Electronic Journal of Qualitative Theory of Differential Equations
2024, No. **49**, 1–16; https://doi.org/10.14232/ejqtde.2024.1.49 www.math.u-szeged.hu/ejqtde/

Symmetric nonlinear solvable system of difference equations

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Received 22 June 2024, appeared 2 September 2024
Communicated by Leonid Berezansky

Abstract. We show the theoretical solvability of the system of difference equations

$$x_{n+k} = \frac{y_{n+l}y_n - cd}{y_{n+l} + y_n - c - d}, \quad y_{n+k} = \frac{x_{n+l}x_n - cd}{x_{n+l} + x_n - c - d}, \quad n \in \mathbb{N}_0,$$

where $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, $l < k$, $c, d \in \mathbb{C}$ and $x_j, y_j \in \mathbb{C}$, $j = \overline{0, k-1}$. For several special cases of the system, we give some detailed explanations on how some formulas for their general solutions can be found in closed form, that is, we show their practical solvability. To do this, among other things, we use the theory of homogeneous linear difference equations with constant coefficients and the product-type difference equations with integer exponents, which are theoretically solvable.

Keywords: symmetric system of difference equations, solvable system, solution in closed form.

2020 Mathematics Subject Classification: 39A20, 39A45.

1 Introduction

Finding general solutions to difference equations and systems of difference equations is a classical problem which can be traced back to the beginning of the 18th century, [5,8,9]. During the century many important results on the problem have been obtained [10, 16, 18–20]. A majority of the results were on linear difference equations and systems of difference equations, but some of them were also on the nonlinear ones (see also [6, 12, 17, 21, 22]). For some later presentations and applications of the equations, see [11, 13, 23, 26, 37]. Since the solvability

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theory for linear difference equations was essentially completed during that time, and since it is practically impossible to find some general methods for solving nonlinear equations, interest in the topic diminished during the 19th century. Although from time to time, some solvable difference equations occurred, for example, in computational mathematics [7], problem books [2,14,24,25] and in some popular journals for a wide audience. Solvable difference equations are also useful in some comparison results [4,15,40]. One can study the invariants of the equations and systems, as it was the case, e.g., in [28–30,32,38,39], but only some of their very special classes were considered therein.

During the last two decades there has been a renewed interest in the area. It seems mostly because of the use of some computer calculations. We have analysed some of the recent papers and given many comments and theoretical explanations related to them (see, for example, [47] where some of the analyses, comments and explanations are given). An interesting fact is that the solvability of almost all of the recently presented classes of solvable difference equations and systems rely on the solvability of some linear ones (see, for example, [3,35,47,49] and the related references therein). However, it is of some interest to enlarge the list of solvable nonlinear difference equations and systems which are not obtained from linear ones in an obvious way.

There has been some interest in systems of difference equations which are close to symmetric ones since the mid of the nineties [27,31,33,34,38,39], which attracted our attention. We have devoted a part of our investigation also in this direction (see, e.g., [41–47]).

Motivated by the equation

$$x_n = \frac{x_{n-s}x_{n-t} + a}{x_{n-s} + x_{n-t}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $s, t \in \mathbb{N}$, $a \in \mathbb{C}$ and $x_{-j} \in \mathbb{C}$, $j = \overline{0, \max\{s,t\}}$ ([36,48]), in [49] we studied the equation

$$x_{n+s} = \frac{x_{n+t}x_n - ab}{x_{n+t} + x_n - a - b}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $s \in \mathbb{N}$, $t \in \mathbb{N}_0$, $t < s$, $a, b \in \mathbb{C}$ and $x_j \in \mathbb{C}$, $j = \overline{0, s-1}$, and showed its theoretical solvability. Equation (1.1) is a natural generalization of its special case with $s = 1$ and $t = 2$, which can be obtained by using the secant method [7]

$$x_n = \frac{x_{n-2}f(x_{n-1}) - x_{n-1}f(x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}, \quad n \in \mathbb{N}_0,$$

for $f(x) = x^2 - a$ (see, e.g., [15]).

On the other hand, motivated by our studies of the systems which stem from equation (1.1) (see the nonlinear systems of difference equations in [41–45]), in [46] we investigated a nonlinear system of difference equations which is related to equation (1.2), showed its solvability and discussed some special cases of the system in detail.

Here, we continue above mentioned investigations on solvability by studying the system

$$x_{n+k} = \frac{y_{n+l}y_n - cd}{y_{n+l} + y_n - c - d}, \quad y_{n+k} = \frac{x_{n+l}x_n - cd}{x_{n+l} + x_n - c - d}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, $l < k$, $c, d \in \mathbb{C}$ and $x_j, y_j \in \mathbb{C}$, $j = \overline{0, k-1}$, which is a symmetric relative to equation (1.2) and has not been considered in the literature yet.

