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Estimates of the numerical radius utilizing various function properties

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Abstract: Numerous inequalities of the vector and numerical radius types have been found. We achieved several extensions of the well-known numerical radius type inequalities by using the function h described in the publication by Stojiljković and Dragomir in [30] and convexity characteristics with standard operator theory approaches. In particular, the following integral type numerical radius inequality has been obtained, that is

$$w(\mathfrak{P}_1^{4k}) \leq h(\iota)\left(\int_0^1 \left\|t|\mathfrak{P}_1|^{4k}+(1-t)|\mathfrak{P}_1^*|^{4k}\right\|^{\frac{1}{2}}dt\right)^2+\frac{h(1-\iota)}{2}w^k(\mathfrak{P}_1^2)\left\||\mathfrak{P}_1|^{2k}+|\mathfrak{P}_1^*|^{2k}\right\| \leq \|\mathfrak{P}_1\|^{4k}.$$

Key words: Numerical radius, norm, integral type inequality

1. Introduction

With the identity operator $1_{\mathfrak{H}}$ in $\mathfrak{B}(\mathfrak{H})$, let $\mathfrak{B}(\mathfrak{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathfrak{H}, \langle .,. \rangle)$.

A relevant and important concept is a numerical radius, which is defined as the following

$$w(\mathfrak{P}) = \sup_{\mathfrak{u}_1\in\mathfrak{H}, \|\mathfrak{u}_1\|=1} |\langle \mathfrak{P}\mathfrak{u}_1, \mathfrak{u}_1\rangle|.$$

It is commonly known that $w(.)$ produces a norm on $\mathfrak{B}(\mathfrak{H})$ that is equal to the norm of an ordinary operator. In other words, the following glaring imbalance is true:

$$\frac{1}{2}\|\mathfrak{P}\| \leq w(\mathfrak{P}) \leq \|\mathfrak{P}\|, \tag{1.1}$$

where the sharpness holds true for the first and second inequalities, respectively, when $\mathfrak{P}^2 = 0$ and $\mathfrak{P}$ is normal. We also have the following power inequality

$$w(\mathfrak{P}^n) \leq w^n(\mathfrak{P}) \tag{1.2}$$

for any $\mathfrak{P} \in \mathfrak{B}(\mathfrak{H})$ and $n \in \mathbb{N}$. In [19], Kittaneh demonstrated that if $\mathfrak{P} \in \mathfrak{B}(\mathfrak{H})$, then

$$w(\mathfrak{P}) \leq \frac{1}{2}\||\mathfrak{P}|+|\mathfrak{P}^*|\| \leq \frac{1}{2}\left(\|\mathfrak{P}\|+\left\|\mathfrak{P}^2\right\|^{\frac{1}{2}}\right), \tag{1.3}$$

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significantly improving the upper bound in (1.1). Refinement of (1.1) has also been given by Najafabadia and Moradi in [25] by

$$w(\mathfrak{P}) \leq \frac{1}{2}\sqrt{\||\mathfrak{P}|^2+|\mathfrak{P}^*|^2\|+\||\mathfrak{P}||\mathfrak{P}^*|+|\mathfrak{P}^*||\mathfrak{P}|\|} \leq \frac{1}{2}\left(\|\mathfrak{P}\|+\left\|\mathfrak{P}^2\right\|^{\frac{1}{2}}\right),$$

for $\mathfrak{P} \in \mathfrak{B}(\mathfrak{H})$.

Kittaneh and Moradi [21] further obtained the inequality for $w^2(.)$, which holds for any $\mathfrak{P} \in \mathfrak{B}(\mathfrak{H})$

$$w^2(\mathfrak{P}) \leq \frac{1}{6}\left\||\mathfrak{P}|^2+|\mathfrak{P}^*|^2\right\|+\frac{1}{3}w(\mathfrak{P})\,\||\mathfrak{P}|+|\mathfrak{P}^*|\|. \tag{1.4}$$

Furthermore, Kittaneh et al. [14] established a few general inequalities that can be expressed as follows:

$$w^{\tau}(\mathfrak{Z}) \leq \frac{1}{2}\left\||\mathfrak{Z}|^{2\tau l}+|\mathfrak{Z}^*|^{2\tau(1-l)}\right\| \tag{1.5}$$

and

$$w^{2\tau}(\mathfrak{Z}) \leq \left\|l|\mathfrak{Z}|^{2\tau}+(1-l)|\mathfrak{Z}^*|^{2\tau}\right\|, \tag{1.6}$$

where $\mathfrak{Z} \in \mathfrak{B}(\mathfrak{H})$, $0 \leq l \leq 1$, and $\tau \geq 1$.

Recently, various interesting inequalities were obtained by Nayak [24], one of them being the following two operator inequality

$$w(\mathfrak{T}^*\mathfrak{S}) \leq \frac{\lambda}{2(\lambda+1)}\left\||\mathfrak{T}|^2+|\mathfrak{S}|^2\right\|+\frac{1}{4(\lambda+1)}\|\mathfrak{T}+\mathfrak{S}\|\left(\|\mathfrak{T}\|+\|\mathfrak{S}\|\right),$$

for $\mathfrak{T}, \mathfrak{S} \in \mathfrak{B}(\mathfrak{H})$ and $\lambda \geq 0$.

Motivated by the recent research conducted in the area of numerical radius inequalities, we obtain further refinements of the well-known inequalities. In particular, (4.1) generalizes the inequality given by Rashid et al. in [26]. We also obtain a boundary for the complex numerical radius, that is (4.21). Variations of (1.1) have been given by (4.12) and (4.15). Generalization of the vector type inequalities has been given by (3.1) and (3.3) of inequalities from Stojiljković [30] and Dragomir in [11]. You may read more about Hilbert space inequalities in the publications that follow [1, 3, 6, 8, 9, 13, 15–18, 27–29, 31, 32].

2. Preliminaries

Lemma 2.1 *Let* $\mathfrak{P}_1 \in \mathfrak{B}(\mathfrak{H})$*, then for* $\zeta \in (0,1]$ *we have*

$$|\langle \mathfrak{P}_1 u_1, u_1\rangle|^2 \leq \langle |\mathfrak{P}_1|^{2\zeta} u_1, u_1\rangle\langle |\mathfrak{P}^*{}_1|^{2(1-\zeta)} u_1, u_1\rangle. \tag{2.1}$$

Lemma 2.2 *If* $\mathfrak{u}_1, \mathfrak{u}_2$, $\mathfrak{u}_3$ *and are all in* $\mathfrak{H}$*, then we have*

$$|\langle \mathfrak{u}_1, \mathfrak{u}_3\rangle\langle \mathfrak{u}_3, \mathfrak{u}_2\rangle| \leq \frac{\|\mathfrak{u}_3\|^2}{2}(\|\mathfrak{u}_1\|\,\|\mathfrak{u}_2\|+|\langle \mathfrak{u}_1, \mathfrak{u}_2\rangle|). \tag{2.2}$$

The final conclusion is the Cauchy-Schwartz inequality extended by Buzano, see (2.2).

Lemma 2.3 *([22]). Suppose that* $\mathfrak{u}_1 \in \mathfrak{H}$ *be any unit vector and that* $\mathfrak{P} \in \mathfrak{B}(\mathfrak{H})$*,* $\mathfrak{P} \geq 0$*. Afterwards, we get*

$$\langle \mathfrak{P}\mathfrak{u}_1, \mathfrak{u}_1\rangle^{\iota} \leq \langle \mathfrak{P}^{\iota}\mathfrak{u}_1, \mathfrak{u}_1\rangle \text{ for } \iota \geq 1, \tag{2.3}$$

$$\langle \mathfrak{P}^{\iota}\mathfrak{u}_1, \mathfrak{u}_1\rangle \leq \langle \mathfrak{P}\mathfrak{u}_1, \mathfrak{u}_1\rangle^{\iota} \text{ for } 0 < \iota \leq 1. \tag{2.4}$$

The following conclusion is found in [5] and is related to nonnegative convex functions.

Lemma 2.4 *Assume that* $\mathfrak{P}_1, \mathfrak{P}_2$ *are positive operators in* $\mathfrak{B}(\mathfrak{H})$ *and that* f *is a nonnegative convex function on* $[0, +\infty)$*. It follows that*

$$\left\| f\left(\frac{\mathfrak{P}_1 + \mathfrak{P}_2}{2}\right)\right\| \leq \left\|\frac{f(\mathfrak{P}_1) + f(\mathfrak{P}_2)}{2}\right\|. \tag{2.5}$$

Dragomir in [12] is the source of another crucial detail on the numerical radius upper boundaries that are relevant to us.

