Algebra Univers. (2024) 85:40

1420-8911/24/040001-9
published online September 23, 2024
https://doi.org/10.1007/s00012-024-00873-5

Algebra Universalis

Nilpotent groups in lattice framework

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This article is dedicated to Reinhard Pöeschel.

Abstract. In the framework of weak congruence lattices, many classes of groups have been characterized up to now, in completely lattice-theoretic terms. In this note, the center of the group is captured lattice-theoretically and nilpotent groups are characterized by lattice properties.

Mathematics Subject Classification. 20F18, 20F99, 20E15, 08A30.

Keywords. Lattice of subgroups, Lattice of weak congruences, Special elements in lattices, Classes of groups.

1. Introduction

It is known that subalgebra and congruence lattices are suitable tools for the structural investigation of algebras. In particular, investigation of groups by properties of their subgroup lattices has been performed since the development of the theory of groups and the theory of lattices. The results on groups in the framework of subgroup lattices are collected in the book [13] by Suzuki, then also in the book [12] by Schmidt (see also a review of this book in [4] by Freese); in addition, there is a survey paper on the topic [11] by Pálfy.

Our research about groups is also situated in the framework of lattices. We use the lattice of weak congruences, which contains, up to the isomorphism, both lattices mentioned above as sublattices (see [14,16]). These lattices are

Presented by F. M. Schneider.
Research of the first author is supported by the Ministry of Science, Technological Development and Innovation, through Faculty of Science, University of Novi Sad. Research of the third author is supported by the Ministry of Science, Technological Development and Innovation, through the Mathematical Institute of the Serbian Academy of Sciences and Arts in Belgrade and Faculty of Science, University of Novi Sad. This research was supported by the Science Fund of the Republic of Serbia, # Grant no 6565, Advanced Techniques of Mathematical Aggregation and Approximative Equations Solving in Digital Operational Research- AT-MATADOR.

particularly convenient when applied to groups, due to the close connection between congruences and normal subgroups.

In the mentioned book [12] by R. Schmidt, the lattice-theoretic characterization of the class of solvable groups is highlighted as one of the "most exciting problems" in this context. This problem is solved in our paper [8], where the class of solvable groups is characterized by weak congruence lattices in a pure lattice theoretic way, together with several other well-known classes of groups (Hamiltonian, Dedekind, abelian, supersolvable, metabelian and others). Although the theory of lattice-theoretic characterization of various classes of groups has been further developed [1,2,5,7,9], a characterization of a nilpotent group G by properties of its subgroup or weak congruence lattice was still an open problem. What is known about these groups is that the nilpotency of G can be characterized within the congruence lattice of the group $G \times G$ (see e.g., [3]). In particular, a characterization of torsion-free nilpotent groups by subgroup lattices was done by Kontorovič and Plotkin in [10]. Further, a characterization of finite nilpotent groups by lower semimodularity of their weak congruence lattice was given in [7].

In the present note, we give necessary and sufficient conditions that the weak congruence lattice of a group G should fulfill to make G nilpotent.

2. Preliminaries

By $H \leqslant G$ (or $H < G$ if $H \neq G$) and $H \lhd G$ we denote that H is a subgroup and a normal subgroup of G, respectively. $\mathsf{Sub}(G)$ is the algebraic lattice of all subgroups of G, ordered by the set inclusion. $\mathsf{Sub_n}(G)$ is the modular sublattice of $\mathsf{Sub}(G)$, consisting of all normal subgroups of G.

A group is said to satisfy the *maximal condition* (the ascending chain condition) if every strictly ascending chain of subgroups is finite. Recall that the *center* of a group G, denoted by $Z(G)$, is the set of elements that commute with every element of G. $Z(G)$ is an *an abelian, normal subgroup.*

A subgroup H of a group G is said to be *central* if it is contained in (is a subgroup of) the center $Z(G)$. Clearly, *central subgroups are abelian and normal in G.*

A group all whose subgroups are normal is a *Dedekind* group, which may be *abelian* and non-abelian, i.e., *Hamiltonian.* A group G is said to be *torsion-free* if the order of each $x \in G$, $x \neq e$ is infinite (i.e., the group $\langle x \rangle$ is infinite).