Definition 1.1. We say that a nonlinear difference equation or system is *theoretically solvable* if by some changes of variables it can be transformed to a linear difference equation or system with constant coefficients. If the general solution to the linear difference equation or system can be found in closed form, we say that the nonlinear difference equation or system is *practically solvable*.

Remark 1.2. Not all linear difference equations with constant coefficients are practically solvable. For example, the difference equation

$$x_{n+5} - 6x_{n+1} + 3x_n = 0, \quad n \in \mathbb{N}_0,$$

is one of them, since we cannot find the roots of the associated characteristic polynomial

$$q_5(\lambda) = \lambda^5 - 6\lambda + 3$$

by radicals (see, e.g., [50]), due to the famous result by Abel and Ruffini [1].

Here we show the theoretical solvability of system (1.3), and give a detailed explanation on how in some cases can be found the general solution, that is, how to show their practical solvability.

2 Main results

Here we state and prove our main results.

Theorem 2.1. *Suppose $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, $l < k$, and $c, d \in \mathbb{C}$. Then, system (1.3) is theoretically solvable.*

Proof. Suppose $c \neq d$. Note that

$$\begin{aligned}
x_{n+k} - d &= \frac{(y_{n+l} - d)(y_n - d)}{y_{n+l} + y_n - c - d}, \\
x_{n+k} - c &= \frac{(y_{n+l} - c)(y_n - c)}{y_{n+l} + y_n - c - d}, \\
y_{n+k} - d &= \frac{(x_{n+l} - d)(x_n - d)}{x_{n+l} + x_n - c - d}, \\
y_{n+k} - c &= \frac{(x_{n+l} - c)(x_n - c)}{x_{n+l} + x_n - c - d},
\end{aligned}$$

for $n \in \mathbb{N}_0$.

Dividing the first two relations we get

$$\frac{x_{n+k} - d}{x_{n+k} - c} = \frac{(y_{n+l} - d)(y_n - d)}{(y_{n+l} - c)(y_n - c)},$$

for $n \in \mathbb{N}_0$, whereas dividing the last two relations we get

$$\frac{y_{n+k} - d}{y_{n+k} - c} = \frac{(x_{n+l} - d)(x_n - d)}{(x_{n+l} - c)(x_n - c)},$$

for $n \in \mathbb{N}_0$.

Now we define two auxiliary sequences as follows

$$\zeta_n = \frac{x_n - d}{x_n - c}, \quad \eta_n = \frac{y_n - d}{y_n - c}, \tag{2.1}$$

for $n \in \mathbb{N}_0$.

They obviously satisfy the relations

$$\zeta_{n+k} = \eta_{n+l}\eta_n, \quad \eta_{n+k} = \zeta_{n+l}\zeta_n, \tag{2.2}$$

for $n \in \mathbb{N}_0$, which yields that ζ_n and η_n are two solutions to the equation

$$\omega_{n+2k} = \omega_{n+2l}\omega_{n+l}^2\omega_n, \quad n \in \mathbb{N}_0, \tag{2.3}$$

a product-type difference equation with integer exponents, which is theoretically solvable. Hence, such one is the system (1.3).

Note also that from (2.1) we have

$$x_n = \frac{c\zeta_n - d}{\zeta_n - 1}, \quad y_n = \frac{c\eta_n - d}{\eta_n - 1}, \tag{2.4}$$

for $n \in \mathbb{N}_0$.

Now suppose $c = d$. Note that

$$x_{n+k} - c = \frac{(y_{n+l} - c)(y_n - c)}{y_{n+l} + y_n - 2c}, \tag{2.5}$$

$$y_{n+k} - c = \frac{(x_{n+l} - c)(x_n - c)}{x_{n+l} + x_n - 2c}, \tag{2.6}$$

for $n \in \mathbb{N}_0$.

Now we define the two auxiliary sequences

$$\zeta_n = \frac{1}{x_n - c}, \quad \eta_n = \frac{1}{y_n - c}, \tag{2.7}$$

for $n \in \mathbb{N}_0$.