Lemma 2.5 *In the event when* $\iota \geq 1$ *and* $\mathfrak{P}_1, \mathfrak{P}_2 \in \mathfrak{B}(\mathfrak{H})$*,*

$$w^{\iota}(\mathfrak{P}_2^*\mathfrak{P}_1) \leq \frac{1}{2}\left\| |\mathfrak{P}_1|^{2\iota} + |\mathfrak{P}_2|^{2\iota}\right\|. \tag{2.6}$$

Lemma 2.6 ([11]) *Let* $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{P}_4 \in \mathfrak{B}(\mathfrak{H})$*. Then for* $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{H}$ *we have the inequality*

$$|\langle \mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1\mathfrak{u}_1, \mathfrak{u}_1\rangle|^2 \leq \langle \mathfrak{P}_1^*|\mathfrak{P}_2|^2\mathfrak{P}_1\mathfrak{u}_1, \mathfrak{u}_1\rangle\langle \mathfrak{P}_4|\mathfrak{P}_3^*|\mathfrak{P}_4^*\mathfrak{u}_2, \mathfrak{u}_2\rangle. \tag{2.7}$$

Lemma 2.7 ([23]) *Suppose* $\mathfrak{u}_1 \in H$ *is a unit vector and* $\mathfrak{P} \in \mathfrak{B}(\mathfrak{H})$ *is a self adjoint operator. If* f *is a convex function on an interval that contains* $\mathfrak{P}_1$*'s spectrum, then we have*

$$f(\langle \mathfrak{P}_1\mathfrak{u}_1, \mathfrak{u}_1\rangle) \leq \langle f(\mathfrak{P}_1)\mathfrak{u}_1, \mathfrak{u}_1\rangle. \tag{2.8}$$

Lemma 2.8 *Assume that* $\min\{\mu_1, \mu_2\} \leq \xi_1 \leq \xi_2 \leq \max\{\mu_1, \mu_2\}$ *is satisfied by positive real numbers* ξ_1, ξ_2 *and* $\mu_1, \mu_2 > 0$*. Then we get*

$$\frac{\xi_2 + \xi_1}{2\sqrt{\xi_1\xi_2}}\sqrt{\mu_1, \mu_2} \leq \frac{\mu_2 + \mu_1}{2}. \tag{2.9}$$

Lemma 2.9 *Given a nonnegative increasing convex function* ψ *on the interval* $[0, \infty)$*, with* $\psi(0) = 0$ *and* $0 \leq \tau \leq 1$*. Then we have the following inequality:*

$$\psi(\tau t) \leq \tau\psi(t). \tag{2.10}$$

Lemma 2.10 ([7]) *If* $\mathfrak{u}_1, \mathfrak{u}_2,$ $\mathfrak{u}_3$ *and are all in* $\mathfrak{H}$*, then we have*

$$|\langle \mathfrak{u}_3, \mathfrak{u}_1\rangle|^2 + |\langle \mathfrak{u}_3, \mathfrak{u}_2\rangle|^2 \leq \|\mathfrak{u}_3\|^2\left(|\langle \mathfrak{u}_1, \mathfrak{u}_2\rangle| + \max(\|\mathfrak{u}_1\|^2, \|\mathfrak{u}_2\|^2)\right). \tag{2.11}$$

Lemma 2.11 *The following relation for* max *holds*

$$\max(\mathfrak{a}_1, \mathfrak{a}_2) = \frac{1}{2}[\mathfrak{a}_1 + \mathfrak{a}_2 + |\mathfrak{a}_2 - \mathfrak{a}_1|] \text{ for any } \mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{R}. \tag{2.12}$$

Lemma 2.12 ([30]) *If h is a non negative mapping with the following characteristics: $h:(0,1)\subset J\to\mathbb{R}^+$ such that $h(\iota)+h(1-\iota)=1$, and $\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3$, and $\mathfrak{P}_4\in\mathfrak{B}(\mathfrak{H})$, $k\geq 1$, then we have*

$$\begin{aligned}|\langle\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_2\rangle|^{2k}&\leq h(\iota)\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^k\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^k\\&\quad+h(1-\iota)|\langle\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_2\rangle|^k\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^{\frac{k}{2}}\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^{\frac{k}{2}}\\&\leq\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^k\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^k.\end{aligned}\tag{2.13}$$

3. Main results

Lemma 3.1 *Let the conditions from (2.13) hold, also if f is an increasing convex function such that $f:[0,+\infty)\to\mathbb{R}$, then the following inequality holds*

$$\begin{aligned}f(|\langle\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_2\rangle|^{2k})&\leq h(\iota)f(\langle\mathfrak{P}_1^*|\mathfrak{P}_2|^2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_1\rangle^k\langle\mathfrak{P}_4|\mathfrak{P}_3^*|^2\mathfrak{P}_4^*\mathfrak{u}_2,\mathfrak{u}_2\rangle^k)\\&\quad+h(1-\iota)f(|\langle\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_2\rangle|^k\langle\mathfrak{P}_1^*|\mathfrak{P}_2|^2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_1\rangle^{\frac{k}{2}}\langle\mathfrak{P}_4|\mathfrak{P}_3^*|^2\mathfrak{P}_4^*\mathfrak{u}_2,\mathfrak{u}_2\rangle^{\frac{k}{2}})\\&\leq f(\langle\mathfrak{P}_1^*|\mathfrak{P}_2|^2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_1\rangle^k\langle\mathfrak{P}_4|\mathfrak{P}_3^*|^2\mathfrak{P}_4^*\mathfrak{u}_2,\mathfrak{u}_2\rangle^k)\\&\leq h(p)\langle f(|\mathfrak{P}_2\mathfrak{P}_1|^{\frac{2k}{h(p)}})\mathfrak{u}_1,\mathfrak{u}_1\rangle+h(1-p)\langle f(|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{\frac{2k}{h(1-p)}})\mathfrak{u}_2,\mathfrak{u}_2\rangle.\end{aligned}\tag{3.1}$$

Proof Think about the following,

$$\begin{aligned}&|\langle\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_2\rangle|^{2k}\\\leq\ &h(\iota)\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^k\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^k\ \text{(by (2.13))}\\&+h(1-\iota)|\langle\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_2\rangle|^k\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^{\frac{k}{2}}\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^{\frac{k}{2}}.\end{aligned}$$

Applying f on both sides and using the fact that f is convex we get

$$\begin{aligned}&f(|\langle\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_2\rangle|^{2k})\\\leq\ &h(\iota)f(\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^k\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^k)\\&+h(1-\iota)f(|\langle\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_2\rangle|^k\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^{\frac{k}{2}}\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^{\frac{k}{2}}).\end{aligned}$$

Using the four operator inequality on $\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1$ term and using the reality that f is increasing, that is for $a<b$ we have $f(a)<f(b)$, we get

$$\leq f(\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^k\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^k)\ (2.7).$$

We now turn to obtaining the right hand side chain of inequalities

$$\begin{aligned}&f(\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^k\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^k)\\&\leq f(\langle|\mathfrak{P}_2\mathfrak{P}_1|^{2k}\mathfrak{u}_1,\mathfrak{u}_1\rangle\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{2k}\mathfrak{u}_2,\mathfrak{u}_2\rangle)\ (2.3)\\&\leq f(h(p)\langle|\mathfrak{P}_2\mathfrak{P}_1|^{\frac{2k}{h(p)}}\mathfrak{u}_1,\mathfrak{u}_1\rangle+h(1-p)\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{\frac{2k}{h(1-p)}}\mathfrak{u}_2,\mathfrak{u}_2\rangle)\\&\text{((2.4) with weighted AG inequality)}\\&\leq h(p)f(\langle|\mathfrak{P}_2\mathfrak{P}_1|^{\frac{2k}{h(p)}}\mathfrak{u}_1,\mathfrak{u}_1\rangle)+h(1-p)f(\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{\frac{2k}{h(1-p)}}\mathfrak{u}_2,\mathfrak{u}_2\rangle)\\&\leq h(p)\langle f(|\mathfrak{P}_2\mathfrak{P}_1|^{\frac{2k}{h(p)}})\mathfrak{u}_1,\mathfrak{u}_1\rangle+h(1-p)\langle f(|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{\frac{2k}{h(1-p)}})\mathfrak{u}_2,\mathfrak{u}_2\rangle\ (2.8).\end{aligned}$$

□

The following inequalities are consequences of (3.1).