A *central series* of a group G is a finite sequence of normal subgroups of G

$$\{e\} = H_0 \leqslant H_1 \leqslant \cdots \leqslant H_n = G,$$

such that all factors are central, i.e., for every i

$$H_{i+1}/H_i \leqslant Z(G/H_i).$$

A group G is *nilpotent* if it has a central series. The smallest k so that G has a central series of length k (meaning that there are $k+1$ different members in the series) is the *nilpotency class* of G, with this k, we say that G is *nilpotent of class* k.

Another characterization of a nilpotent group:

A group G is nilpotent if there exists an upper central series terminating with the whole group; in other words, there is *a finite sequence of subgroups of G, $\{e\} = H_0 \leqslant H_1 \leqslant \cdots \leqslant H_{n+1} = G$ such that*

(a) $H_i \lhd G$ *for all* i;
(b) $H_1 = Z(G)$, *and for all* i, $H_{i+1} \leqslant G$ such that $H_{i+1}/H_i = Z(G/H_i)$ (the subgroups H_i are called *iterated centers* of G).

If L is a lattice, and $a \in L$, we use notation $\downarrow a$ for the ideal generated with a, i.e., $\downarrow a = \{x \in L \mid x \leqslant a\}$. The dual notion, the filter generated by a is denoted with $\uparrow a$.

A *weak congruence* of a group G is a congruence on a subgroup of G, considered as a relation on G. Weak congruences of a group G form a lattice under inclusion denoted by $(\mathsf{Wcon}(G), \vee, \wedge)$, where $\Delta_G \in \mathsf{Wcon}(G)$ is a special element representing the diagonal relation on G. *The weak congruence lattice of a group G is an algebraic lattice.*

For $\theta \in \mathsf{Wcon}(G)$, $\theta = \Delta_H$ for a subgroup H of G if and only if $\theta \leqslant \Delta_G$. For $\theta \leq \Delta$ we define $\overline{\theta}$ as the largest element τ of the weak congruence lattice of G such that $\tau \cap \Delta = \theta$. In other words $\overline{\theta} = H^2$ where H is the unique subgroup of G such that $\Delta_H = \theta$. We extend this definition and for all $\rho \in [\Delta_H, H^2]$, we say that $\overline{\rho} = H^2$. Furthermore, we can identify "squares" of subgroups in $\mathsf{Wcon}(G)$: for $\theta \in \mathsf{Wcon}(G)$, $\theta = H^2$ for a subgroup H of G if and only if θ is the largest element of $\mathsf{Wcon}(G)$ such that $\theta \wedge \Delta_G = \Delta_H$. For a subgroup H of G, $\mathsf{Wcon}(H)$ is, as a lattice, the ideal $\downarrow H^2$ in $\mathsf{Wcon}(G)$, endowed with the constant Δ_H. Also, $\mathsf{Con}(H)$ is, as a lattice, the interval $[\Delta_H, H^2]$ in $\mathsf{Wcon}(G)$, and $\mathsf{Sub}(H)$ is isomorphic to the ideal $\downarrow \Delta_H$ of $\mathsf{Wcon}(G)$. Moreover, $H^2 \vee \Delta_G$ is the smallest weak congruence on G containing both H^2 and Δ_G - this is actually the smallest congruence on G containing H^2.

We denote by 0 the bottom element of the lattice $\mathsf{Wcon}(G)$, $0 = \{(e, e)\}$.

In a complete lattice L, an element $a \in L$ is called *cyclic* in L if $\downarrow a$ is a distributive lattice satisfying the ascending chain condition. Denote by $\mathsf{C}(L)$ the *set of all cyclic elements in* L.

