

Magic Billiards: the Case of Elliptical Boundaries

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Dedicated to Academician A. T. Fomenko on the occasion of his 80th anniversary.

Abstract

In this work, we introduce a novel concept of magic billiards, which can be seen as an umbrella, unifying several well-known generalisations of mathematical billiards. We analyse properties of magic billiards in the case of elliptical boundaries. We provide explicit conditions for periodicity in algebro-geometric, analytic, and polynomial forms. A topological description of those billiards is given using Fomenko graphs.

Keywords: integrable systems, elliptical billiards, topological billiards, Fomenko graphs, periodic trajectories, Cayley's conditions, divisors on elliptic curves, polynomial Pell's equations

MSC2020: 37C83, 37J35, 37J39

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1 Introduction

The elliptic billiard [Bir1927, Bol1990, KT1991, Tab2005] is a notable example of a completely Liouville integrable system. Thus, the Liouville-Arnold theorem implies that the phase space of that particular billiard is foliated into invariant Liouville tori [Arn1989].

Fomenko and his school developed a beautiful theory for topological description and classification of integrable systems using what are now known as *Fomenko graphs* and *Fomenko-Zieschang invariants*, see [Fom1987, FZ1991] and, in particular, the book [BF2004], with the fundamentals of that theory, including a large list of well-known integrable systems, such as the integrable cases of rigid body motion and geodesic flows on surfaces.

The use of topological tools in the study of integrable billiards was initiated by the authors in [DR2009], see also [DR2010, DR2011]. Further details and applications to other integrable systems can be found in the literature related to billiards [Fok2014, Rad2015, DR2017, VK2018, FV2019b, FV2019a, PRK2021, DGR2022b, DGR2022a, BF2024]. For the applications in the broader theory of Hamiltonian systems with two degrees of freedom see [BMF1990, RRK2008, BBM2010].

An important milestone of this theory is the so-called *Fomenko conjecture*, emphasizing a surprising universality of billiard dynamics. This conjecture is about realization of topology of Liouville foliations of smooth and real-analytic integrable Hamiltonian systems by integrable billiards, see e.g. [FKK2020] and [FV2023] and references therein.

In this paper, we introduce a general concept of *magic billiards*, where after hitting the boundary the particle is magically transported to another point of the boundary and continues motion from there. A formal definition of such class of systems is given in Section 2. Magic billiards can be seen as an umbrella, unifying several well-known generalisations of mathematical billiards (Example 2.2), see e.g. Examples 2.3 and 2.4.

In Section 3, we focus to magic billiards within an ellipse, and among them only to those ones where the equations of motion in elliptic coordinates remain the same as for the standard billiard, which will mean that the obtained system will be integrable. We provide conditions for periodicity of such systems and give topological description using Fomenko graphs. We note that one of the cases we consider, so called *billiards with slipping* was recently introduced and studied by Fomenko and his school [FVZ2021, FV2021, VZ2022, Fom2023, Zav2023]. In Section 4, we consider magic billiard within elliptic annulus. The last Section 5 contains discussion.

2 Definition of magic billiards

In this section, we will introduce a new class of dynamics, where a particle moves along straight segments by constant speed within a given domain in the plane, and when it reaches its boundary, it is transported to another point of the domain boundary from where it continues the motion within the domain.

More formally, we will introduce that dynamics as follows.

Let D be a given domain in the plane, which is bounded by a smooth closed curve. Suppose that φ is a continuous bijective mapping of the boundary ∂D onto itself. Now, consider the circle bundle $S^1(\partial D)$ with the base ∂D , such that the fiber over any point $p \in \partial D$ consists of the unit vectors in the tangent space to the plane at p . Let φ^* be a continuous bijective mapping of $S^1(\partial D)$ onto itself, satisfying the following:

- $\pi \circ \varphi^* = \varphi \circ \pi$, where $\pi : S^1(\partial D) \rightarrow \partial D$ is the projection to the base points;
- φ^* maps tangent vectors to the boundary ∂D to tangent vectors to that boundary curve;

- vectors pointing outwards D are mapped to vectors pointing inwards D .

Definition 2.1. A *magic billiard* (D, φ, φ^*) is a dynamical system where a particle moves with a unit speed along straight lines in the interior of D , and when it hits the boundary at a point A with velocity \vec{v} , it will bounce off at point $\varphi(A)$ with velocity $\varphi^*(\vec{v})$.

Example 2.2. Notice that the standard billiard in D belongs to the class of just defined magic billiards. There, mapping φ is the identity and φ^* is the billiard reflection, i.e. reflection with respect to the direction of the tangent line to the boundary at each point.

Example 2.3. Similarly, projective, Finsler and Minkowski billiards [Tab1997, GT2002, Rad2003, KT2009, DR2012, DR2013, DR2017, ADR2019], see also [GM2024], are classes of magic billiards with φ being the identity map.

Example 2.4. Another class of magic billiards are billiards with slipping [FVZ2021] (see also [DGK2024]). Namely, in that case, φ is an isometry of the boundary ∂D , while φ^* is defined as follows.

First, choose a direction of the boundary ∂D and note that the isometry φ either preserves or reverses it. If the direction is preserved, then we set that φ^* maps the vectors tangent to ∂D and pointing in the direction of ∂D to the tangent vectors also pointing in the direction of the boundary. If the direction is reversed, then such vectors are mapped to the vectors pointing in the opposite direction.

Now, suppose that \vec{v} belongs to the fiber over point $p \in \partial D$ of the circle bundle $S^1(\partial D)$. Denote by \vec{t} one of the two unit tangent vectors to ∂D at p . Then $\varphi^*(\vec{v})$ is the unique vector satisfying the following equality of oriented angles $\angle(\vec{t}, \vec{v}) = 2\pi - \angle(\varphi^*(\vec{t}), \varphi^*(\vec{v}))$.

3 Magic billiards within an ellipse

Before starting the analysis of novel examples of magic elliptical billiards, we will review the standard billiard within an ellipse.