Remark 3.2 *The following Furuta [15] type inequality for* $\alpha, \beta \geq 0$ *with* $\alpha + \beta \geq 1$ *is a particular case of (3.1), namely*

$$\begin{aligned} & f(|\langle \mathfrak{T}|\mathfrak{T}|^{\alpha+\beta-1}\mathfrak{u}_1, \mathfrak{u}_2\rangle|^{2k}) \leq h(\iota) f(\langle |\mathfrak{T}|^{2\alpha}\mathfrak{u}_1, \mathfrak{u}_1\rangle^k \langle |\mathfrak{T}^*|^{2\beta}\mathfrak{u}_2, \mathfrak{u}_2\rangle^k) \\ & + h(1-\iota) f(|\langle \mathfrak{T}|\mathfrak{T}|^{\alpha+\beta-1}\mathfrak{u}_1, \mathfrak{u}_2\rangle|^k \langle |\mathfrak{T}|^{2\alpha}\mathfrak{u}_1, \mathfrak{u}_1\rangle^{\frac{k}{2}} \langle |\mathfrak{T}^*|^{2\beta}\mathfrak{u}_2, \mathfrak{u}_2\rangle^{\frac{k}{2}}) \\ & \leq f(\langle |\mathfrak{T}|^{2\alpha}\mathfrak{u}_1, \mathfrak{u}_1\rangle^k \langle |\mathfrak{T}^*|^{2\beta}\mathfrak{u}_2, \mathfrak{u}_2\rangle^k) \\ & \leq h(p)\langle f(|\mathfrak{T}|^{\frac{2k\alpha}{h(p)}})\mathfrak{u}_1, \mathfrak{u}_1\rangle + h(1-p)\langle f(|\mathfrak{T}^*|^{\frac{2k\beta}{h(1-p)}})\mathfrak{u}_2, \mathfrak{u}_2\rangle. \end{aligned} \tag{3.2}$$

Proof Following the proof of (eq. 3.6 in [30]) the result follows, therefore we omit the details. □

It is also evident that the Kato's inequality [18] follows from (3.1).

We also obtain the following inequality as a consequence of (3.1),

$$\begin{aligned} & f(|\langle \mathfrak{T}|\mathfrak{T}|^{\omega-1}\mathfrak{T}|\mathfrak{T}|^{\tau-1}\mathfrak{u}_1, \mathfrak{u}_2\rangle|^{2k}) \leq h(\iota) f(\langle |\mathfrak{T}|^{2\tau}\mathfrak{u}_1, \mathfrak{u}_1\rangle^k \langle |\mathfrak{T}^*|^{2\omega}\mathfrak{u}_2, \mathfrak{u}_2\rangle^k) \\ & + h(1-\iota) f(|\langle \mathfrak{T}|\mathfrak{T}|^{\omega-1}\mathfrak{T}|\mathfrak{T}|^{\tau-1}\mathfrak{u}_1, \mathfrak{u}_2\rangle|^k \langle |\mathfrak{T}|^{2\tau}\mathfrak{u}_1, \mathfrak{u}_1\rangle^{\frac{k}{2}} \langle |\mathfrak{T}^*|^{2\omega}\mathfrak{u}_2, \mathfrak{u}_2\rangle^{\frac{k}{2}}) \\ & \leq f(\langle |\mathfrak{T}|^{2\tau}\mathfrak{u}_1, \mathfrak{u}_1\rangle^k \langle |\mathfrak{T}^*|^{2\omega}\mathfrak{u}_2, \mathfrak{u}_2\rangle^k) \\ & \leq \langle h(p) f(|\mathfrak{T}|^{\frac{2k\tau}{h(p)}})\mathfrak{u}_1, \mathfrak{u}_1\rangle + \langle h(1-p) f(|\mathfrak{T}^*|^{\frac{2k\omega}{h(1-p)}})\mathfrak{u}_2, \mathfrak{u}_2\rangle \end{aligned} \tag{3.3}$$

which holds for every operator $\mathfrak{T} \in \mathfrak{B}(\mathfrak{H})$ *and any* $\tau, \omega \geq 1$.

Proof Following the proof of (eq. 3.7 in [30]) the result follows, we omit the details. □

Further setting $\omega = 1, \tau = 1, k = 1$ *in* (3.3) *we obtain the following interesting inequality*

$$\begin{aligned} & f(|\langle \mathfrak{T}^2\mathfrak{u}_1, \mathfrak{u}_2\rangle|^2) \leq h(\iota) f(\langle |\mathfrak{T}|^2\mathfrak{u}_1, \mathfrak{u}_1\rangle \langle |\mathfrak{T}^*|^2\mathfrak{u}_2, \mathfrak{u}_2\rangle) \\ & + h(1-\iota) f(|\langle \mathfrak{T}^2\mathfrak{u}_1, \mathfrak{u}_2\rangle| \langle |\mathfrak{T}|^2\mathfrak{u}_1, \mathfrak{u}_1\rangle^{\frac{1}{2}} \langle |\mathfrak{T}^*|^2\mathfrak{u}_2, \mathfrak{u}_2\rangle^{\frac{1}{2}}) \\ & \leq f(\langle |\mathfrak{T}|^2\mathfrak{u}_1, \mathfrak{u}_1\rangle \langle |\mathfrak{T}^*|^2\mathfrak{u}_2, \mathfrak{u}_2\rangle) \\ & \leq h(p)\langle f(|\mathfrak{T}|^{\frac{2}{h(p)}})\mathfrak{u}_1, \mathfrak{u}_1\rangle + h(1-p)\langle f(|\mathfrak{T}^*|^{\frac{2}{h(1-p)}})\mathfrak{u}_2, \mathfrak{u}_2\rangle. \end{aligned} \tag{3.4}$$

In particular, setting $f = I, h = \frac{1}{2}$*, we obtain the following interesting inequality*

$$\begin{aligned} & |\langle \mathfrak{T}^2\mathfrak{u}_1, \mathfrak{u}_2\rangle|^2 \leq \frac{1}{2}\langle |\mathfrak{T}|^2\mathfrak{u}_1, \mathfrak{u}_1\rangle^k \langle |\mathfrak{T}^*|^2\mathfrak{u}_2, \mathfrak{u}_2\rangle^k + \frac{1}{2}|\langle \mathfrak{T}^2\mathfrak{u}_1, \mathfrak{u}_2\rangle|^k \langle |\mathfrak{T}|^2\mathfrak{u}_1, \mathfrak{u}_1\rangle^{\frac{k}{2}} \langle |\mathfrak{T}^*|^2\mathfrak{u}_2, \mathfrak{u}_2\rangle^{\frac{1}{2}} \\ & \leq \langle |\mathfrak{T}|^2\mathfrak{u}_1, \mathfrak{u}_1\rangle \langle |\mathfrak{T}^*|^2\mathfrak{u}_2, \mathfrak{u}_2\rangle \\ & \leq \frac{1}{2}\left(\langle |\mathfrak{T}|^4\mathfrak{u}_1, \mathfrak{u}_1\rangle + \langle |\mathfrak{T}^*|^4\mathfrak{u}_2, \mathfrak{u}_2\rangle\right). \end{aligned} \tag{3.5}$$

The following Lemma is utilized to obtain a refinement of the inequality given by Kittaneh et al. (1.5).