3. Characterization of nilpotent groups

Lemma 3.1. *If N is a subgroup of G then $N \lhd G$ if and only if $N^2 \leqslant H^2 \leqslant N^2 \vee \Delta_G$ implies $N^2 = H^2$.*

Proof. If $N \lhd G$ then N is the equivalence class of the neutral element of G for a congruence θ on G, $N = [e]_\theta$, and θ is the smallest congruence containing N^2, hence $\theta = N^2 \vee \Delta_G$ in $\mathsf{Wcon}(G)$. If $N^2 \leqslant H^2 \leqslant N^2 \vee \Delta_G$ for a subgroup H of G, then for $x \in H$, $(x, e) \in \theta$, hence $x \in N$. For the opposite direction, if N is not a normal subgroup of G then its normal closure H in G satisfies $N^2 < H^2 \leqslant N^2 \vee \Delta_G$. $\square$

If $N \lhd G$ we say that Δ_N is *normal* in Δ_G in $\mathsf{Wcon}(G)$; we denote it by $\Delta_N \blacktriangleleft \Delta_G$. By Lemma 3.1 and the preceding remarks, this (unary) relation is defined within the structure $\mathsf{Wcon}(G)$.

If $N \lhd G$ and H a subgroup of G such that $N \subseteq H$, then by the Correspondence Theorem, there is an isomorphism ϕ_H between $\mathsf{Con}(H/N)$ and the filter $\uparrow(N^2 \vee \Delta_H)$ in the sublattice $[\Delta_H, H^2]$ of $\mathsf{Wcon}(G)$ (that is, between $\mathsf{Con}(H/N)$ and the interval $[N^2 \vee \Delta_H, H^2]$ of $\mathsf{Wcon}(G)$). $\mathsf{Wcon}(G/N)$ is a disjoint union of $\mathsf{Wcon}(H/N)$, for $N \leqslant H \leqslant G$. Therefore, the union of the maps Φ_H is an isomorphism Φ between $\mathsf{Wcon}(G/N)$ and the filter $\uparrow N^2$ in $\mathsf{Wcon}(G)$, and it maps $\Delta_{G/N}$ onto $N^2 \vee \Delta_G$.

Next, we recall a result by Ore (see, e.g. [12], p.12).

Proposition 3.2 [12]. *A group G is cyclic if and only if the lattice $\mathsf{Sub}(G)$ is distributive and satisfies the maximal condition.*

The following lemma is a direct consequence of Proposition 3.2:

Lemma 3.3. *A subgroup H of G is cyclic if and only if the ideal $\downarrow\Delta_H$ of $\mathsf{Wcon}(G)$ is a distributive lattice satisfying the maximal condition.*

Further, we characterize commutativity of a subgroup of G within $\mathsf{Wcon}(G)$. For this, we need the following characterization of Dedekind groups.

Theorem 3.4 [2,8]. *The following are equivalent for a group G:*
- (i) *G is a Dedekind group;*
- (ii) *the lattice $\mathsf{Wcon}(G)$ of weak congruences of G is modular;*
- (iii) *the diagonal relation Δ is a neutral element in the lattice $\mathsf{Wcon}(G)$.*

Abelian groups are characterized in [8], here we formulate the characterization of a subgroup which is abelian.

Proposition 3.5. *A subgroup H of a group G is abelian if and only if the ideal $\downarrow H^2$ in $\mathsf{Wcon}(G)$ is a modular lattice and the ideal $\downarrow\Delta_H$ (in $\mathsf{Wcon}(G)$) does not have a subinterval isomorphic to Q in Figure 1.*

Proof. If the ideal $\downarrow H^2$ in $\mathsf{Wcon}(G)$ is a modular lattice, H is a Dedekind group by Theorem 3.4. Moreover, since the ideal $\downarrow\Delta_H$ does not have a subinterval isomorphic to Q in Figure 1, H does not have a quaternion group as a subgroup; hence it is abelian. The opposite direction is obvious. □

In the sequel we use also the following [12, p. 360]:

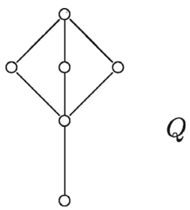

FIGURE 1. Subgroup lattice of the quaternion group.

Proposition 3.6. *A subgroup K of a group G is central if and only if the subgroup $K \vee \langle x\rangle$ is abelian for every $x \in G$.*

We are ready to characterize central subgroups using weak congruence lattices of groups.