Suppose that the ellipse is given by:

$$\mathcal{E} : \frac{x^2}{a} + \frac{y^2}{b} = 1, \quad a > b > 0. \quad (3.1)$$

Following classical ideas of Jacobi, one can introduce *the elliptic coordinates* (λ_1, λ_2) , which are, for each given point in the plane, the parameters of an ellipse and a hyperbola from the confocal family:

$$\mathcal{C}_\lambda : \frac{x^2}{a - \lambda} + \frac{y^2}{b - \lambda} = 1, \quad (3.2)$$

which intersect at that point. Note that each billiard trajectory within \mathcal{E} has a unique *caustic* \mathcal{C}_α , which is touching each segment of the trajectory. The differential equation of the billiard motion is separated in elliptic coordinates:

$$\frac{d\lambda_1}{\sqrt{(a - \lambda_1)(b - \lambda_1)(\alpha - \lambda_1)}} + \frac{d\lambda_2}{\sqrt{(a - \lambda_2)(b - \lambda_2)(\alpha - \lambda_2)}} = 0. \quad (3.3)$$

We are interested in magic billiards which keep those nice geometric and analytic properties of standard elliptic billiards, so we will focus to the mappings φ and φ^* which preserve the equation (3.3).

In particular, we note that the elliptic coordinates remain unchanged for φ being one of the following:

- the reflection with respect to one of the axes of the billiard boundary \mathcal{E} ; or
- the half-turn around the center of \mathcal{E} .

Notice that such φ is defined as an isometry on the whole plane, not only on the boundary of the billiard table. Thus, the differential map $d\varphi$ will preserve the circle bundle $S^1(\partial D)$: in fact it maps each fiber of $S^1(\partial D)$ isometrically to another fiber. Moreover, because the elliptic coordinates are invariant with respect to reflections with respect to the axes, the equation (3.3) will also be invariant if φ^* is the composition of the billiard reflection at the point of impact and the differential map $d\varphi$. Thus, for such φ and φ^* , the dynamics of the magic billiard $(\mathcal{E}, \varphi, \varphi^*)$ in elliptic coordinates will be identical to the usual billiard motion. In particular, each trajectory of such magic billiards will have a unique caustic from the family of conics which are confocal with the boundary \mathcal{E} .

Remark 3.1. Note that for such φ , the configuration space can be defined as \mathcal{D}/\sim , where \mathcal{D} is the billiard table, and $x \sim \varphi(x)$, for $x \in \mathcal{E} = \partial\mathcal{D}$. The phase space is defined, analogously, by identifying the velocity vectors mapped into each other by φ^* .

Remark 3.2. Since φ and φ^* are involutions, one can see that every second segment of a trajectory of the corresponding magic billiard coincides with every second segment of the standard billiard, as illustrated in Figure 1.

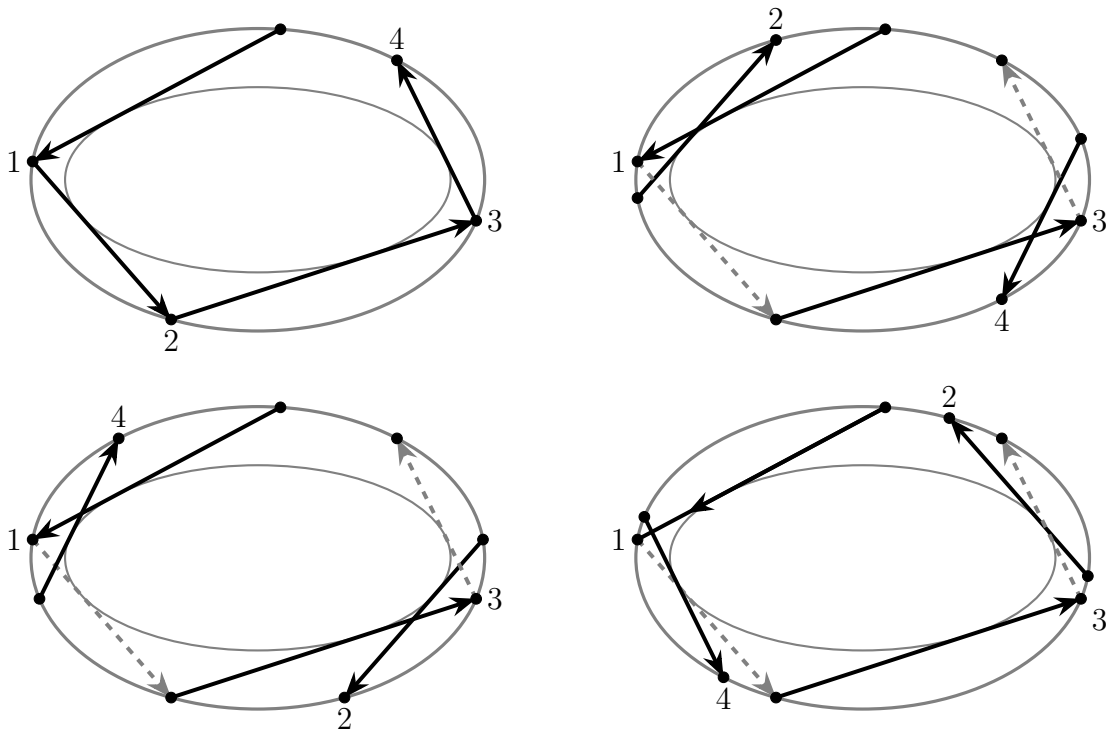


Figure 1: A standard billiard trajectory (upper left) and three trajectories of the magic billiard with the same initial segment. Upper right: flipping over the long axis; lower left: flipping over the short axis; lower right: half-turn around the center. In each case, the consecutive points of impact with the boundary are numerated.

Remark 3.3. As a consequence of Remark 3.2, we note that a magic billiard trajectory in those cases is periodic if and only if the corresponding standard billiard trajectory is periodic.

Moreover, the n -periodicity conditions for even n will be the same as for the standard elliptical billiard. For odd periods, the conditions will be different, and will be distinct for each case that we consider.