Lemma 3.3 *Let f be a nonnegative increasing convex function on $[0,+\infty)$ such that $f(0)=0$ and $\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3$ and $\mathfrak{P}_4\in\mathfrak{B}(\mathfrak{H})$, also let the conditions from (2.13) hold. If*

$$0<|\mathfrak{P}_2\mathfrak{P}_1|^2\leq\xi\leq\eta\leq|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2$$

or

$$0<|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\leq\xi\leq\eta\leq|\mathfrak{P}_2\mathfrak{P}_1|^2,$$

then

$$\begin{aligned}
&f(|\langle\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_2\rangle|^{2k})\leq h(\iota)f(\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^k\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^k)\\
&+h(1-\iota)f(|\langle\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_2\rangle|^k\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^{\frac{k}{2}}\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^{\frac{k}{2}})\\
&\leq f(\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^k\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^k)\\
&\leq\frac{2\sqrt{\xi\eta}}{\xi+\eta}\left(\frac{\langle f(|\mathfrak{P}_2\mathfrak{P}_1|^{4k})\mathfrak{u}_1,\mathfrak{u}_1\rangle+\langle f(|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{4k})\mathfrak{u}_2,\mathfrak{u}_2\rangle}{2}\right).
\end{aligned}\tag{3.6}$$

Proof Using the similar reasoning as in (3.1), we continue with the following

$$\begin{aligned}
&\langle|\mathfrak{P}_2\mathfrak{P}_1|^{2k}\mathfrak{u}_1,\mathfrak{u}_1\rangle\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{2k}\mathfrak{u}_2,\mathfrak{u}_2\rangle\\
\leq\;&\frac{2\sqrt{\xi\eta}}{\xi+\eta}\left(\frac{\langle|\mathfrak{P}_2\mathfrak{P}_1|^{4k}\mathfrak{u}_1,\mathfrak{u}_1\rangle+\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{4k}\mathfrak{u}_2,\mathfrak{u}_2\rangle}{2}\right)\ (2.9).
\end{aligned}$$

By using the convexity of f, we arrive

$$\begin{aligned}
&f(\langle|\mathfrak{P}_2\mathfrak{P}_1|^{2k}\mathfrak{u}_1,\mathfrak{u}_1\rangle\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{2k}\mathfrak{u}_2,\mathfrak{u}_2\rangle)\\
\leq\;&\frac{2\sqrt{\xi\eta}}{\xi+\eta}f\left(\frac{\langle|\mathfrak{P}_2\mathfrak{P}_1|^{4k}\mathfrak{u}_1,\mathfrak{u}_1\rangle+\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{4k}\mathfrak{u}_2,\mathfrak{u}_2\rangle}{2}\right)\ (2.10).
\end{aligned}$$

So,

$$\begin{aligned}
&\leq\frac{2\sqrt{\eta\xi}}{\xi+\eta}\left(\frac{f(\langle|\mathfrak{P}_2\mathfrak{P}_1|^{4k}\mathfrak{u}_1,\mathfrak{u}_1\rangle)+f(\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{4k}\mathfrak{u}_2,\mathfrak{u}_2\rangle)}{2}\right)\\
&\leq\frac{2\sqrt{\xi\eta}}{\xi+\eta}\left(\frac{\langle f(|\mathfrak{P}_2\mathfrak{P}_1|^{4k})\mathfrak{u}_1,\mathfrak{u}_1\rangle+\langle f(|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{4k})\mathfrak{u}_2,\mathfrak{u}_2\rangle}{2}\right)\ (2.8)
\end{aligned}$$

is obtained by once more utilizing the fact that f is convex. □

Remark 3.4 *Setting $f(\mathfrak{x})=\mathfrak{x}$, we obtain*

$$\begin{aligned}
|\langle\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_2\rangle|^{2k}&\leq h(\iota)\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^k\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^k\\
&+h(1-\iota)|\langle\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_2\rangle|^k\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^{\frac{k}{2}}\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^{\frac{k}{2}}\\
&\leq\langle|\mathfrak{P}_2\mathfrak{P}_1|^2\mathfrak{u}_1,\mathfrak{u}_1\rangle^k\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^2\mathfrak{u}_2,\mathfrak{u}_2\rangle^k\\
&\leq\frac{2\sqrt{\xi\eta}}{\xi+\eta}\left(\frac{\langle|\mathfrak{P}_2\mathfrak{P}_1|^{4k}\mathfrak{u}_1,\mathfrak{u}_1\rangle+\langle|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{4k}\mathfrak{u}_2,\mathfrak{u}_2\rangle}{2}\right).
\end{aligned}\tag{3.7}$$

Setting $\mathfrak{P}_4 = \mathfrak{S}^*, \mathfrak{P}_1 = \mathfrak{T}, \mathfrak{P}_2 = \mathfrak{P}_3 = I$*, we obtain*

$$\begin{aligned}|\langle \mathfrak{S}^*\mathfrak{T}\mathfrak{u}_1, \mathfrak{u}_2\rangle|^{2k} &\leq h(\iota)\langle |\mathfrak{T}|^2\mathfrak{u}_1, \mathfrak{u}_1\rangle^k \langle |\mathfrak{S}|^2\mathfrak{u}_2, \mathfrak{u}_2\rangle^k \\ &\quad + h(1-\iota)|\langle \mathfrak{S}^*\mathfrak{T}\mathfrak{u}_1, \mathfrak{u}_2\rangle|^k \langle |\mathfrak{T}|^2\mathfrak{u}_1, \mathfrak{u}_1\rangle^{\frac{k}{2}} \langle |\mathfrak{S}|^2\mathfrak{u}_2, \mathfrak{u}_2\rangle^{\frac{k}{2}} \\ &\leq \langle |\mathfrak{T}|^2\mathfrak{u}_1, \mathfrak{u}_1\rangle^k \langle |\mathfrak{S}|^2\mathfrak{u}_2, \mathfrak{u}_2\rangle^k \\ &\leq \frac{2\sqrt{\xi_1\xi_2}}{\xi_1+\xi_2}\left(\frac{\langle |\mathfrak{T}|^{4k}\mathfrak{u}_1, \mathfrak{u}_1\rangle + \langle |\mathfrak{S}|^{4k}\mathfrak{u}_2, \mathfrak{u}_2\rangle}{2}\right), \end{aligned} \tag{3.8}$$

when

$$0 < |\mathfrak{T}|^2 \leq \xi_1 \leq \xi_2 \leq |\mathfrak{S}|^2$$

or

$$0 < |\mathfrak{S}|^2 \leq \xi_1 \leq \xi_2 \leq |\mathfrak{T}|^2.$$

4. Numerical radius type inequalities

Theorem 4.1 *Suppose* $k \geq 1$ *and* $\mathfrak{P}_4, \mathfrak{P}_3, \mathfrak{P}_2,$ *and* $\mathfrak{P}_1 \in \mathfrak{B}(\mathfrak{H})$*. Let the conditions from (3.1) hold. Then, the inequality shown below is true:*

$$f(w^{2k}(\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1)) \leq \left\| h(p)f(|\mathfrak{P}_2\mathfrak{P}_1|^{\frac{2k}{h(p)}}) + h(1-p)f(|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{\frac{2k}{h(1-p)}})\right\|. \tag{4.1}$$

Proof We get the necessary inequality by setting $\mathfrak{u}_1 = \mathfrak{u}_2$ and obtaining the supremum over unit vectors $\mathfrak{u}_1 \in \mathfrak{H}$ in (3.1). □

Remark 4.2 *The inequality that was previously discovered extends Rashid's finding found in [26, eq. 2.6]. When we set* $k = 1, h = I$*, we get*

$$f(w^2(\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1)) \leq \left\| pf(|\mathfrak{P}_2\mathfrak{P}_1|^{\frac{2}{p}}) + (1-p)f(|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{\frac{2}{1-p}})\right\|,$$

which is the inequality that Rashid provided. In particular, by setting $f = I, h = I$ *we obtain the inequality (eq. (2.7)) in the paper given by Rashid [26]*

$$w^{2k}(\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1) \leq \left\| p|\mathfrak{P}_2\mathfrak{P}_1|^{\frac{2k}{p}} + (1-p)|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{\frac{2k}{1-p}}\right\|.$$

Setting $f = I, h = I, \mathfrak{P}_2 = I, \mathfrak{P}_4 = I, \mathfrak{P}_3 = \mathfrak{Q}^*, p = \frac{1}{2}$ *we obtain the (2.6) inequality for the case when the powers are even, that is*

$$w^{2k}(\mathfrak{Q}^*\mathfrak{P}_1) \leq \frac{1}{2}\left\| |\mathfrak{P}_1|^{4k} + |\mathfrak{Q}|^{4k}\right\|. \tag{4.2}$$

Theorem 4.3 *Let* $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{P}_4 \in \mathfrak{B}(\mathfrak{H})$ *such that* $f(0) = 0$ *and let* f *be a nonnegative increasing convex function on* $[0, +\infty)$*. If*

$$0 < |\mathfrak{P}_2\mathfrak{P}_1|^2 \leq \xi \leq \eta \leq |(\mathfrak{P}_4\mathfrak{P}_3)^*|^2$$

or

$$0 < |(\mathfrak{P}_4\mathfrak{P}_3)^*|^2 \leq \xi \leq \eta \leq |\mathfrak{P}_2\mathfrak{P}_1|^2,$$

then

$$f(w^{2k}(\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1)) \leq \frac{2\sqrt{\xi\eta}}{\xi+\eta}\left(\left\|\frac{f(|\mathfrak{P}_2\mathfrak{P}_1|^{4k})+f(|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{4k})}{2}\right\|\right). \tag{4.3}$$