Proposition 3.7. *Let K be a subgroup of G. K is a central subgroup of G if and only if the following holds in $\mathsf{Wcon}(G)$: for every $\Delta_X \in \mathsf{C}(\downarrow\Delta)$, the ideal $\downarrow\overline{\Delta_K \vee \Delta_X}$ is a modular lattice and $\downarrow(\Delta_K \vee \Delta_X)$ does not have a subinterval isomorphic to Q in Figure 1.*

Proof. As we said above, $(K \vee X)^2 = \overline{\Delta_K \vee \Delta_X}$ is the largest element ρ in $\mathsf{Wcon}(G)$ such that $\rho \wedge \Delta_G = \Delta_{K\vee X}$; also, $\Delta_K \vee \Delta_X = \Delta_{K\vee X}$ in $\mathsf{Wcon}(G)$. Now we use Lemma 3.3 and Propositions 3.5 and 3.6. □

Let us denote by S the set of diagonals of central subgroups of G,

$$S = \{\Delta_K \in \downarrow\Delta \mid \text{K is a central subgroup of G}\}.$$

$S \subseteq \downarrow\Delta$ in $\mathsf{Wcon}(G)$ and by Proposition 3.7, it can be defined in strictly lattice-theoretic terms.

Corollary 3.8. *A subgroup H of a group G is the center of G ($H = Z(G)$) if and only if $\Delta_H = \bigvee S$ in the lattice $\mathsf{Wcon}(G)$.*

Proof. Straightforwardly, since by Proposition 3.7, K is a central subgroup of G if and only if $\Delta_K \in S$, and the center of G is the greatest central subgroup. □

Corollary 3.8 shows that the center of a group G can be determined in $\mathsf{Wcon}(G)$ by purely lattice-theoretic methods.

The element $\Delta_H = \bigvee S$ from Corollary 3.8 is called *w-center* of the lattice $\mathsf{Wcon}(G)$.

Next, we use the isomorphism Φ mentioned above Proposition 3.2 to characterize cyclic, abelian, and central subgroups of a quotient group of G within the lattice $\mathsf{Wcon}(G)$; $\Phi : \mathsf{Wcon}(G/N) \longrightarrow [N^2, G^2]$. (For a subgroup H of G such that $N \leqslant H$, it maps $\mathsf{Con}(H/N)$ onto $[N^2 \vee \Delta_H, H^2]$, and $\mathsf{Wcon}(H/N)$ onto $[N^2, H^2]$ in $\mathsf{Wcon}(G)$).

Proposition 3.9. *If $N \lhd G$, and $N \leqslant H \leqslant G$, then H/N is a cyclic subgroup of G/N if and only if the interval $[N^2, N^2 \vee \Delta_H]$ in $\mathsf{Wcon}(G)$ is a distributive lattice and satisfies the maximal condition.*

Proof. It is easy to see that $\downarrow\Delta_{H/N}$ in $\mathsf{Wcon}(G/N)$ is isomorphic to the interval $[N^2, N^2 \vee \Delta_H]$ in $\mathsf{Wcon}(G)$, with $\downarrow\Delta_{H/N}$ mapped onto $N^2 \vee \Delta_H$. Now we use Lemma 3.3. □

Proposition 3.10. *If $N \lhd G$, and $N \leqslant H \leqslant G$, then H/N is an abelian subgroup of G/N if and only if the interval $[N^2, H^2]$ in $\mathsf{Wcon}(G)$ is a modular lattice and the interval $[N^2, N^2 \vee \Delta_H]$ in $\mathsf{Wcon}(G)$ does not have a subinterval isomorphic to Q in Figure 1.*

Proof. The isomorphism Φ maps $(H/N)^2$ to H^2 and $\Delta_{H/N}$ to $N^2 \vee \Delta_H$ in $\mathsf{Wcon}(G)$. The proof now follows from Proposition 3.5. □