Remark 3.4. Note that the construction of the magic billiard trajectories reminds of the construction of the standard billiard trajectories within a half-ellipse or a quarter-ellipse. Namely, in all of those cases any trajectory in part coincides with the trajectory of elliptic billiard with the same initial conditions, while in part is symmetric to that trajectory, as illustrated in Figures 1 and 2. Note that the trajectories of each magic billiard that we consider and the standard billiards within ellipse, half-ellipse, and quarter-ellipse satisfy the equation (3.3), thus they all look the same in elliptic coordinates.

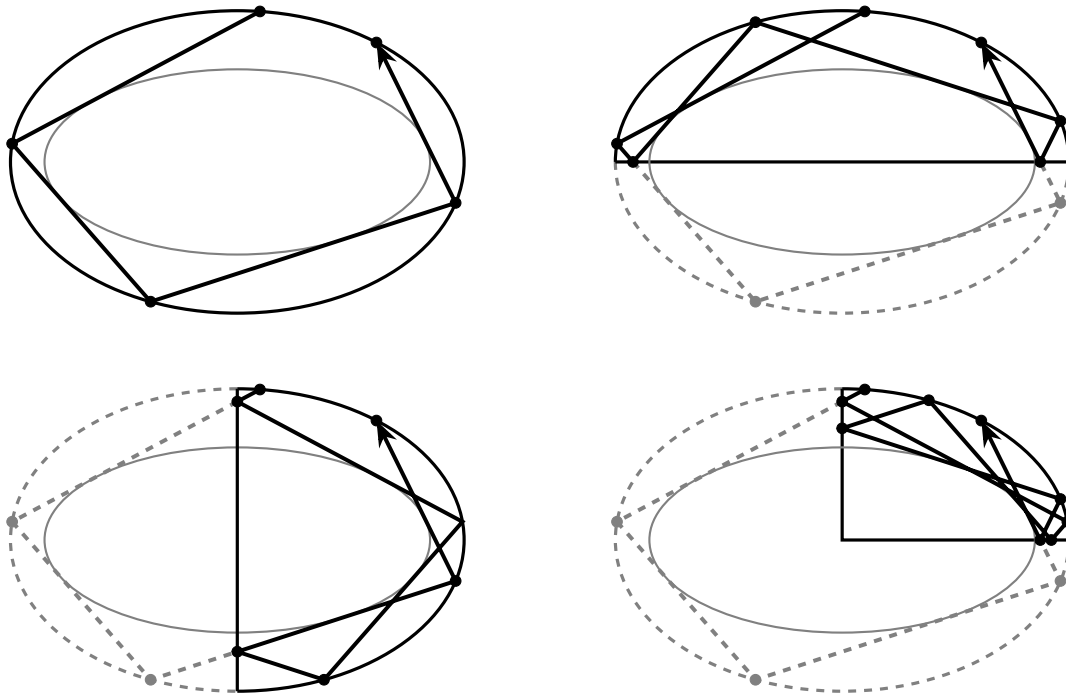


Figure 2: Billiards within ellipse (upper left), half-ellipses (upper right and lower left), and quarter-ellipse (lower right). All trajectories have the same initial conditions. Unfolding the trajectories within the last three domains about the axes give the trajectory within the ellipse.

Remark 3.5. The billiards with flipping over one of the axes can be related to topological billiards $\Delta_\beta(2A'_2)$ and $\Delta_\beta(2A_1)$ from [Fok2015], by reflecting half of the table relative to the corresponding axis and then gluing the two halves of the disk along the isometry of the elliptic boundary arcs.

In the following subsections, we will consider each of the three magic billiards separately. We will derive periodicity conditions in three forms: algebro-geometric, analytic, and polynomial. We will also present Fomenko graphs corresponding to the Liouville foliation of the phase space.

3.1 Elliptical billiard with flipping over long axis

In this system, φ is the reflection with respect to the longer axis. Examples of its trajectories are depicted in Figure 3.

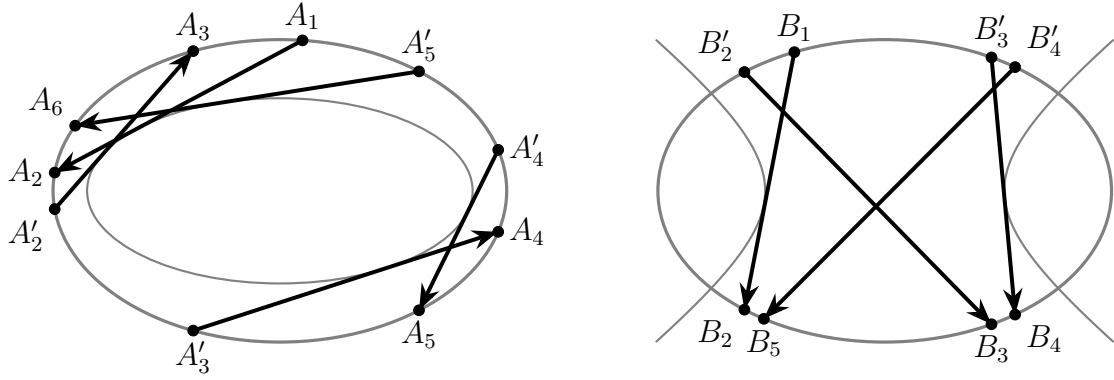


Figure 3: Magic billiard with flipping over the long axis: each time when the particle hits the boundary, it is reflected off the boundary and immediately magically flipped over the longer axis.

Periodicity conditions for such trajectories can be obtained by applying the results from [DR2019a]. In the following theorem, we represent such conditions in the algebro-geometric form, from which the classical form of Cayley's and in the polynomial form can be derived.

Theorem 3.6. *In the billiard table bounded by the ellipse \mathcal{E} given by (3.1), consider the magic billiard with flipping over the long axis. Consider a trajectory of such billiard with the caustic \mathcal{C}_β from the confocal family (3.2). Such a trajectory is n -periodic if and only if one of the following conditions is satisfied:*

- n is even and $nQ_0 \sim nQ_\infty$;
- n is odd, \mathcal{C}_β is hyperbola, and $nQ_0 \sim nQ_b$.