Combining (2.5)–(2.7), we get

$$\zeta_{n+k} = \eta_{n+l} + \eta_n, \quad \eta_{n+k} = \zeta_{n+l} + \zeta_n, \tag{2.8}$$

for $n \in \mathbb{N}_0$, which implies that ζ_n and η_n satisfy the equation

$$\omega_{n+2k} - \omega_{n+2l} - 2\omega_{n+l} - \omega_n = 0, \tag{2.9}$$

for $n \in \mathbb{N}_0$, which according to Definition 1.1 means that system (1.3) is theoretically solvable in this case. □

A natural problem is to find special cases of system (1.3) for which it is possible to find some closed-form formulas for their general solutions.

The polynomial

$$q_{2k}(\lambda) = \lambda^{2k} - \lambda^{2l} - 2\lambda^l - 1,$$

is the characteristic one associated to equation (2.9) [11,13,23,25]. Note that

$$q_{2k}(\lambda) = \lambda^{2k} - (\lambda^l + 1)^2 = (\lambda^k - \lambda^l - 1)(\lambda^k + \lambda^l + 1). \tag{2.10}$$

In some cases it is possible to find its roots (but, of course, not always [1]), for instance, if $0 \le l < k \le 4$ (there are ten cases). In all these cases, among other ones, equation (2.9) is practically solvable. Now we present the general solution to system (1.3) in some of these cases.

Theorem 2.2. *Consider the system* (1.3) *with* $k = 1$, $l = 0$ *and* $c, d \in \mathbb{C}$.

(a) If $c = d$, $x_0 \ne c \ne y_0$, *then*

$$x_{2m} = c + \frac{x_0 - c}{4^m}, \tag{2.11}$$

$$x_{2m+1} = c + \frac{y_0 - c}{2 \cdot 4^m}, \tag{2.12}$$

$$y_{2m} = c + \frac{y_0 - c}{4^m}, \tag{2.13}$$

$$y_{2m+1} = c + \frac{x_0 - c}{2 \cdot 4^m}, \tag{2.14}$$

for $m \in \mathbb{N}_0$.

(b) If $c \ne d$, *then well-defined solutions to the system are given by*

$$x_{2m} = \frac{c\left(\frac{x_0-d}{x_0-c}\right)^{4^m} - d}{\left(\frac{x_0-d}{x_0-c}\right)^{4^m} - 1}, \tag{2.15}$$

$$x_{2m+1} = \frac{c\left(\frac{y_0-d}{y_0-c}\right)^{2\cdot 4^m} - d}{\left(\frac{y_0-d}{y_0-c}\right)^{2\cdot 4^m} - 1}, \tag{2.16}$$

$$y_{2m} = \frac{c\left(\frac{y_0-d}{y_0-c}\right)^{4^m} - d}{\left(\frac{y_0-d}{y_0-c}\right)^{4^m} - 1}, \tag{2.17}$$

$$y_{2m+1} = \frac{c\left(\frac{x_0-d}{x_0-c}\right)^{2\cdot 4^m} - d}{\left(\frac{x_0-d}{x_0-c}\right)^{2\cdot 4^m} - 1}, \tag{2.18}$$

for $m \in \mathbb{N}_0$.

Proof. *(a)* First, note that (2.8) is

$$\zeta_{n+1} = 2\eta_n, \quad \eta_{n+1} = 2\zeta_n,$$

for $n \in \mathbb{N}_0$. Thus

$$\zeta_{n+2} = 4\zeta_n, \quad \eta_{n+2} = 4\eta_n,$$

for $n \in \mathbb{N}_0$, which yields

$$\zeta_{2m} = 4^m\zeta_0, \quad \zeta_{2m+1} = 4^m\zeta_1, \quad \eta_{2m} = 4^m\eta_0, \quad \eta_{2m+1} = 4^m\eta_1,$$

for $m \in \mathbb{N}_0$. This and (2.7) imply (2.11)–(2.14), under the assumption $c = d$.

(b) If we assume that $c \ne d$, then from (2.3) we have

$$\zeta_{n+2} = \zeta_n^4, \quad \eta_{n+2} = \eta_n^4, \quad n \in \mathbb{N}_0.$$

Therefore

$$\zeta_{2m} = \zeta_0^{4^m}, \quad \zeta_{2m+1} = \zeta_1^{4^m}, \quad \eta_{2m} = \eta_0^{4^m}, \quad \eta_{2m+1} = \eta_1^{4^m},$$

for $m \in \mathbb{N}_0$.

These four relations, the transformation in (2.1), and (2.4), imply (2.15)–(2.18), completing the proof. □

Remark 2.3. Assume that in Theorem 2.2, $c = d$, and that $x_0 = c$ or $y_0 = c$. Note that from (1.3) we have

$$x_1 = \frac{y_0^2 - c^2}{2(y_0 - c)} \tag{2.19}$$

and

$$y_1 = \frac{x_0^2 - c^2}{2(x_0 - c)}. \tag{2.20}$$

Hence, if $x_0 = c$, then from (2.20) we see that y_1 is not defined, whereas if $y_0 = c$, then from (2.19) we see that x_1 is not defined.

