or one of the Banach bimodules from Proposition 3.1 and let $\widetilde{\Phi} : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{Y}$ be a bounded left-invariant positive sesquilinear map. Define the map $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathcal{S}_{\mathfrak{Y}}(\mathfrak{A}_0)$ by

$$\Phi(a, b)(c, d) = \widetilde{\Phi}(ac, bd), \quad \forall a, b \in \mathfrak{A}, c, d \in \mathfrak{A}_0.$$

Then Φ is sesquilinear, bounded, left-invariant and completely positive.

Example 4.10 Let $\mathfrak{M}$ be a von Neumann algebra with a finite trace ρ and let W and $\eta_x(W)$ be as in Example 3.11. Consider a separable Hilbert space $\mathcal{H}$ and the space $L^2([0, \|W\|], \mathfrak{B}(\mathcal{H}))$ w.r.t. the Gel'fand-Pettis integral (see [13] and [6, Example 2.26]). Fix $F \in L^2([0, \|W\|], \mathfrak{B}(\mathcal{H}))$ with $F(t) \geq 0$ for a.e. $t \in [0, \|W\|]$ and $T \in \mathfrak{B}_2(\mathcal{H})$ and consider $\Phi : L^2(\rho) \times L^2(\rho) \to \mathcal{S}_{\mathfrak{B}_1(\mathcal{H})}(L^\infty(\rho))$ given by

$$\Phi(A, B)(X_1, X_2) = T^* \left(\int_0^{\|W\|} \rho(X_2^* B^* A X_1 \eta_x(W))) F(x) dx \right) T,$$

for every $A, B \in L^2(\rho)$, $X_1, X_2 \in L^2(\rho)$. Then Φ is a bounded, left-invariant, completely positive sesquilinear map.

4.1 Applications to operator-valued maps

Definition 4.11 Let $\mathfrak{X}$ be a Banach space, $\Gamma : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ a bounded sesquilinear map. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit e. A sesquilinear map $\Phi : \mathfrak{A} \times \mathfrak{A} \to$

$\mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$ will be called *completely positive w.r. to* Γ if, for each $N \in \mathbb{N}$ and for every $a_1, \ldots, a_N \in \mathfrak{A}$, $x_1, \ldots, x_N \in \mathfrak{X}$, it is

$$\sum_{i,j=1}^{N} \Phi(a_i, a_j)(\Gamma(x_i, x_j)) \in \mathfrak{K}.$$

Proposition 4.12 *Let $\mathfrak{X}$ be a Banach space, $\Gamma : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ a bounded sesquilinear map. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a normed quasi *-algebra with unit e and $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$ be a bounded, left-invariant, completely positive sesquilinear map. Then, there exist a dense subspace $\mathcal{D}_\Phi$ of a $B_{\mathfrak{Y}}$-space $\mathcal{K}_\Phi$, a closed *-representation π of $\mathfrak{A}$ in $\mathcal{L}^\dagger(\mathcal{D}_\Phi, \mathcal{K}_\Phi)$ and a bounded linear operator $V : \mathfrak{X} \to \mathcal{D}_\Phi$ with $\|V\|^2 \leq \|\Phi(\mathsf{e}, \mathsf{e})\| \|\Gamma\|$ such that we have*

$$\Phi(a, b)(\Gamma(x_1, x_2)) = \langle \pi(a) V x_1 | \pi(b) V x_2 \rangle_{\mathcal{K}_\Phi}, \quad \forall a, b \in \mathfrak{A}, \; x_1, x_2 \in \mathfrak{X}.$$

Moreover, for all $a, b \in \mathfrak{A}$ and $x_1, x_2 \in \mathfrak{X}$, we have

$$\begin{aligned} &\|\Phi(a, b)(\Gamma(x_1, x_2))\|_{\mathfrak{Y}} \\ &\leq \|\Phi(a, a)(\Gamma(x_1, x_1))\|_{\mathfrak{Y}}^{\frac{1}{2}} \|\Phi(b, b)(\Gamma(x_2, x_2))\|_{\mathfrak{Y}}^{\frac{1}{2}}. \end{aligned} \tag{4.4}$$