Here Q_0, Q_b, Q_∞ denote the points with coordinates $(0, \sqrt{ab\beta}), (b, 0), (\infty, \infty)$ on the elliptic curve:

$$y^2 = (a-x)(b-x)(\beta-x). \quad (3.4)$$

Proof. Integrating (3.3) along a trajectory of the magic billiard, we get that the n -periodicity condition will be equivalent to the following divisor condition:

$$n(Q_0 - Q_{c_1}) + m(Q_{c_2} - Q_a) \sim 0 \quad (3.5)$$

on the elliptic curve (3.4). Here, we have $\{c_1, c_2\} = \{\beta, b\}$ and $c_1 < c_2$, while Q_β, Q_a denote the points of the curve with coordinates $(\beta, 0), (a, 0)$ respectively. Note that, on the trajectory, the elliptic coordinates λ_1 and λ_2 belong to the intervals $[0, c_1]$ and $[c_2, a]$ respectively. Natural numbers n and m represent the numbers of times each of them traces back and forth its respective interval along the closed trajectory.

Note that the short axis of the ellipse is the degenerate conic \mathcal{C}_a of the confocal family. Since that axis must be crossed even number of times along a closed trajectory, we have that m is even, thus the periodicity condition (3.5) reduces to $n(Q_0 - Q_{c_1}) \sim 0$.

On the other hand, the sum of the number of the flippings over the long axis and the number of times when the billiard particle actually crossed the long axis must also be even. That sum equals $n + m$ if \mathcal{C}_β is an ellipse, and $2n$ if \mathcal{C}_β is hyperbola. From there, we conclude that all closed trajectories with ellipse as caustic must have even period.

Now, the required conditions follow from the fact that $c_1 = b$ when \mathcal{C}_β is hyperbola and $2Q_\beta \sim 2Q_\infty$. \square

Now, analogously as in [DR2019a], we can get the conditions of Cayley's type.

Corollary 3.7. *A trajectory of the billiard within \mathcal{E} with flipping over the long axis is n -periodic if and only if one of the following conditions is satisfied:*

- n is even and

$$\begin{vmatrix} B_3 & B_4 & \dots & B_{n/2+1} \\ B_4 & B_5 & \dots & B_{n/2+2} \\ \dots & & & \\ B_{n/2+1} & B_{n/2+2} & \dots & B_{n-1} \end{vmatrix} = 0; \quad (3.6)$$

- n is odd, the caustic is hyperbola, and

$$\begin{vmatrix} C_2 & C_3 & \dots & C_{(n+1)/2} \\ C_3 & C_4 & \dots & C_{(n+1)/2+1} \\ \dots & & & \\ C_{(n+1)/2} & C_{(n+1)/2+1} & \dots & C_{n-1} \end{vmatrix} = 0. \quad (3.7)$$

Here, we denoted:

$$\begin{aligned} \sqrt{(a-x)(b-x)(\beta-x)} &= B_0 + B_1x + B_2x^2 + \dots, \\ \frac{\sqrt{(a-x)(b-x)(\beta-x)}}{b-x} &= C_0 + C_1x + C_2x^2 + \dots, \end{aligned}$$

the Taylor expansions around $x = 0$, and β is the parameter of the caustic from (3.2).

The periodicity conditions can be equivalently stated in the form of polynomial equations. Again, for details on how to obtain those equations, refer to [DR2019a]. A more general theory connecting polynomial Pell's equations and integrable billiards in arbitrary dimension is given in [DR2019b].

Corollary 3.8. *The trajectories of the magic billiard with flipping over the long axis with caustic C_β are n -periodic if and only if there exists a pair of real polynomials p_{d_1} , q_{d_2} of degrees d_1 , d_2 respectively, and satisfying the following:*

- (a) if $n = 2m$ is even, then $d_1 = m$, $d_2 = m - 2$, and

$$p_m^2(s) - s \left(s - \frac{1}{a}\right) \left(s - \frac{1}{b}\right) \left(s - \frac{1}{\beta}\right) q_{m-2}^2(s) = 1; \quad (3.8)$$

- (b) if $n = 2m + 1$ is odd, then $d_1 = m$, $d_2 = m - 1$, and

$$\left(s - \frac{1}{b}\right) p_m^2(s) - s \left(s - \frac{1}{a}\right) \left(s - \frac{1}{\beta}\right) q_{m-1}^2(s) = -1.$$

We conclude this section by a topological description of the system.

Theorem 3.9. *The Fomenko-Zieschang invariant of the magic billiard with flipping over the long axis is shown in Figure 4.*

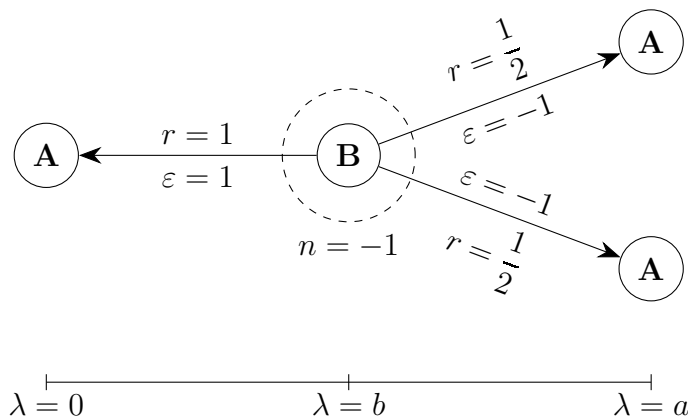


Figure 4: Theorem 3.9: Fomenko graph for the magic billiard with flipping over the long axis.

Proof. The billiard table consists of the boundary ellipse \mathcal{E} and its interior. According to Remark 3.1, the configuration space is obtained when the points of \mathcal{E} which are symmetric to each other with respect to the long axis are identified. Thus, the configuration space is homeomorphic to a sphere, where the boundary of the billiard table is represented by an arc.

Any level set corresponding to the trajectories having a fixed ellipse as caustic consists of a single Liouville torus, see Figure 5.