Corollary 2.4. *The system (1.3) with $c, d \in \mathbb{C}$, $l = 0$ and $k \in \mathbb{N} \setminus \{1\}$ is practically solvable.*

Proof. Under these conditions, we have

$$x_{n+k} = \frac{y_n^2 - cd}{2y_n - c - d}, \quad y_{n+k} = \frac{x_n^2 - cd}{2x_n - c - d}, \quad n \in \mathbb{N}_0,$$

which is a system with interlacing indices ([47]).

Let

$$x_m^{(j)} = x_{mk+j}, \quad y_m^{(j)} = y_{mk+j},$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, k-1}$.

Then, $(x_m^{(j)}, y_m^{(j)})_{m\in\mathbb{N}_0}$, $j = \overline{0, k-1}$, are k solutions to the system

$$x_{m+1} = \frac{y_m^2 - cd}{2y_m - c - d}, \quad y_{m+1} = \frac{x_m^2 - cd}{2x_m - c - d}, \quad m \in \mathbb{N}_0.$$

Note that it is the system (1.3) with $k = 1$ and $l = 0$.

Thus, if $c = d$, $x_0^{(j)} \neq c \neq y_0^{(j)}$, $j = \overline{0, k-1}$, by Theorem 2.2 we get

$$x_{2m}^{(j)} = c + \frac{x_0^{(j)} - c}{4^m},$$
$$x_{2m+1}^{(j)} = c + \frac{y_0^{(j)} - c}{2 \cdot 4^m},$$
$$y_{2m}^{(j)} = c + \frac{y_0^{(j)} - c}{4^m},$$
$$y_{2m+1}^{(j)} = c + \frac{x_0^{(j)} - c}{2 \cdot 4^m},$$

for $m \in \mathbb{N}_0$, $j = \overline{0, k-1}$, whereas if $c \neq d$, then well-defined solutions to the system are

$$x_{2m}^{(j)} = \frac{c\left(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\right)^{4^m} - d}{\left(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\right)^{4^m} - 1},$$

$$x_{2m+1}^{(j)} = \frac{c\left(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\right)^{2\cdot 4^m} - d}{\left(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\right)^{2\cdot 4^m} - 1},$$

$$y_{2m}^{(j)} = \frac{c\left(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\right)^{4^m} - d}{\left(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\right)^{4^m} - 1},$$

$$y_{2m+1}^{(j)} = \frac{c\left(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\right)^{2\cdot 4^m} - d}{\left(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\right)^{2\cdot 4^m} - 1}$$

for $m \in \mathbb{N}$, $j = \overline{0, k-1}$.

Hence, if $c = d$ we have

$$x_{2mk+j} = c + \frac{x_j - c}{4^m},$$

$$x_{(2m+1)k+j} = c + \frac{y_j - c}{2 \cdot 4^m},$$

$$y_{2mk+j} = c + \frac{y_j - c}{4^m},$$

$$y_{(2m+1)k+j} = c + \frac{x_j - c}{2 \cdot 4^m},$$

for $m \in \mathbb{N}_0$, $j = \overline{0, k-1}$, whereas if $c \neq d$ we have

$$x_{2mk+j} = \frac{c\left(\frac{x_j-d}{x_j-c}\right)^{4^m} - d}{\left(\frac{x_j-d}{x_j-c}\right)^{4^m} - 1},$$

$$x_{(2m+1)k+j} = \frac{c\left(\frac{y_j-d}{y_j-c}\right)^{2\cdot 4^m} - d}{\left(\frac{y_j-d}{y_j-c}\right)^{2\cdot 4^m} - 1},$$

$$y_{2mk+j} = \frac{c\left(\frac{y_j-d}{y_j-c}\right)^{4^m} - d}{\left(\frac{y_j-d}{y_j-c}\right)^{4^m} - 1},$$

$$y_{(2m+1)k+j} = \frac{c\left(\frac{x_j-d}{x_j-c}\right)^{2\cdot 4^m} - d}{\left(\frac{x_j-d}{x_j-c}\right)^{2\cdot 4^m} - 1}$$

for $m \in \mathbb{N}$, $j = \overline{0, k-1}$. □

Theorem 2.5. *The system* (1.3) *with* $c, d \in \mathbb{C}$, $l = 1$ *and* $k = 2$ *is practically solvable.*