If B_1 denotes the unit ball in $\mathfrak{X}$ and $\Gamma(B_1 \times B_1) = B_1$, then for all $a, b \in \mathfrak{A}$ we have

$$\|\Phi(a, b)\| \leq \|\Phi(a, a)\|^{1/2} \|\Phi(b, b)\|^{1/2}. \tag{4.5}$$

Proof Let $\mathcal{S}_{\mathfrak{Y}}(\mathfrak{X})$ be the space of all $\mathfrak{Y}$-valued sesquilinear maps on $\mathfrak{X}$ and $\widetilde{\Phi} : \mathfrak{A} \times \mathfrak{A} \to \mathcal{S}_{\mathfrak{Y}}(\mathfrak{X})$ be given by:

$$\widetilde{\Phi}(a, b)(x_1, x_2) := \Phi(a, b)(\Gamma(x_1, x_2)), \quad \forall a, b \in \mathfrak{A}, \; x_1, x_2 \in \mathfrak{X}.$$

Then, $\widetilde{\Phi}$ is a left-invariant, completely positive sesquilinear map on $\mathfrak{Y}$ since it inherits these properties from Φ. Moreover $\widetilde{\Phi}$ is bounded. By Theorem 4.2, the statement follows. Since π is a *-representation, $\pi(\mathsf{e}) = \mathbb{I}_{\mathcal{K}_\Phi}$, the identity operator on the space $\mathcal{K}_\Phi$, hence, for every $x \in \mathfrak{X}$, we have that

$$\begin{aligned} \|\langle Vx | Vx \rangle_{\mathcal{K}_\Phi}\|_{\mathfrak{Y}} &= \|\langle \pi(\mathsf{e}) V x | \pi(\mathsf{e}) V x \rangle_{\mathcal{K}_\Phi}\|_{\mathfrak{Y}} = \|\langle \mathsf{e} \otimes x | \mathsf{e} \otimes x \rangle_{\mathcal{K}_\Phi}\|_{\mathfrak{Y}} \\ &= \|(\Phi(\mathsf{e}, \mathsf{e}))(\Gamma(x, x))\|_{\mathfrak{Y}} \leq \|\Phi(\mathsf{e}, \mathsf{e})\| \|\Gamma(x, x)\| \\ &\leq \|\Phi(\mathsf{e}, \mathsf{e})\| \|\Gamma\| \|x\|^2. \end{aligned}$$

The inequality (4.4) follows from Proposition 3.1 applied to the obtained representation, whereas (4.5) follows by taking supremums over B_1 in (4.4) and using that $\Gamma(B_1 \times B_1) = B_1$. □

The following example shows maps Φ and Γ satisfying the hypotheses of Proposition 4.12.

Example 4.13 Take $\mathfrak{X} = \mathfrak{M}$ a von Neumann algebra with a finite trace ρ and $\Gamma(T_1, T_2) = T_1 T_2^*$, for all $T_1, T_2 \in \mathfrak{M}$. Since $\mathfrak{M}$ is unital, it follows that $\Gamma(B_1 \times B_1) = B_1$. Let W and $\eta_x(W)$ be as in Example 3.11. For every $T \in \mathfrak{M}$ and $x \in [0, \|W\|]$, let

$$(\varphi(A, B)(T))(x) = \rho(A\, \eta_x(W)\, T\, \eta_x(W)\, B^*), \quad A, B \in L^2(\rho).$$