The level set corresponding to the trajectories having a fixed hyperbola as caustic consists of two Liouville tori: one torus contains the trajectories where motion along each segment is downwards (as shown in the lefthandside of Figure 3), and on the other torus the motion is upwards.

The level set corresponding to the caustic $\mathcal{E} = \mathcal{C}_0$ consists of the limit motion back and forth along the boundary. This is a single closed trajectory, thus corresponding to the Fomenko atom **A**.

The level set with the caustic \mathcal{C}_a corresponds to the motion along the short axis of the ellipse \mathcal{E} . When the particle hits the boundary, it is magically flipped to the opposite point on the axis. Thus, there are two closed trajectories there: the downwards one and the upwards one. Each of those trajectories corresponds to one Fomenko atom of type **A**.

The level set \mathcal{C}_b consists of the trajectories that contain the foci of the ellipse \mathcal{E} . Exactly one of those trajectories is closed, corresponding to horizontal motion along the long axis, i.e. the level set is of complexity 1. All other trajectories alternately pass through the one and the other focus, and on each such trajectory, because of the flipping along the long axis, the motion is either upwards along each segment or downwards along each segment. Thus, there are two separatrices on that level set.

According to the classification of Fomenko atoms from [BF2004], there are three types of atoms of complexity 1: atom **A** consisting of a single closed orbit without separatrices, atom **A*** which has one separatrix, and atom **B** with two separatrices. From there, the level set \mathcal{C}_b must be **B**, which is consistent with the number of connected components of regular level sets close to \mathcal{C}_b – that number is 2 for $\lambda > b$ and 1 for $\lambda < b$. Thus, the rough Liouville equivalence class of this billiard is represented by the graph in Figure 4.

In order to get the numerical invariants, notice that the Liouville torus corresponding to the motion where an ellipse is a caustic is obtained by gluing four copies of the annulus between the billiard boundary and the the caustic, as shown in Figure 5, while the two Liouville tori on the level set corresponding to a hyperbola as caustic are obtained by gluing two pairs of regions within the ellipse which is between the branches of hyperbola. Choosing the coordinate

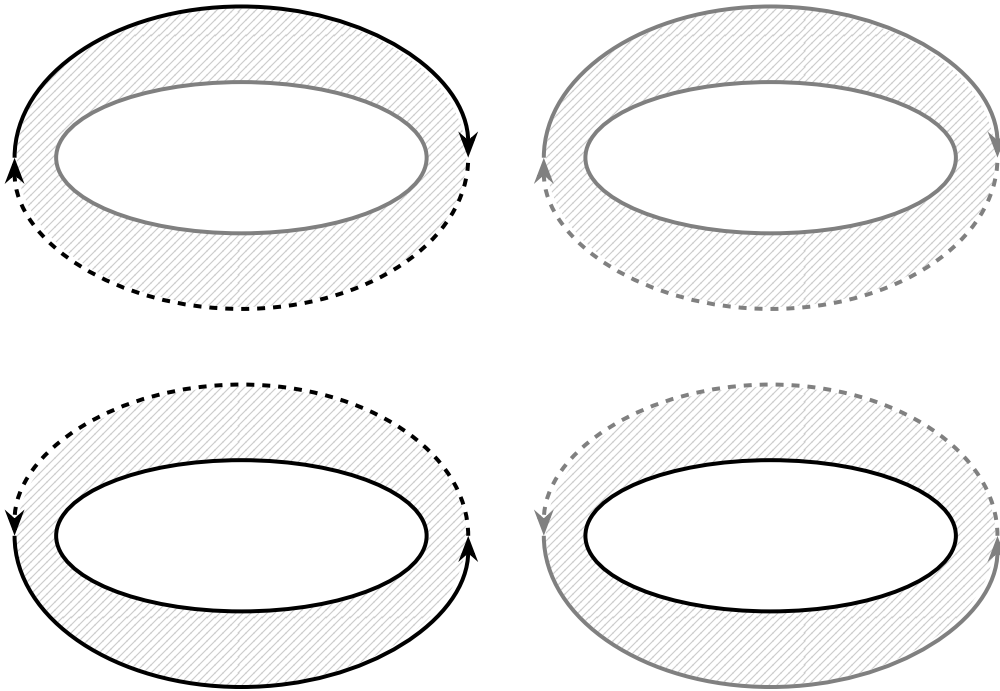


Figure 5: The Liouville torus corresponding to an ellipse as caustic of magic billiard with flipping over the long axis is obtained by gluing four annuli along congruent arcs of the same color and texture.

systems on those tori as explained in [BF2004], we calculate the numerical marks which are shown in Figure 4. \square

3.2 Elliptic billiard with flipping over short axis

In this case, with φ is the reflection with respect to the short axis. Examples of its trajectories are depicted in Figure 6.

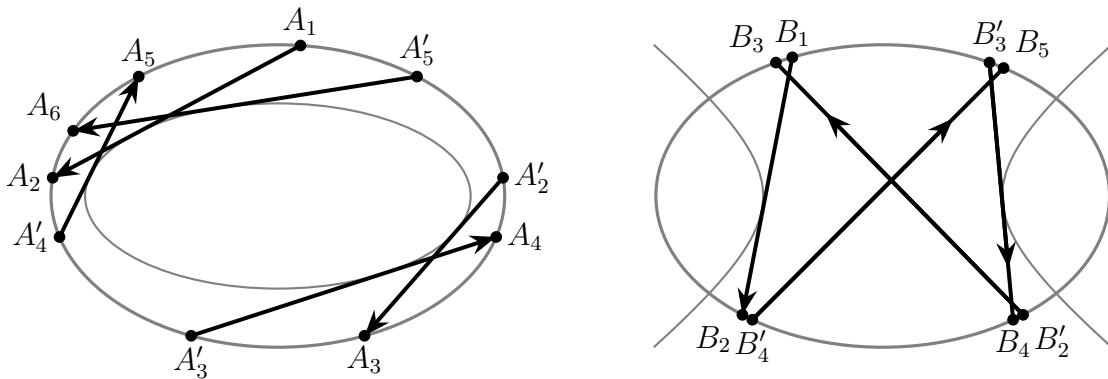


Figure 6: Magic billiard with flipping over the short axis of the ellipse: each time when the particle hits the boundary, it is reflected with respect to the boundary and immediately magically flipped over the short axis.