Proof. Suppose $c \neq d$. From (2.3) we get that $(\zeta_n)_{n\in\mathbb{N}_0}$ and $(\eta_n)_{n\in\mathbb{N}_0}$ are the solutions to

$$\omega_{n+4} = \omega_{n+2}\omega_{n+1}^2\omega_n, \quad n \in \mathbb{N}_0, \tag{2.21}$$

with the initial values

$$\zeta_0, \quad \zeta_1, \quad \zeta_2 = \eta_1\eta_0, \quad \zeta_3 = \zeta_1\zeta_0\eta_1, \tag{2.22}$$

$$\eta_0, \quad \eta_1, \quad \eta_2 = \zeta_1\zeta_0, \quad \eta_3 = \eta_1\eta_0\zeta_1, \tag{2.23}$$

respectively.

Rewrite (2.21) as follows

$$\omega_n = \omega_{n-2}^{a_1}\omega_{n-3}^{b_1}\omega_{n-4}^{c_1}\omega_{n-5}^{d_1}, \quad n \geq 5, \tag{2.24}$$

where

$$a_1 := 1, \quad b_1 := 2, \quad c_1 := 1, \quad d_1 := 0. \tag{2.25}$$

Further, we have

$$\begin{aligned}\omega_n &= (\omega_{n-4}\omega_{n-5}^2\omega_{n-6})^{a_1}\omega_{n-3}^{b_1}\omega_{n-4}^{c_1}\omega_{n-5}^{d_1}\\ &= \omega_{n-3}^{b_1}\omega_{n-4}^{a_1+c_1}\omega_{n-5}^{2a_1+d_1}\omega_{n-6}^{a_1}\\ &= \omega_{n-3}^{a_2}\omega_{n-4}^{b_2}\omega_{n-5}^{c_2}\omega_{n-6}^{d_2},\end{aligned} \tag{2.26}$$

for $n \geq 6$, where $a_2 := b_1$, $b_2 := a_1 + c_1$, $c_2 := 2a_1 + d_1$ and $d_2 := a_1$.

A simple inductive argument shows that

$$\omega_n = \omega_{n-k-1}^{a_k}\omega_{n-k-2}^{b_k}\omega_{n-k-3}^{c_k}\omega_{n-k-4}^{d_k}, \tag{2.27}$$

for $n \geq k + 4$, and

$$a_k = b_{k-1}, \quad b_k = a_{k-1} + c_{k-1}, \quad c_k = 2a_{k-1} + d_{k-1}, \quad d_k = a_{k-1} \tag{2.28}$$

for $k \geq 2$.

For $k = n - 4$ from (2.27) and (2.28), we get

$$\begin{aligned}\omega_n &= \omega_3^{a_{n-4}}\omega_2^{b_{n-4}}\omega_1^{c_{n-4}}\omega_0^{d_{n-4}}\\ &= \omega_3^{a_{n-4}}\omega_2^{a_{n-3}}\omega_1^{a_{n-2}-a_{n-4}}\omega_0^{a_{n-5}},\end{aligned} \tag{2.29}$$

for $n \geq 6$, whereas from (2.28), we get

$$a_k = a_{k-2} + 2a_{k-3} + a_{k-4}, \tag{2.30}$$

for $k \geq 5$, and

$$a_1 = 1, \quad a_2 = 2, \quad a_3 = 2, \quad a_4 = 4. \tag{2.31}$$

The polynomial

$$q_4(\lambda) = \lambda^4 - \lambda^2 - 2\lambda - 1 = (\lambda^2 - \lambda - 1)(\lambda^2 + \lambda + 1), \tag{2.32}$$

is the characteristic one associated to (2.30), with the zeros

$$\lambda_{1,2}=\frac{1\pm\sqrt{5}}{2} \quad \text{and} \quad \lambda_{3,4}=\frac{-1\pm i\sqrt{3}}{2}. \tag{2.33}$$

Since

$$a_{k-4}=a_k-a_{k-2}-2a_{k-3}, \tag{2.34}$$

by using (2.31) we can find a_k for $k\le 0$, and get

$$a_{-4}=a_{-3}=a_{-2}=0, \quad a_{-1}=1 \quad \text{and} \quad a_0=0. \tag{2.35}$$

Using [41, Lemma 1] we obtain

$$a_n=\frac{\lambda_1^{n+4}}{q_4'(\lambda_1)}+\frac{\lambda_2^{n+4}}{q_4'(\lambda_2)}+\frac{\lambda_3^{n+4}}{q_4'(\lambda_3)}+\frac{\lambda_4^{n+4}}{q_4'(\lambda_4)}, \tag{2.36}$$

for $n\in\mathbb{Z}$.