By some calculations it is not hard to see that $\varphi(A, B)(T)(\cdot)$ is a continuous function on $[0, \|W\|]$ and that φ is a bounded, left-invariant, sesquilinear map from $L^2(\rho) \times L^2(\rho)$ into $\mathfrak{B}(\mathfrak{M}, C([0, \|W\|]))$, which is completely positive w.r.t. to Γ. Moreover, if $\widetilde{\mathfrak{M}}$ is another von Neumann algebra with a semi-finite trace $\widetilde{\rho}$ and if we choose some $\widetilde{W} \in \widetilde{\mathfrak{M}}$, with $\widetilde{W} \geq 0$ and $\|\widetilde{W}\| = \|W\|$, then by the functional calculus, $\varphi(A, B)(T)(\widetilde{W}) \in \widetilde{\mathfrak{M}}$, for all $A, B \in L^2(\rho)$ and $T \in \mathfrak{M}$. Hence, the map $\Phi : L^2(\rho) \times L^2(\rho) \to \mathfrak{B}(\mathfrak{M}, \widetilde{\mathfrak{M}}^*)$ given by

$$(\Phi(A, B)(T))(S) = \tilde{\rho}(S(\varphi(A, B)(T))(\widetilde{W}))$$

for all $T \in \mathfrak{M}$, $A, B \in L^2(\rho)$ and $S \in \widetilde{\mathfrak{M}}$ is another example of a bounded, left-invariant sesquilinear map which is completely positive w.r.t. to Γ. Further, if $G \in L^1(\widetilde{\rho})$, then the map $\widetilde{\Phi} : L^2(\rho) \times L^2(\rho) \to \mathfrak{B}(\mathfrak{M}, L^1(\widetilde{\rho}))$ given by

$$\widetilde{\Phi}(A, B)(T) = G(\varphi(A, B)(T)(\widetilde{W}))G^*$$

is also an example of bounded left-invariant sesquilinear map which is completely positive w.r.t. to Γ.

Motivated by the classical notion of complete positivity, we introduce now the following generalization.

Definition 4.14 Let $\mathfrak{X}, \mathfrak{Z}$ be Banach spaces, $\Psi : \mathfrak{Z} \times \mathfrak{X} \to \mathfrak{Y}$ be a map which is linear in the first entry and conjugate linear in the second one and $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{B}(\mathfrak{X}, \mathfrak{Z})$ be a sesquilinear map. The map Φ is said to be *completely positive w.r. to* Ψ if for every $N \in \mathbb{N}$ and all $a_1, \dots, a_N \in \mathfrak{A}$, $x_1, x_2, \dots, x_N \in \mathfrak{X}$ it holds that

$$\sum_{i,j=1}^{N} \Psi(\Phi(a_i, a_j)x_i, x_j) \in \mathfrak{K}.$$

Remark 4.15 If $\mathfrak{X} = \mathfrak{Z} = \mathcal{H}$ is a Hilbert space, $\mathfrak{Y} = \mathbb{C}$, $\mathfrak{A}$ is a unital C^*-algebra, Ψ is inner product on $\mathcal{H}$, $\phi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ is a linear map and $\Phi(a, b) = \phi(b^*a)$, for all $a, b \in \mathfrak{A}$, then Definition 4.14 reduces to the definition of the classical complete positivity of ϕ and Φ.

Proposition 4.16 *Let $\mathfrak{X}, \mathfrak{Z}$ be Banach spaces, $\Psi : \mathfrak{Z} \times \mathfrak{X} \to \mathfrak{Y}$ and $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{B}(\mathfrak{X}, \mathfrak{Z})$ be sesquilinear maps. If Φ and Ψ are bounded and Φ is left-invariant and completely positive w.r. to Ψ, then there exists a dense subspace $\mathcal{D}_\Phi$ of a $B_{\mathfrak{Y}}$-space $\mathcal{K}_\Phi$, a closed *-representation π of $\mathfrak{A}$ in $\mathcal{L}^\dagger(\mathcal{D}_\Phi, \mathcal{K}_\Phi)$ and a bounded linear operator*