Now, we derive the algebro-geometric condition for periodicity of the such magic billiard.

Theorem 3.10. *In the billiard table bounded by the ellipse \mathcal{E} given by (3.1), consider the magic billiard with flipping over the short axis. Consider a trajectory of such billiard with the caustic*

\mathcal{C}_β from the confocal family (3.2). Such a trajectory is n -periodic if and only if n is even and $nQ_0 \sim nQ_\infty$.

Here Q_0 and Q_∞ are as in Theorem 3.6.

Proof. We will use notation as in Theorem 3.6 and its proof. In the same way as explained there, we can derive the divisor condition (3.5).

In addition to that condition, we need the following:

- the number of times that the particle crossed the long axis must be even: that number equals m if the caustic is ellipse, or n if it is hyperbola;
- the sum $n + m$ must be even, since that is the sum of the number of the flippings over the short axis and the number of times when the billiard particle actually crossed that axis.

In any case, we get that both m and n must be even, so the conditions for closure reduce to $n(Q_0 - Q_{c_1}) \sim 0$, which is equivalent to the stated relation. \square

Remark 3.11. It is interesting to note that in this case there are no odd-periodic trajectories. Notice that, as announced in Remark 3.3, the conditions for trajectories of even period are the same as in the case of flipping over the long axis, see Theorem 3.6, and as in the case of standard billiard, see [DR2019a, Theorem 2].

As a consequence, the analytic and polynomial conditions can be obtained as follows.

Corollary 3.12. *In the billiard table bounded by the ellipse \mathcal{E} given by (3.1), consider the magic billiard with flipping over the short axis. Consider a trajectory of such billiard with the caustic \mathcal{C}_β from the confocal family (3.2). Such a trajectory is n -periodic if and only if n is even and the following equivalent conditions are true:*

- relation (3.6) is satisfied;
- there are polynomials satisfying the polynomial equation (3.8).

We use Fomenko graph to give a topological description of the system.

Theorem 3.13. *The Liouville equivalence class of the magic billiard with flipping over the long axis is given by the Fomenko graph in Figure 7.*

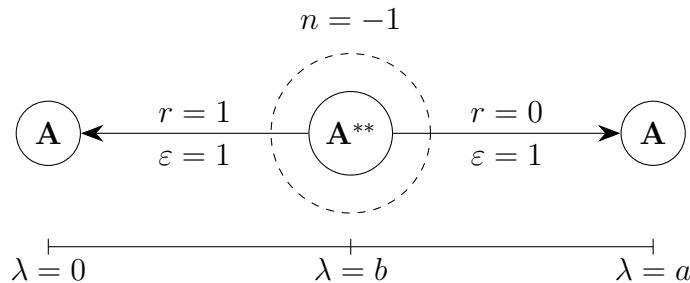


Figure 7: Theorem 3.13: Fomenko graph for the magic billiard with flipping over the short axis.

Proof. Similarly as in the proof of Theorem 3.9, the configuration space is homeomorphic to a sphere, where the boundary of the billiard table is represented by an arc.

In the phase space, any level set corresponding to the trajectories having a fixed non-degenerate conic as caustic consists of a single Liouville torus.

The level set corresponding to the caustic $\mathcal{E} = \mathcal{C}_0$ corresponds to the Fomenko atom \mathbf{A} , similarly as in the proof of Theorem 3.9.

The level set with the caustic \mathcal{C}_a corresponds to the motion back and forth along the short axis of the ellipse \mathcal{E} , thus it corresponds to the Fomenko atom of type \mathbf{A} .

The level set with the caustic \mathcal{C}_b consists of the trajectories that contain the foci of the ellipse \mathcal{E} . First, we will show that the level set is of complexity 2, i.e. that it contains exactly two closed orbits. One such orbit corresponds to the motion from left to the right along the long axis: when the particle hits the boundary, it is magically flipped to the other endpoint of the diameter, from where it continues its motion to the right. The second closed trajectory corresponds to the motion from right to the left along the long axis. For any other trajectory on that level set, each segment will pass through the same focus, so there are two separatrices on that level set.

The classification of Fomenko atoms of complexity 2 from [BF2004] determines that only one of them, \mathbf{A}^{**} , has two separatrices. We note that this is consistent with the numbers of connected components of non-degenerate level sets which are close to \mathcal{C}_b : only 1 for $\lambda < b$ and also 1 for $\lambda > b$. \square

3.3 Elliptic billiard with half-turn around the center

In this case, with φ is the half-turn about the center of the ellipse. Such billiards are a case of the billiards with slipping, see [FVZ2021].

Examples of its trajectories are depicted in Figure 8.

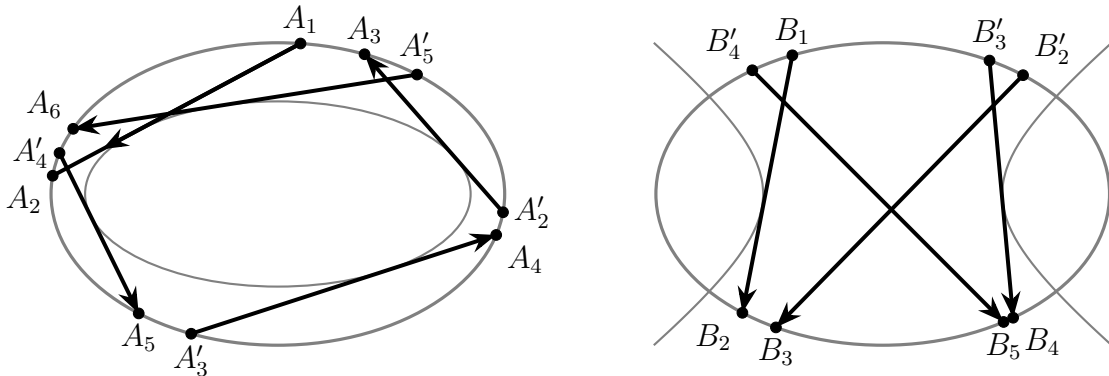


Figure 8: Magic billiard with slipping by half-ellipse: each time when the particle hits the boundary, it is reflected with respect to the boundary and immediately magically transported to the diametrically opposite point.