Since

$$q_4'(\lambda)=4\lambda^3-2\lambda-2=2(2\lambda^3-\lambda-1)=2(\lambda-1)(2\lambda^2+2\lambda+1),$$

we have

$$q_4'(\lambda_1)=5+3\sqrt{5}, \quad q_4'(\lambda_2)=5-3\sqrt{5}, \tag{2.37}$$

$$q_4'(\lambda_3)=3-i\sqrt{3}, \quad q_4'(\lambda_4)=3+i\sqrt{3}. \tag{2.38}$$

Using (2.37) and (2.38) in (2.36) imply

$$\begin{aligned} a_n&=\frac{\lambda_1^{n+4}}{5+3\sqrt{5}}+\frac{\lambda_2^{n+4}}{5-3\sqrt{5}}+\frac{\lambda_3^{n+4}}{3-i\sqrt{3}}+\frac{\lambda_4^{n+4}}{3+i\sqrt{3}}\\ &=\frac{\lambda_1^{n+2}-\lambda_2^{n+2}}{2\sqrt{5}}+\frac{\lambda_3^{n+2}-\lambda_4^{n+2}}{2i\sqrt{3}}, \end{aligned} \tag{2.39}$$

for $n\in\mathbb{Z}$. Employing this formula it is not difficult to check that (2.29) holds for all $n\in\mathbb{N}_0$.

Relations (2.22) and (2.29) imply

$$\begin{aligned} \zeta_n&=\zeta_3^{a_{n-4}}\zeta_2^{a_{n-3}}\zeta_1^{a_{n-2}-a_{n-4}}\zeta_0^{a_{n-5}}\\ &=(\eta_1\zeta_0\zeta_1)^{a_{n-4}}(\eta_1\eta_0)^{a_{n-3}}\zeta_1^{a_{n-2}-a_{n-4}}\zeta_0^{a_{n-5}}\\ &=\zeta_0^{a_{n-4}+a_{n-5}}\zeta_1^{a_{n-2}}\eta_0^{a_{n-3}}\eta_1^{a_{n-3}+a_{n-4}}, \end{aligned} \tag{2.40}$$

for $n\in\mathbb{N}_0$, and due to the symmetry

$$\eta_n=\eta_0^{a_{n-4}+a_{n-5}}\eta_1^{a_{n-2}}\zeta_0^{a_{n-3}}\zeta_1^{a_{n-3}+a_{n-4}}, \tag{2.41}$$

for $n\in\mathbb{N}_0$.

By some simple calculation and use of the Viète formulas we get

$$\begin{aligned} a_n+a_{n-1}&=\frac{(\lambda_1+1)\lambda_1^{n+1}-(\lambda_2+1)\lambda_2^{n+1}}{2\sqrt{5}}+\frac{(\lambda_3+1)\lambda_3^{n+1}-(\lambda_4+1)\lambda_4^{n+1}}{2i\sqrt{3}}\\ &=\frac{\lambda_1^{n+3}-\lambda_2^{n+3}}{2\sqrt{5}}-\frac{\lambda_3^{n+3}-\lambda_4^{n+3}}{2i\sqrt{3}}, \end{aligned} \tag{2.42}$$

for $n \in \mathbb{Z}$.

From (2.39)–(2.42) we get

$$\zeta_n = \zeta_0^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \zeta_1^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \eta_0^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \eta_1^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}},$$
$$\eta_n = \eta_0^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \eta_1^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \zeta_0^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \zeta_1^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}},$$

for $n \in \mathbb{N}_0$, from which together with (2.1) with $n = 0, 1$, we get

$$\begin{aligned} \zeta_n = {} & \left(\frac{x_0-d}{x_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{x_1-d}{x_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \\ & \times \left(\frac{y_0-d}{y_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{y_1-d}{y_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}, \end{aligned} \tag{2.43}$$

$$\begin{aligned} \eta_n = {} & \left(\frac{x_0-d}{x_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{x_1-d}{x_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \\ & \times \left(\frac{y_0-d}{y_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{y_1-d}{y_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}, \end{aligned} \tag{2.44}$$

for $n \in \mathbb{N}_0$.