Proof We obtain the requisite inequality by taking the supremum over all unit vectors $\mathfrak{u}_1 \in \mathfrak{H}$ with $\mathfrak{u}_1 = \mathfrak{u}_2$ in (3.6). □

Remark 4.4 *Setting* $f(\mathfrak{u}_1) = \mathfrak{u}_1$ *in (4.3), we obtain*

$$w^{2k}(\mathfrak{P}_4\mathfrak{P}_3\mathfrak{P}_2\mathfrak{P}_1) \leq \frac{2\sqrt{\xi\eta}}{\xi+\eta}\left(\left\|\frac{|\mathfrak{P}_2\mathfrak{P}_1|^{4k}+|(\mathfrak{P}_4\mathfrak{P}_3)^*|^{4k}}{2}\right\|\right). \tag{4.4}$$

Setting $\mathfrak{P}_4 = \mathfrak{S}^*, \mathfrak{P}_1 = \mathfrak{T}, \mathfrak{P}_2 = \mathfrak{P}_3 = I$, *we obtain*

$$w^{2k}(\mathfrak{S}^*\mathfrak{T}) \leq \frac{2\sqrt{\xi\eta}}{\xi+\eta}\left(\left\|\frac{|\mathfrak{T}|^{4k}+|\mathfrak{S}|^{4k}}{2}\right\|\right), \tag{4.5}$$

when

$$0 < |\mathfrak{T}|^2 \leq \xi \leq \eta \leq |\mathfrak{S}|^2$$

or

$$0 < |\mathfrak{S}|^2 \leq \xi \leq \eta \leq |\mathfrak{T}|^2.$$

Setting $\mathfrak{P}_3 = \mathfrak{P}_4 = \mathfrak{P}_2 = I$ *and* $\mathfrak{P}_1 = \mathfrak{T}, f = I$ *in (4.3) we obtain*

$$w^{2k}(\mathfrak{T}) \leq \frac{2\sqrt{\xi\eta}}{\xi+\eta}\left(\left\|\frac{|\mathfrak{T}|^{4k}+I}{2}\right\|\right), \tag{4.6}$$

when

$$0 < |\mathfrak{T}|^2 \leq \xi \leq \eta \leq I$$

or

$$0 < I \leq \xi \leq \eta \leq |\mathfrak{T}|^2.$$

Using the same configuration but setting $\xi = 3, \eta = 5$ *in (4.5), we obtain a refinement of (2.6) in the case of even powers, namely*

$$w^{2k}(\mathfrak{S}^*\mathfrak{T}) \leq \frac{\sqrt{15}}{8}\left\||\mathfrak{T}|^{4k}+|\mathfrak{S}|^{4k}\right\|, \tag{4.7}$$

which obviously is a refinement of (2.6) since

$$\begin{aligned} w^{2k}(\mathfrak{S}^*\mathfrak{T}) &\leq \frac{\sqrt{15}}{8}\left\||\mathfrak{T}|^{4k}+|\mathfrak{S}|^{4k}\right\| \\ &< \frac{\sqrt{16}}{8}\left\||\mathfrak{T}|^{4k}+|\mathfrak{S}|^{4k}\right\| = \frac{1}{2}\left\||\mathfrak{T}|^{4k}+|\mathfrak{S}|^{4k}\right\|, \end{aligned} \tag{4.8}$$

when

$$0 < |\mathfrak{T}|^2 \leq 3 \leq 5 \leq |\mathfrak{S}|^2$$

or

$$0 < |\mathfrak{S}|^2 \leq 3 \leq 5 \leq |\mathfrak{T}|^2.$$

Theorem 4.5 *Let* $\tau, \omega \geq 1$ *and* $\mathfrak{T} \in \mathfrak{B}(\mathfrak{H})$, *then we get the following inequality*

$$f(w^{2k}(\mathfrak{T}|\mathfrak{T}|^{\omega-1}\mathfrak{T}|\mathfrak{T}|^{\tau-1})) \leq \left\| h(p)f(|\mathfrak{T}|^{\frac{2k\tau}{h(p)}}) + h(1-p)f(|\mathfrak{T}^*|^{\frac{2k\omega}{h(1-p)}}) \right\|. \tag{4.9}$$

Proof We obtain the requisite inequality by taking the supremum over all unit vectors $\mathfrak{u}_1 \in \mathfrak{H}$ with $\mathfrak{u}_1 = \mathfrak{u}_2$ in (3.3). □

Corollary 4.6 *We get*

$$w^{2k}(\mathfrak{T}|\mathfrak{T}|\mathfrak{T}|\mathfrak{T}|) \leq \left\| p|\mathfrak{T}|^{\frac{4k}{p}} + (1-p)|\mathfrak{T}^*|^{\frac{4k}{(1-p)}} \right\|$$

by letting $f(\mathfrak{u}_1) = \mathfrak{u}_1, h = I, \tau = \omega = 2$ *in (4.9). Further, setting* $k = 2, p = \frac{1}{2}$, *we obtain*

$$w^4(\mathfrak{T}|\mathfrak{T}|\mathfrak{T}|\mathfrak{T}|) \leq \frac{1}{2} \left\| |\mathfrak{T}|^{16} + |\mathfrak{T}^*|^{16} \right\|.$$

Theorem 4.7 *In the event where* $\mathfrak{P}_1 \in \mathfrak{B}(\mathfrak{H})$ *and* h *satisfy the criteria in (2.13), and* ψ *is a nonnegative increasing convex submultiplicative function on* $[0, +\infty)$, *the following inequality holds for* $k \geq 1$*:*

$$\psi(w^{2k}(\mathfrak{P}_1^2)) \leq h(\iota)\psi\left(\left(\int_0^1 \left\| t|\mathfrak{P}_1|^{4k} + (1-t)|\mathfrak{P}_1^*|^{4k} \right\|^{\frac{1}{2}} dt\right)^2\right) + h(1-\iota)\psi(w^k(\mathfrak{P}_1^2)) \left\| \frac{\psi(|\mathfrak{P}_1|^{2k}) + \psi(|\mathfrak{P}_1^*|^{2k})}{2} \right\|. \tag{4.10}$$

Proof Let $\mathfrak{u}_1 \in \mathfrak{H}$ represent any unit vector. Then, take into account