As in the previous cases, we will first derive the divisor conditions for periodicity.

Theorem 3.14. *In the billiard table bounded by the ellipse \mathcal{E} given by (3.1), consider the magic billiard with half-turn about the center. Consider a trajectory of such billiard with the caustic \mathcal{C}_β from the confocal family (3.2). Such a trajectory is n -periodic if and only if $nQ_0 \sim nQ_\infty$. Here Q_0 and Q_∞ are as in Theorem 3.6.*

Proof. We follow the notation of Theorem 3.6 and its proof, and, as there, we can derive the divisor condition (3.5).

In addition to that condition, we need that the number of times the particle hit the boundary, the number of times it crossed the long axis, and the number of times it crossed the short axis, are all odd or all even. This is equivalent to m and n being both odd or both even.

Thus, the periodicity condition for even n is equivalent to $n(Q_0 - Q_{c_1}) \sim 0$, and for odd n to $n(Q_0 - Q_{c_1}) + (Q_{c_2} - Q_a) \sim 0$. Using $2Q_\infty \sim 2Q_a \sim 2Q_b \sim 2Q_\beta$ and $3Q_\infty \sim Q_a + Q_b + Q_\beta$, we get that the periodicity condition for any n is $nQ_0 \sim nQ_\infty$. \square

From divisor conditions, one can derive the analytic conditions of Cayley's type and the polynomial conditions, similarly as in [DR2019a]. We present them in the following two corollaries.

Corollary 3.15. *In the billiard table bounded by the ellipse \mathcal{E} given by (3.1), consider the magic billiard with the half-turn about the center. Consider a trajectory of such billiard with the caustic \mathcal{C}_β from the confocal family (3.2). Such a trajectory is n -periodic if and only if:*

- n is even and (3.6) is satisfied; or
- n is odd and

$$\begin{vmatrix} B_2 & B_3 & \cdots & B_{(n+1)/2} \\ B_3 & B_4 & \cdots & B_{(n+1)/2+1} \\ \cdots & & & \\ B_{(n+1)/2} & B_{(n+1)/2+1} & \cdots & B_{n-1} \end{vmatrix} = 0.$$

The coefficients B_2, B_3, \dots are as in the statement of Corollary 3.7.

Corollary 3.16. *The trajectories of the magic billiard with flipping over the long axis with caustic \mathcal{C}_β are n -periodic if and only if there exists a pair of real polynomials p_{d_1}, q_{d_2} of degrees d_1, d_2 respectively, and satisfying the following:*

(a) if $n = 2m$ is even, then $d_1 = m, d_2 = m - 2$, and (3.8);

(b) if $n = 2m + 1$ is odd, then $d_1 = m, d_2 = m - 1$, and

$$sp_m^2(s) - \left(s - \frac{1}{a}\right) \left(s - \frac{1}{b}\right) \left(s - \frac{1}{\beta}\right) q_{m-1}^2(s) = 1.$$

Finally, we give the topological description of this system.

Theorem 3.17 ([FVZ2021]). *The Liouville equivalence class of the magic billiard with flipping through the center is given by the Fomenko graph in Figure 9.*

Proof. According to Remark 3.1, we define the configuration space as the billiard table where the points of \mathcal{E} which are diametrically symmetric to each other are identified. Such configuration space is homeomorphic to the projective plane.

In the phase space, any level set corresponding to a any given ellipse as caustic consists of two Liouville tori: one torus contains the trajectories that are winding in the clockwise direction about the caustic, the other torus contains the counterclockwise trajectories.

The level set corresponding to hyperbola as caustic also consists of two Liouville tori: one torus where all segments of the trajectories point downwards, and the other where they point upwards.

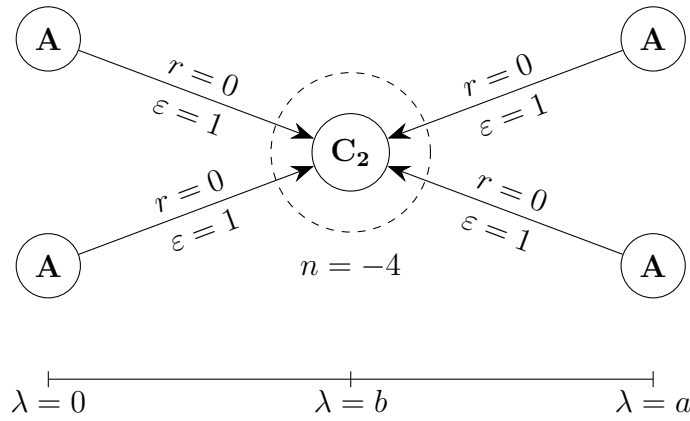


Figure 9: Theorem 3.17: Fomenko graph for the magic billiard with slipping.

The level set corresponding to the caustic $\mathcal{E} = \mathcal{C}_0$ corresponds to a pair of Fomenko atoms **A**: each of them contains a single closed trajectory along the boundary ellipse, winding in one of the two directions.

The level set with the caustic \mathcal{C}_a corresponds to the motion the short axis of the ellipse \mathcal{E} . It consists of two closed trajectories – the upwards and the downwards one. Thus we have two Fomenko atoms of type **A** there.

The level set with the caustic \mathcal{C}_b consists of the trajectories that contain the foci of the ellipse \mathcal{E} . Two of those trajectories are closed: one corresponds to motion from left to the right along the long axis, and the other one to the motion from right to the left, so \mathcal{C}_b is of complexity 2.