Combining (2.4), (2.43) and (2.44), we have

$$x_n = \frac{c\left(\frac{x_0-d}{x_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{x_1-d}{x_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \left(\frac{y_0-d}{y_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{y_1-d}{y_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} - d}{\left(\frac{x_0-d}{x_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{x_1-d}{x_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \left(\frac{y_0-d}{y_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{y_1-d}{y_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} - 1}$$
$$y_n = \frac{c\left(\frac{y_0-d}{y_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{y_1-d}{y_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \left(\frac{x_0-d}{x_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{x_1-d}{x_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} - d}{\left(\frac{y_0-d}{y_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{y_1-d}{y_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \left(\frac{x_0-d}{x_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{x_1-d}{x_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} - 1},$$

for $n \in \mathbb{N}_0$.

Now assume that $c = d$. In this case, we have

$$\zeta_{n+2} = \eta_{n+1} + \eta_n, \quad \eta_{n+2} = \zeta_{n+1} + \zeta_n, \tag{2.45}$$

for $n \in \mathbb{N}_0$, implying that $(\zeta_n)_{n\in\mathbb{N}_0}$ and $(\eta_n)_{n\in\mathbb{N}_0}$ are the two solutions to the equation

$$\omega_{n+4} - \omega_{n+2} - 2\omega_{n+1} - \omega_n = 0, \tag{2.46}$$

for $n \in \mathbb{N}_0$, with the initial values

$$\zeta_0, \quad \zeta_1, \quad \zeta_2 = \eta_1 + \eta_0, \quad \zeta_3 = \eta_1 + \zeta_1 + \zeta_0, \tag{2.47}$$

$$\eta_0, \quad \eta_1, \quad \eta_2 = \zeta_1 + \zeta_0, \quad \eta_3 = \zeta_1 + \eta_1 + \eta_0, \tag{2.48}$$

respectively (see (2.45)).

If we write equation (2.46) in the form

$$\begin{aligned} \omega_n &= \omega_{n-2} + 2\omega_{n-3} + \omega_{n-4} + 0 \cdot \omega_{n-5} \\ &= a_1\omega_{n-2} + b_1\omega_{n-3} + c_1\omega_{n-4} + d_1\omega_{n-5}, \end{aligned}$$

where a_1, b_1, c_1, d_1 are given in (2.25), then by a simple inductive argument we can prove that

$$\omega_n = a_k\omega_{n-k-1} + b_k\omega_{n-k-2} + c_k\omega_{n-k-3} + d_k\omega_{n-k-4}, \tag{2.49}$$

for $n \geq k+4$, where $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$, satisfy (2.28). Thus (2.39) holds.

For $k = n-4$ we have

$$\omega_n = a_{n-4}\omega_3 + a_{n-3}\omega_2 + (a_{n-2} - a_{n-4})\omega_1 + a_{n-5}\omega_0, \tag{2.50}$$

for $n \geq 6$, from which along with (2.47), we get

$$\begin{aligned} \zeta_n &= a_{n-4}(\eta_1 + \zeta_1 + \zeta_0) + a_{n-3}(\eta_1 + \eta_0) + (a_{n-2} - a_{n-4})\zeta_1 + a_{n-5}\zeta_0 \\ &= (a_{n-4} + a_{n-5})\zeta_0 + a_{n-2}\zeta_1 + a_{n-3}\eta_0 + (a_{n-3} + a_{n-4})\eta_1, \end{aligned} \tag{2.51}$$

for $n \in \mathbb{N}_0$. Therefore

$$\eta_n = (a_{n-4} + a_{n-5})\eta_0 + a_{n-2}\eta_1 + a_{n-3}\zeta_0 + (a_{n-3} + a_{n-4})\zeta_1, \tag{2.52}$$

for $n \in \mathbb{N}_0$.

Combining (2.39), (2.42), (2.51) and (2.52) it follows that

$$\zeta_n = \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - c} + \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - c},$$

$$\eta_n = \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - c} + \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - c},$$

for $n \in \mathbb{N}_0$.

Thus

$$x_n = \frac{c\left(\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - c} + \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - c}\right) + 1}{\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - c} + \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - c}},$$

$$y_n = \frac{c\left(\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - c} + \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - c}\right) + 1}{\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - c} + \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - c}},$$

for $n \in \mathbb{N}_0$, where we have used the change of variables (2.7). □

Corollary 2.6. *The system* (1.3) *with $c, d \in \mathbb{C}$, $k = 2s$, $l = s$, for some $s \in \mathbb{N}$, is practically solvable.*

Proof. Under these conditions we have

$$x_{n+2s} = \frac{y_{n+s}y_n - cd}{y_{n+s} + y_n - c - d}, \quad y_{n+2s} = \frac{x_{n+s}x_n - cd}{x_{n+s} + x_n - c - d}, \quad n \in \mathbb{N}_0,$$

which is a system with interlacing indices ([47]).