$$
\begin{aligned}
&\psi(|\langle \mathfrak{P}_1\mathfrak{u}_1, \mathfrak{P}_1^*\mathfrak{u}_1\rangle|^{2k}) \\
&\leq h(\iota)\psi(\|\mathfrak{P}_1\mathfrak{u}_1\|^{2k}\|\mathfrak{P}_1^*\mathfrak{u}_1\|^{2k}) \\
&+ h(1-\iota)\psi(\|\mathfrak{P}_1\mathfrak{u}_1\|^{k}\|\mathfrak{P}_1^*\mathfrak{u}_1\|^{k}|\langle \mathfrak{P}_1\mathfrak{u}_1, \mathfrak{P}_1^*\mathfrak{u}_1\rangle|^{k}) \ (2.13) \\
&\leq h(\iota)\psi\left(\left(\int_0^1 \|\mathfrak{P}_1\mathfrak{u}_1\|^{2tk}\|\mathfrak{P}_1^*\mathfrak{u}_1\|^{2k(1-t)}\,dt\right)^2\right) \ \text{(Logarithmic mean)} \\
&+ h(1-\iota)\psi(|\langle \mathfrak{P}_1^2\mathfrak{u}_1, \mathfrak{u}_1\rangle|^{k})\psi(\|\mathfrak{P}_1\mathfrak{u}_1\|^{k}\|\mathfrak{P}_1^*\mathfrak{u}_1\|^{k}) \ \text{(submultiplicavity of } \psi) \\
&\leq h(\iota)\psi\left(\left(\int_0^1 \|\mathfrak{P}_1\mathfrak{u}_1\|^{2tk}\|\mathfrak{P}_1^*\mathfrak{u}_1\|^{2k(1-t)}\,dt\right)^2\right) \\
&+ h(1-\iota)\psi(|\langle \mathfrak{P}_1^2\mathfrak{u}_1, \mathfrak{u}_1\rangle|^{k})\psi\left(\frac{\langle |\mathfrak{P}_1|^{2k}\mathfrak{u}_1, \mathfrak{u}_1\rangle + \langle |\mathfrak{P}_1^*|^{2k}\mathfrak{u}_1, \mathfrak{u}_1\rangle}{2}\right) \ \text{(AG)} \\
&\leq h(\iota)\psi\left(\left(\int_0^1 \left(\langle |\mathfrak{P}_1|^{4k}\mathfrak{u}_1, \mathfrak{u}_1\rangle^{t}\langle |\mathfrak{P}_1^*|^{4k}\mathfrak{u}_1, \mathfrak{u}_1\rangle^{1-t}\right)^{\frac{1}{2}}dt\right)^2\right) \\
&+ h(1-\iota)\psi(|\langle \mathfrak{P}_1^2\mathfrak{u}_1, \mathfrak{u}_1\rangle|^{k})\frac{\psi(\langle |\mathfrak{P}_1|^{2k}\mathfrak{u}_1, \mathfrak{u}_1\rangle) + \psi(\langle |\mathfrak{P}_1^*|^{2k}\mathfrak{u}_1, \mathfrak{u}_1\rangle)}{2} \\
&\leq h(\iota)\psi\left(\left(\int_0^1 \left(t\langle |\mathfrak{P}_1|^{4k}\mathfrak{u}_1, \mathfrak{u}_1\rangle + (1-t)\langle |\mathfrak{P}_1^*|^{4k}\mathfrak{u}_1, \mathfrak{u}_1\rangle\right)^{\frac{1}{2}}dt\right)^2\right) \ \text{(weighted AG inequality)} \\
&+ h(1-\iota)\psi(|\langle \mathfrak{P}_1^2\mathfrak{u}_1, \mathfrak{u}_1\rangle|^{k})\frac{\langle(\psi(|\mathfrak{P}_1|^{2k}) + \psi(|\mathfrak{P}_1^*|^{2k}))\mathfrak{u}_1, \mathfrak{u}_1\rangle}{2} \ (2.8) \\
&\leq h(\iota)\psi\left(\left(\int_0^1 \left\|t|\mathfrak{P}_1|^{4k} + (1-t)|\mathfrak{P}_1^*|^{4k}\right\|^{\frac{1}{2}}dt\right)^2\right) \\
&+ h(1-\iota)\psi(|\langle \mathfrak{P}_1^2\mathfrak{u}_1, \mathfrak{u}_1\rangle|^{k})\left\|\frac{\psi(|\mathfrak{P}_1|^{2k}) + \psi(|\mathfrak{P}_1^*|^{2k})}{2}\right\|.
\end{aligned}
$$

We get the required inequality when we take sup over all the unit vectors. □

Corollary 4.8 *The following chain of inequalities for $\mathfrak{P}_1 \in \mathfrak{B}(\mathfrak{H})$ when $\psi = I$ holds:*

$$
\begin{aligned}
w^{2k}(\mathfrak{P}_1^2) &\leq h(\iota)\left(\int_0^1 \left\|t|\mathfrak{P}_1|^{4k} + (1-t)|\mathfrak{P}_1^*|^{4k}\right\|^{\frac{1}{2}}dt\right)^2 \\
&+ \frac{h(1-\iota)}{2}w^{k}(\mathfrak{P}_1^2)\left\||\mathfrak{P}_1|^{2k} + |\mathfrak{P}_1^*|^{2k}\right\| \\
&\leq \|\mathfrak{P}_1\|^{4k}.
\end{aligned} \tag{4.11}
$$

Proof We will bound two quantities, first the term inside the integral, that is

$$\left\|t|\mathfrak{P}_1|^{4k} + (1-t)|\mathfrak{P}_1^*|^{4k}\right\| \leq \|\mathfrak{P}_1\|^{4k},$$

we will also use the following bound $w^k(\mathfrak{P}_1^2) \leq \left\|\mathfrak{P}_1^2\right\|^k \leq \|\mathfrak{P}_1\|^{2k}$ then using the elementary inequality $\left\||\mathfrak{P}_1|^{2k} + |\mathfrak{P}_1^*|^{2k}\right\| \leq 2\|\mathfrak{P}_1\|^{2k}$. Combining everything, we get

$$h(\iota)\|\mathfrak{P}_1\|^{4k} + h(1-\iota)\|\mathfrak{P}_1\|^{4k} = \|\mathfrak{P}_1\|^{4k}.$$

□

The inequality (4.11) is sharp. To see this, let us suppose that $\mathfrak{P}_1$ is a normal operator, that is $\mathfrak{P}_1^*\mathfrak{P}_1 = \mathfrak{P}_1\mathfrak{P}_1^*$, we also have $w(\mathfrak{P}_1^2) = w^2(\mathfrak{P}_1) = \|\mathfrak{P}_1\|^2 = \left\|\mathfrak{P}_1^2\right\|$, from that we get

$$\begin{aligned}
\|\mathfrak{P}_1\|^{4k} &\leq h(\iota)\left(\int_0^1 \left\|t|\mathfrak{P}_1|^{4k} + (1-t)|\mathfrak{P}_1^*|^{4k}\right\|^{\frac{1}{2}} dt\right)^2 \\
&\quad + \frac{h(1-\iota)}{2} w^k(\mathfrak{P}_1^2)\left\||\mathfrak{P}_1|^{2k} + |\mathfrak{P}_1^*|^{2k}\right\| \\
&= h(\iota)\|\mathfrak{P}_1\|^{4k} + \frac{h(1-\iota)\|\mathfrak{P}_1\|^{2k}}{2} \cdot \left(2\left\||\mathfrak{P}_1|^2\right\|^k\right) \\
&= h(\iota)\|\mathfrak{P}_1\|^{4k} + h(1-\iota)\|\mathfrak{P}_1\|^{2k} \cdot \|\mathfrak{P}_1\|^{2k} \\
&= \|\mathfrak{P}_1\|^{4k}.
\end{aligned}$$

Further, from (1.2) we obtain that

$$\begin{aligned}
w(\mathfrak{P}_1^{4k}) \leq w^{2k}(\mathfrak{P}_1^2) &\leq h(\iota)\left(\int_0^1 \left\|t|\mathfrak{P}_1|^{4k} + (1-t)|\mathfrak{P}_1^*|^{4k}\right\|^{\frac{1}{2}} dt\right)^2 \\
&\quad + \frac{h(1-\iota)}{2} w^k(\mathfrak{P}_1^2)\left\||\mathfrak{P}_1|^{2k} + |\mathfrak{P}_1^*|^{2k}\right\| \\
&\leq \|\mathfrak{P}_1\|^{4k}.
\end{aligned} \tag{4.12}$$

We can also obtain the following inequality by fixing the function h.

Setting $h(\iota) = \frac{1}{2}\left(\frac{1}{2} + \iota\right), \iota \in [0,1], k = 1$, we obtain

$$\begin{aligned}
w(\mathfrak{P}_1^4) \leq w^2(\mathfrak{P}_1^2) &\leq \frac{1}{2}\left(\frac{1}{2} + \iota\right)\left(\int_0^1 \left\|t|\mathfrak{P}_1|^4 + (1-t)|\mathfrak{P}_1^*|^4\right\|^{\frac{1}{2}} dt\right)^2 \\
&\quad + \frac{\frac{1}{2}\left(\frac{3}{2} - \iota\right)}{2} w(\mathfrak{P}_1^2)\left\||\mathfrak{P}_1|^2 + |\mathfrak{P}_1^*|^2\right\| \\
&\leq \|\mathfrak{P}_1\|^4
\end{aligned} \tag{4.13}$$

which is clearly a variation of (1.1).