$V : \mathfrak{X} \to \mathcal{D}_\Phi$ *with* $\|V\|^2 \leq \|\Phi(\mathsf{e}, \mathsf{e})\| \|\Psi\|$ *such that we have*

$$\Psi(\Phi(a, b)x_1, x_2) = \langle \pi(a)Vx_1 | \pi(b)Vx_2 \rangle_{\mathcal{K}_\Phi}, \quad \forall a, b \in \mathfrak{A}, \ x_1, x_2 \in \mathfrak{X}.$$

Moreover,

$$\|\Psi(\Phi(a, b)x_1, x_2)\|_{\mathfrak{Y}} \leq \|\Psi(\Phi(a, a)x_1, x_2)\|_{\mathfrak{Y}}^{1/2} \|\Psi(\Phi(b, b)x_1, x_2)\|_{\mathfrak{Y}}^{1/2},$$

for all $a, b \in \mathfrak{A}, \ x_1, x_2 \in \mathfrak{X}$.

If $\mathfrak{Y} = L^1(\rho)$, *the statements hold under the additional assumption that* $\Psi(\Phi(a, b)x_1, x_2) \in L^1(\rho) \bigcap L^\infty(\rho)$, *for all* $a, b \in \mathfrak{A}$ *and* $x_1, x_2 \in \mathfrak{X}$.

Finally, if If $\|\Psi\| = 1$ *and for all* $z \in \mathfrak{Z}$ *we have that* $\|z\|_{\mathfrak{Z}} = \sup_{\substack{x \in \mathfrak{X} \\ \|x\|_{\mathfrak{X}} \leq 1}} \|\Psi(z, x)\|_{\mathfrak{Y}}$, *then for all* $a, b \in \mathfrak{A}$, *we have*

$$\|\Phi(a, b)\| \leq \|\Phi(a, a)\|^{1/2} \|\Phi(b, b)\|^{1/2}.$$

Proof As in the proof of Proposition 4.12, we can consider the induced sesquilinear map $\widetilde{\Phi} : \mathfrak{A} \times \mathfrak{A} \to \mathcal{S}_{\mathfrak{Y}}(\mathfrak{X})$ given by

$$\widetilde{\Phi}(a, b)(x_1, x_2) = \Psi(\Phi(a, b)x_1, x_2), \quad \forall x_1, x_2 \in \mathfrak{X}$$

and deduce the first two statements by applying Theorem 4.2 and the obtained *-representation. The last inequality in the statements follows from the first one, by taking the supremums over the unit ball in $\mathfrak{Z}$. □

Example 4.17 Let $\mathfrak{M}$ be a von Neumann algebra with a finite trace ρ and let $W \in L^\infty(\rho)$. Put $\mathfrak{Z} = \mathfrak{Y} = L^1(\rho)$, $\mathfrak{X} = L^\infty(\rho)$. If $\Psi : \mathfrak{Z} \times \mathfrak{X} \to \mathfrak{Y}$ is given by $\Psi(Z, X) = X^* Z$, then Ψ is bounded, $\|\Psi\| = 1$ and $\|Z\|_{\mathfrak{Z}} = \sup_{\substack{X \in \mathfrak{X} \\ \|X\|_{\mathfrak{X}} \leq 1}} \|\Psi(Z, X)\|_{\mathfrak{Y}}$, due to the fact that $L^\infty(\rho) \subset L^1(\rho)$ and that $L^\infty(\rho) = \mathfrak{M}$ is unital. Let $\Phi : L^2(\rho) \times L^2(\rho) \to \mathfrak{B}(\mathfrak{X}, \mathfrak{Z})$ be given by

$$\Phi(A, B)(X) = W^* B^* A W X, \quad \forall A, B \in L^2(\rho), \ X \in \mathfrak{X} = L^\infty(\rho).$$

Then Φ is a bounded, left-invariant sesquilinear map which is completely positive w.r. to Ψ.

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