Proposition 3.11. *If $N \lhd G$, and $N \leqslant K \leqslant G$, then K/N is a central subgroup of G/N if and only if the following holds in $\mathsf{Wcon}(G)$: For every $\delta \in \mathsf{C}([N^2, N^2 \vee \Delta_G])$, the interval $[N^2, \overline{N^2 \vee \Delta_K \vee \delta}]$ is a modular lattice, and the interval $[N^2, N^2 \vee \Delta_K \vee \delta]$ does not have a subinterval isomorphic to Q in Figure 1.*

Proof. The proof follows from Propositon 3.10 and Proposition 3.7. □

We can now characterize the center of a quotient group G/N by the lattice $\mathsf{Wcon}(G)$:

The center of G/N is the subgroup X of G/N such that

$\Delta_X = \bigvee\{\Delta_{K/N} \mid K/N$ is a central subgroup of $G/N\}$,

or, equivalently,

$\Delta_X = \Phi^{-1}(\bigvee\{\Phi(\Delta_{K/N}) \mid K/N$ is a central subgroup of $G/N\}) =$
$\Phi^{-1}(\bigvee\{N^2 \vee \Delta_K \mid K/N$ is a central subgroup of $G/N\})$.

From everything said above, it follows that one can characterize nilpotent groups only by the properties of their weak congruence lattices. The following theorems are therefore consequences of the above-listed propositions.

Theorem 3.12. *A group G is nilpotent if and only if there is a finite series of elements in $\mathsf{Wcon}(G)$:*

$$0 = \Delta_{H_0}, \Delta_{H_1}, \ldots, \Delta_{H_k}, \Delta_G, \textit{such that}\, \Delta_{H_0} < \Delta_{H_1} < \cdots < \Delta_{H_k} < \Delta_G.$$

so that for every $i \in \{0, 1, \ldots, k\}$ the following holds:

(a) $\Delta_{H_i} \blacktriangleleft \Delta_G$;
(b) *For every $\delta \in \mathsf{C}([H_i^2, H_i^2 \vee \Delta_G])$, the interval $[H_i^2, \overline{H_i^2 \vee \Delta_{H_{i+1}} \vee \delta}]$ is a modular lattice, and the interval $[H_i^2, H_i^2 \vee \Delta_{H_{i+1}} \vee \delta]$ does not have a subinterval isomorphic to Q in Figure 1.*

Since we have characterized the center of a group (and the center of a quotient group) by means of weak congruence lattices, using an upper central series we obtain one more lattice theoretical characterization of nilpotent groups.

Theorem 3.13. *A group G is nilpotent if and only if there is a finite series of elements in $\mathsf{Wcon}(G)$:*

$$0 = \Delta_{H_0}, \Delta_{H_1}, \ldots, \Delta_{H_k}, \Delta_G, \textit{such that}\, \Delta_{H_0} < \Delta_{H_1} < \cdots < \Delta_{H_k} < \Delta_G.$$

so that for every $i \in \{0, 1, \ldots, k\}$ the following holds:

(a) $\Delta_{H_i} \blacktriangleleft \Delta_G$;
(b) *Δ_{H_1} is the w-center of $\mathsf{Wcon}(G)$, and $H_i^2 \vee \Delta_{H_{i+1}}$ is the w-center of $[H_i^2, G^2]$ (considered as the weak congruence lattice of G/H_i).*

4. Conclusion

In the series of papers including this one, we have characterized many known classes of groups in lattice terms, using their weak congruence lattices. However, the weak congruence lattices of arbitrary algebras may not have relevant properties as those for groups. Namely, lattices of congruences and subalgebras are generally independent. Algebras for which congruences and subalgebras are related similarly as for groups are those having a *good ideal theory* in the sense of [15], see also [6]. There are also *group-like structures*, which have already been investigated in [2] by properties of their weak congruence lattices. Therefore, as our future task, we intend to further characterize those classes of algebras in the same lattice framework.

Acknowledgements

The authors express their gratitude to the anonymous referees whose comments improved the general quality of this paper.

Data availability Data sharing is not applicable to this article as datasets were neither generated nor analyzed.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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Received: 8 March 2023.
Accepted: 28 August 2024.