Theorem 4.9 *Let $\mathfrak{P}_1 \in \mathfrak{B}(\mathfrak{H})$ and the criteria from (2.13) are met, then the following inequality holds for $k \geq 1, \tau \in (0,1]$*

$$\begin{aligned}
w^{2k}(\mathfrak{P}_1) &\leq h(\iota)\int_0^1 \left\|t|\mathfrak{P}_1|^{4\tau k} + (1-t)|\mathfrak{P}_1^*|^{4k(1-\tau)}\right\| dt + h(1-\iota) \cdot \frac{\left\||\mathfrak{P}_1|^{2k} + |\mathfrak{P}_1^*|^{2k}\right\|}{4} \\
&\quad + \frac{h(1-\iota)}{2} w^k(\mathfrak{P}_1^2).
\end{aligned} \tag{4.14}$$

Proof Consider the following for the unit vector $\mathfrak{u}_1$

$$\begin{aligned}
|\langle \mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_1\rangle|^{2k} &= h(\iota)|\langle \mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_1\rangle|^{2k} + h(1-\iota)|\langle \mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_1\rangle\langle \mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_1\rangle|^{k} \\
&\le h(\iota)\langle |\mathfrak{P}_1|^{2\tau k}\mathfrak{u}_1,\mathfrak{u}_1\rangle\langle |\mathfrak{P}_1^*|^{2k(1-\tau)}\mathfrak{u}_1,\mathfrak{u}_1\rangle (2.1) \\
&+ \frac{h(1-\iota)}{2}(\|\mathfrak{P}_1\mathfrak{u}_1\|^k\,\|\mathfrak{P}_1^*\mathfrak{u}_1\|^k + |\langle \mathfrak{P}_1\mathfrak{u}_1,\mathfrak{P}_1^*\mathfrak{u}_1\rangle|^k)(2.2) \\
&\le h(\iota)\int_0^1 \langle |\mathfrak{P}_1|^{2\tau k}\mathfrak{u}_1,\mathfrak{u}_1\rangle^{2t}\langle |\mathfrak{P}_1^*|^{2k(1-\tau)}\mathfrak{u}_1,\mathfrak{u}_1\rangle^{2(1-t)}dt \\
&+ \frac{h(1-\iota)}{2}|\langle \mathfrak{P}_1^2\mathfrak{u}_1,\mathfrak{u}_1\rangle|^k + \frac{h(1-\iota)}{4}\langle (|\mathfrak{P}_1|^{2k}+|\mathfrak{P}_1^*|^{2k})\mathfrak{u}_1,\mathfrak{u}_1\rangle \\
&\le h(\iota)\int_0^1 (t\langle |\mathfrak{P}_1|^{4\tau k}\mathfrak{u}_1,\mathfrak{u}_1\rangle + (1-t)\langle |\mathfrak{P}_1^*|^{4k(1-\tau)}\mathfrak{u}_1,\mathfrak{u}_1\rangle)dt \quad \text{(Weighted AG)} \\
&+ \frac{h(1-\iota)}{2}|\langle \mathfrak{P}_1^2\mathfrak{u}_1,\mathfrak{u}_1\rangle|^k + \frac{h(1-\iota)}{4}\langle (|\mathfrak{P}_1|^{2k}+|\mathfrak{P}_1^*|^{2k})\mathfrak{u}_1,\mathfrak{u}_1\rangle \\
&\le h(\iota)\int_0^1 \left\| t|\mathfrak{P}_1|^{4\tau k} + (1-t)|\mathfrak{P}_1^*|^{4k(1-\tau)}\right\| dt \\
&+ \frac{h(1-\iota)}{2}|\langle \mathfrak{P}_1^2\mathfrak{u}_1,\mathfrak{u}_1\rangle|^k + \frac{h(1-\iota)}{4}\langle (|\mathfrak{P}_1|^{2k}+|\mathfrak{P}_1^*|^{2k})\mathfrak{u}_1,\mathfrak{u}_1\rangle.
\end{aligned}$$

Taking sup over $\mathfrak{u}_1$, we obtain the desired inequality. □

It can be shown that (4.14) is sharp for $\tau=\frac{1}{2}$ when assuming that $\mathfrak{P}_1$ is normal, it can also be shown that for $\tau=\frac{1}{2}$

$$\begin{aligned}
w^{2k}(A) &\le \int_0^1 \left\| t|A|^{2k} + (1-t)|A^*|^{2k}\right\| dt + h(1-\iota)\cdot\frac{\left\| |A|^{2k}+|A^*|^{2k}\right\|}{4} \\
&+ \frac{h(1-\iota)}{2}w^k(A^2) \\
&\le \|A\|^{2k} \qquad (4.15)
\end{aligned}$$

which is a refinement of (1.1).

The following Theorem gives us a boundary for $w^2(\mathfrak{P}_1)$ and as a corollary, we obtain an inequality for a complex numerical radius.

Theorem 4.10 *Suppose that $\mathfrak{P}_1\in\mathfrak{B}(\mathfrak{H})$. Let ψ be a nonnegative increasing convex function on $[0,+\infty)$ and let the mapping h satisfy the conditions of (2.13). Then for any $\tau\in[0,1]$ the following inequality holds*

$$\psi(w^2(\mathfrak{P}_1)) \le \frac{1}{2}\psi(w(|\mathfrak{P}_1^*|^{2(1-\tau)}|\mathfrak{P}_1|^{2\tau})) + \frac{1}{4}\left(\psi(\left\| |\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)}\right\|) + \psi(w(|\mathfrak{P}_1|^{4\tau} - |\mathfrak{P}_1^*|^{4(1-\tau)}))\right). \quad (4.16)$$

Proof Setting $\mathfrak{u}_1 = |\mathfrak{P}_1|^{2\tau}, \mathfrak{u}_2 = |\mathfrak{P}_1^*|^{2(1-\tau)}$ and $\mathfrak{u}_3 = \mathfrak{u}_1$ in (2.11) while taking $\mathfrak{u}_3$ to be unit vector, we obtain

$$|\langle|\mathfrak{P}_1|^{2\tau}\mathfrak{u}_1,\mathfrak{u}_1\rangle|^2 + |\langle|\mathfrak{P}_1^*|^{2(1-\tau)}\mathfrak{u}_1,\mathfrak{u}_1\rangle|^2 \leq |\langle|\mathfrak{P}_1^*|^{2(1-\tau)}|\mathfrak{P}_1|^{2\tau}\mathfrak{u}_1,\mathfrak{u}_1\rangle| + \max(\left\||\mathfrak{P}_1|^{2\tau}\mathfrak{u}_1\right\|^2, \left\||\mathfrak{P}_1^*|^{2(1-\tau)}\mathfrak{u}_1\right\|^2).$$

Now realizing the following

$$2|\langle\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_1\rangle|^2 \leq 2\langle|\mathfrak{P}_1|^{2\tau}\mathfrak{u}_1,\mathfrak{u}_1\rangle\langle|\mathfrak{P}_1^*|^{2(1-\tau)}\mathfrak{u}_1,\mathfrak{u}_1\rangle \leq \langle|\mathfrak{P}_1|^{2\tau}\mathfrak{u}_1,\mathfrak{u}_1\rangle^2 + \langle|\mathfrak{P}_1^*|^{2(1-\tau)}\mathfrak{u}_1,\mathfrak{u}_1\rangle^2,$$

and using (2.12), we obtain

$$2\langle\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_1\rangle|^2 \leq |\langle|\mathfrak{P}_1^*|^{2(1-\tau)}|\mathfrak{P}_1|^{2\tau}\mathfrak{u}_1,\mathfrak{u}_1\rangle| + \frac{1}{2}\left[\langle(|\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)})\mathfrak{u}_1,\mathfrak{u}_1\rangle + |\langle(|\mathfrak{P}_1|^{4\tau} - |\mathfrak{P}_1^*|^{4(1-\tau)})\mathfrak{u}_1,\mathfrak{u}_1\rangle|\right].$$

It gives rise to the inequality

$$|\langle\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_1\rangle|^2 \leq \frac{1}{2}|\langle|\mathfrak{P}_1^*|^{2(1-\tau)}|\mathfrak{P}_1|^{2\tau}\mathfrak{u}_1,\mathfrak{u}_1\rangle| + \frac{1}{4}\left[\langle(|\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)})\mathfrak{u}_1,\mathfrak{u}_1\rangle + |\langle(|\mathfrak{P}_1|^{4\tau} - |\mathfrak{P}_1^*|^{4(1-\tau)})\mathfrak{u}_1,\mathfrak{u}_1\rangle|\right]$$

as follows. Taking ψ on both sides and using the convexity properties, we obtain the following