Let

$$x_m^{(j)} = x_{ms+j}, \quad y_m^{(j)} = y_{ms+j},$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$. Then, $(x_m^{(j)}, y_m^{(j)})_{m\in\mathbb{N}_0}$, $j = \overline{0, s-1}$, are s solutions to the system

$$x_{m+2} = \frac{y_{m+1}y_m - cd}{y_{m+1} + y_m - c - d},$$

$$y_{m+2} = \frac{x_{m+1}x_m - cd}{x_{m+1} + x_m - c - d},$$

for $m \in \mathbb{N}_0$. Note that it is the system (1.3) with $k = 2$ and $l = 1$.

Thus, Theorem 2.5 can be applied, and if $c \neq d$, we get

$$x_m^{(j)} = \frac{c\Big(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{x_1^{(j)}-d}{x_1^{(j)}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\Big(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{y_1^{(j)}-d}{y_1^{(j)}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-d}{\Big(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{x_1^{(j)}-d}{x_1^{(j)}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\Big(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{y_1^{(j)}-d}{y_1^{(j)}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-1},$$

$$y_m^{(j)} = \frac{c\Big(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{y_1^{(j)}-d}{y_1^{(j)}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\Big(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{x_1^{(j)}-d}{x_1^{(j)}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-d}{\Big(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{y_1^{(j)}-d}{y_1^{(j)}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\Big(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{x_1^{(j)}-d}{x_1^{(j)}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-1},$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$, whereas if $c = d$, we get

$$x_m^{(j)} = \frac{c\left(\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_1^{(j)}-c}+\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_1^{(j)}-c}\right)+1}{\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_1^{(j)}-c}+\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_1^{(j)}-c}},$$

$$y_m^{(j)} = \frac{c\left(\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_1^{(j)}-c}+\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_1^{(j)}-c}\right)+1}{\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_1^{(j)}-c}+\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_1^{(j)}-c}},$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$, that is, if $c \neq d$, we have

$$x_{ms+j} = \frac{c\Big(\frac{x_j-d}{x_j-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{x_{s+j}-d}{x_{s+j}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\Big(\frac{y_j-d}{y_j-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{y_{s+j}-d}{y_{s+j}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-d}{\Big(\frac{x_j-d}{x_j-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{x_{s+j}-d}{x_{s+j}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\Big(\frac{y_j-d}{y_j-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{y_{s+j}-d}{y_{s+j}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-1},$$

$$y_{ms+j} = \frac{c\Big(\frac{y_j-d}{y_j-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{y_{s+j}-d}{y_{s+j}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\Big(\frac{x_j-d}{x_j-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{x_{s+j}-d}{x_{s+j}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-d}{\Big(\frac{y_j-d}{y_j-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{y_{s+j}-d}{y_{s+j}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\Big(\frac{x_j-d}{x_j-c}\Big)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\Big(\frac{x_{s+j}-d}{x_{s+j}-c}\Big)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-1},$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$, whereas if $c = d$, we get

$$x_{ms+j} = \frac{c\left(\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_{s+j} - c} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_{s+j} - c}\right) + 1}{\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_{s+j} - c} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_{s+j} - c}},$$

$$y_{ms+j} = \frac{c\left(\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_{s+j} - c} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_{s+j} - c}\right) + 1}{\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_{s+j} - c} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_{s+j} - c}},$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$. □

Remark 2.7. Theorem 2.2, Corollary 2.4, Theorem 2.5 and Corollary 2.6, show the practical solvability of system (1.3) in the following six cases: $k = 1$, $l = 0$; $k = 2$, $l = 0$; $k = 2$, $l = 1$; $k = 3$, $l = 0$; $k = 4$, $l = 0$ and $k = 4$, $l = 2$. Practical solvability of the system (1.3) in the cases: $k = 3$, $l = 1$; $k = 3$, $l = 2$; $k = 4$, $l = 1$ and $k = 4$, $l = 3$ is shown similarly, but with more technical details.

Remark 2.8. Employing the formulas for the general solutions to system (1.3), one can describe their well-defined solutions. The standard problem is left to the reader as an exercise.

Author Contributions. Stevo Stević proposed this research project. All the authors contributed to the study. All authors have read and agreed to this version of the manuscript.

Funding. This work was partially supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia under contract number: 451-03-65/2024-03/200103. This research is also partially supported by the Science Fund of the Republic of Serbia, #GRANT No 7632, Project "Mathematical Methods in Image Processing under Uncertainty"– MaMIPU.

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