$$\begin{aligned}
&\psi(|\langle\mathfrak{P}_1\mathfrak{u}_1,\mathfrak{u}_1\rangle|^2)\\
&\leq \psi\left(\frac{(|\langle|\mathfrak{P}_1^*|^{2(1-\tau)}|\mathfrak{P}_1|^{2\tau}\mathfrak{u}_1,\mathfrak{u}_1\rangle|)}{2} + \frac{\frac{\langle(|\mathfrak{P}_1|^{4\tau}+|\mathfrak{P}_1^*|^{4(1-\tau)})\mathfrak{u}_1,\mathfrak{u}_1\rangle + |\langle(|\mathfrak{P}_1|^{4\tau}-|\mathfrak{P}_1^*|^{4(1-\tau)})\mathfrak{u}_1,\mathfrak{u}_1\rangle|}{2}}{2}\right)\\
&\leq \frac{1}{2}\psi(|\langle|\mathfrak{P}_1^*|^{2(1-\tau)}|\mathfrak{P}_1|^{2\tau}\mathfrak{u}_1,\mathfrak{u}_1\rangle|)\\
&+ \frac{1}{2}\psi\left(\frac{\langle(|\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)})\mathfrak{u}_1,\mathfrak{u}_1\rangle + |\langle(|\mathfrak{P}_1|^{4\tau} - |\mathfrak{P}_1^*|^{4(1-\tau)})\mathfrak{u}_1,\mathfrak{u}_1\rangle|}{2}\right)\\
&\leq \frac{1}{2}\psi(|\langle|\mathfrak{P}_1^*|^{2(1-\tau)}|\mathfrak{P}_1|^{2\tau}\mathfrak{u}_1,\mathfrak{u}_1\rangle|)\\
&+ \frac{\psi(\langle(|\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)})\mathfrak{u}_1,\mathfrak{u}_1\rangle) + \psi(|\langle(|\mathfrak{P}_1|^{4\tau} - |\mathfrak{P}_1^*|^{4(1-\tau)})\mathfrak{u}_1,\mathfrak{u}_1\rangle|)}{4}\\
&\leq \frac{1}{2}\psi(|\langle|\mathfrak{P}_1^*|^{2(1-\tau)}|\mathfrak{P}_1|^{2\tau}\mathfrak{u}_1,\mathfrak{u}_1\rangle|)\\
&+ \frac{\psi(\left\||\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)}\right\|) + \psi(|\langle(|\mathfrak{P}_1|^{4\tau} - |\mathfrak{P}_1^*|^{4(1-\tau)})\mathfrak{u}_1,\mathfrak{u}_1\rangle|)}{4}.
\end{aligned}$$

Taking sup over the unit vectors, the proof is finished. □

Remark 4.11 *Using (2.6) and the fact that* $w(|\mathfrak{P}_1|^{4\tau} - |\mathfrak{P}_1^*|^{4(1-\tau)}) \leq w(|\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)})$, *we can obtain* *the following bound while setting* $\psi = I$

$$w^2(\mathfrak{P}_1) \leq \frac{1}{2}w(|\mathfrak{P}_1^*|^{2(1-\tau)}|\mathfrak{P}_1|^{2\tau}) + \frac{1}{4}\left(\left\||\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)}\right\| + w(|\mathfrak{P}_1|^{4\tau} - |\mathfrak{P}_1^*|^{4(1-\tau)})\right) \tag{4.17}$$

$$\leq \frac{3}{4}\left\||\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)}\right\|. \tag{4.18}$$

Remark 4.12 *The inequality (4.17) is sharp for* $\tau = \frac{1}{2}$. *To see this, let us assume that* $\mathfrak{P}_1$ *is a normal operator, that is* $\mathfrak{P}_1^*\mathfrak{P}_1 = \mathfrak{P}_1\mathfrak{P}_1^*$, *we also have* $w(\mathfrak{P}_1^2) = w^2(\mathfrak{P}_1) = \|\mathfrak{P}_1\|^2 = \left\|\mathfrak{P}_1^2\right\|$, *from that we get*

$$\begin{aligned}\|\mathfrak{P}_1\|^2 &\leq \frac{1}{2}\||\mathfrak{P}_1^*||\mathfrak{P}_1|\| + \frac{1}{4}\left(\left\||\mathfrak{P}_1|^2 + |\mathfrak{P}_1^*|^2\right\|\right) \\ &= \frac{\left\|(\mathfrak{P}_1^*)^2\right\|}{2} + \frac{\left\||\mathfrak{P}_1|^2\right\|}{2} \\ &= \frac{\|\mathfrak{P}_1\|^2}{2} + \frac{\|\mathfrak{P}_1\|^2}{2}.\end{aligned}$$

Further, we obtain the following inequality concerning the invertible operator.

Corollary 4.13 *Assume that* $\mathfrak{P}_1 \in \mathfrak{B}(\mathfrak{H})$ *is an invertible operator, then we have the following inequality*

$$w^2(\mathfrak{P}_1) \leq \frac{\left\||\mathfrak{P}_1|^2 + |\mathfrak{P}_1^*|^2\right\|}{2} + \frac{\|\mathfrak{P}_1\|^2 - \left\|\mathfrak{P}_1^{-1}\right\|^{-2}}{4}.$$

Proof Setting $\tau = \frac{1}{2}$ in (4.16) and utilizing the basic inequalities, (2.6) and [20, eq. 34], we achieve the intended disparity. □

Setting $\psi = \mathfrak{u}_1^p, p \geq 1$ *in (4.16) we obtain the following*

$$w^{2p}(\mathfrak{P}_1) \leq \frac{1}{2}w^p(|\mathfrak{P}_1^*|^{2(1-\tau)}|\mathfrak{P}_1|^{2\tau}) + \frac{1}{4}\left(\left\||\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)}\right\|^p + w^p(|\mathfrak{P}_1|^{4\tau} - |\mathfrak{P}_1^*|^{4(1-\tau)})\right). \tag{4.19}$$

Setting $p = 2$, *we obtain the following interesting result*

$$w^4(\mathfrak{P}_1) \leq \frac{1}{2}w^2(|\mathfrak{P}_1^*|^{2(1-\tau)}|\mathfrak{P}_1|^{2\tau}) + \frac{1}{4}\left(\left\||\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)}\right\|^2 + w^2(|\mathfrak{P}_1|^{4\tau} - |\mathfrak{P}_1^*|^{4(1-\tau)})\right). \tag{4.20}$$

Following the proof of (4.16) and using the fact $\langle|\mathfrak{T}|^{2\tau}\mathfrak{u}_1, \mathfrak{u}_1\rangle^2 + \langle|\mathfrak{T}^*|^{2(1-\tau)}\mathfrak{u}_1, \mathfrak{u}_1\rangle^2 = |\langle|\mathfrak{T}|^{2\tau}\mathfrak{u}_1, \mathfrak{u}_1\rangle + i\langle|\mathfrak{T}^*|^{2(1-\tau)}\mathfrak{u}_1, \mathfrak{u}_1\rangle|^2 = |\langle(|\mathfrak{T}|^{2\tau} + i|\mathfrak{T}^*|^{2(1-\tau)})\mathfrak{u}_1, \mathfrak{u}_1\rangle|^2$ *and taking* sup *over all the unit vectors, we get*

$$\begin{aligned}&w^2(|\mathfrak{P}_1|^{2\tau} + i|\mathfrak{P}_1^*|^{2(1-\tau)}) \\ &\leq w(|\mathfrak{P}_1^*|^{2(1-\tau)}|\mathfrak{P}_1|^{2\tau}) + \frac{1}{2}\left(\left\||\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)}\right\| + w(|\mathfrak{P}_1|^{4\tau} - |\mathfrak{P}_1^*|^{4(1-\tau)})\right).\end{aligned} \tag{4.21}$$

Corollary 4.14 *Using a similar reasoning to the one in (4.18), we obtain the following chain of inequalities*

$$\begin{aligned} & w^2(|\mathfrak{P}_1|^{2\tau} + i|\mathfrak{P}_1^*|^{2(1-\tau)}) \\ & \leq w(|\mathfrak{P}_1^*|^{2(1-\tau)}|\mathfrak{P}_1|^{2\tau}) + \frac{1}{2}\left(\left\||\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)}\right\| + w(|\mathfrak{P}_1|^{4\tau} - |\mathfrak{P}_1^*|^{4(1-\tau)})\right) \\ & \leq \frac{3}{2}\left\||\mathfrak{P}_1|^{4\tau} + |\mathfrak{P}_1^*|^{4(1-\tau)}\right\|. \end{aligned} \tag{4.22}$$

5. Conclusion

Various vector and numerical radius-type inequalities have been obtained. Utilizing the function h defined in the paper written by Stojiljković and Dragomir in [30] and convexity properties with standard operator theory techniques, we obtained various refinements of the well-known numerical radius type inequalities. Namely, we have generalized the inequality given by Rashid in [26], and we also have sharpened the inequality given by Dragomir (2.6) by (4.8). Further, inequality for the complex numerical radius of the absolute value of a bounded operator has been given by (4.22). Variation of (1.1) has been given by (4.12) and (4.14).

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