For any other trajectory on that level set, each segment will pass through the same focus, and each segment points in the same direction: upwards or downwards. Moreover, such a trajectory is heteroclinic, i.e. it approaches one of the closed orbits as $t \rightarrow +\infty$ and the other one as $t \rightarrow -\infty$. Namely, the trajectories containing the left focus will approach the closed orbit where the motion is from right to the left along the axis in the forward direction and the other closed orbit in the backward direction of time, and vice versa for the trajectories through the right focus.

Thus, there are four separatrices on that level set, all consisting of heteroclinic trajectories.

Now, following the classification of Fomenko atoms from [BF2004], we find that the following ones are of complexity 2 with four separatrices: \mathbf{C}_1 , \mathbf{C}_2 , \mathbf{D}_1 , \mathbf{D}_2 . The last two atoms have two homoclinic and two heteroclinic separatrices, thus they do not correspond to this case. The atom \mathbf{C}_1 also can be eliminated, since level sets close to that one consist of only one Liouville torus. Since non-degenerate level sets close to \mathcal{C}_b have two connected components for both cases $\lambda < b$ and $\lambda > b$, we conclude that it is represented by the Fomenko atom \mathbf{C}_2 . \square

Remark 3.18. For a more detailed proof of Theorem 3.17, together with the calculation of the numerical invariants, see [FVZ2021].

4 Magic billiards in elliptic annulus

In this section, we discuss magic billiards in the annulus between two confocal ellipses \mathcal{E}_1 and \mathcal{E}_2 , see Figure 10. In order to preserve integrability and caustics, we will assume that the mapping φ is defined as follows:

- the restriction of φ to \mathcal{E}_1 is the reflection with respect to one of the axes or the central symmetry;

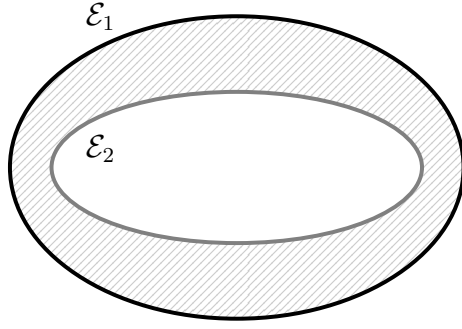


Figure 10: The boundary of the billiard table consists of two confocal ellipses.

- the restriction to \mathcal{E}_2 is the identity.

The mapping φ^* will be defined on the velocity vectors at the points of \mathcal{E}_1 in the same way as explained in Section 3. On \mathcal{E}_2 , mapping φ^* is the ordinary billiard reflection.

Remark 4.1. If magic reflection is introduced on the inner boundary, then the dynamics does not any more depend continuously on the initial conditions. Namely, consider motion parallel to a tangent line to \mathcal{E}_2 , close to the point of tangency. If the particle does not cross \mathcal{E}_2 , the motion is continued straight along that line. If the particle reaches \mathcal{E}_2 , it will be, according to the mapping φ magically transported to another point of \mathcal{E}_2 and continue motion from there. Thus, the continuity close to tangency to \mathcal{E}_2 is lost.

Example 4.2. Two trajectories when φ on the outer boundary is flipping over the long axis are shown in Figure 11.

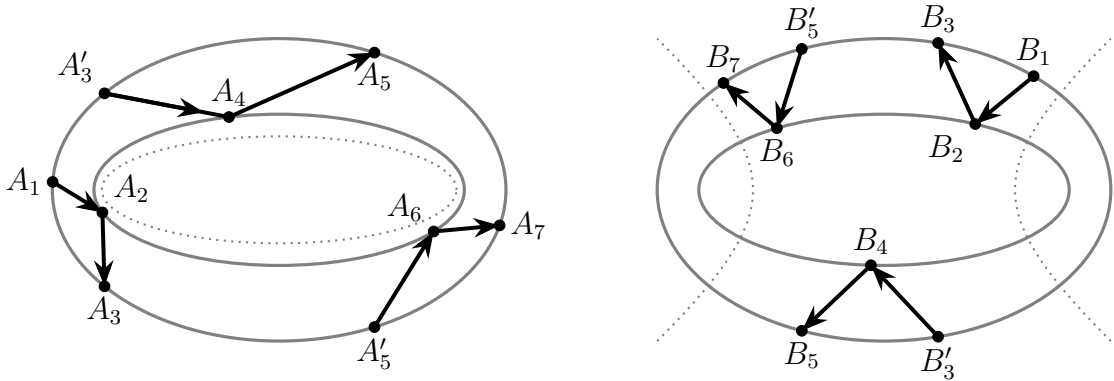


Figure 11: Billiard between two ellipses with magic flipping over the long axis when the particle hits the outer boundary. The dotted curves are caustics.

Theorem 4.3. *The magic billiard in the annulus between two confocal ellipses with flipping over the long axis on the outer boundary is roughly Liouville equivalent to the magic billiard within an ellipse with flipping over the short axis.*

Proof. In the phase space, any level set corresponding to the trajectories having a fixed non-degenerate conic as caustic consists of a single Liouville torus.

The level set corresponding to the caustic $\mathcal{E} = \mathcal{C}_0$ corresponds to the Fomenko atom \mathbf{A} , similarly as in the proof of Theorem 3.9.

The level set with the caustic \mathcal{C}_a corresponds to the motion along two segments of the short axis which are within the annulus. Each time when the particle hits the outer boundary, it is magically flipped to the other segment, thus this level set consists of a single closed trajectory, so we have the Fomenko atom of type **A** there.

The level set with the caustic \mathcal{C}_b consists of the trajectories that contain the foci of the ellipse \mathcal{E} . Two of those trajectories is closed: each trajectory corresponds to the motion along one of the segments of the long axis which are within the annulus. For any other trajectory on that level set, the extensions of the segments will pass alternately through the two foci. In one class of those trajectories, the segments that point from the outer boundary to the inner one contain the left focus, and their limit as time goes to $+\infty$ is the left segment on the long axis, while the time limit to $-\infty$ is the right segment. In the other class, everything is opposite. Thus, there are two separatrices on that level set, so it is represented by the Fomenko atom **A****. \square

Example 4.4. Two trajectories when φ on the outer boundary is flipping over the short axis are shown in Figure 12.

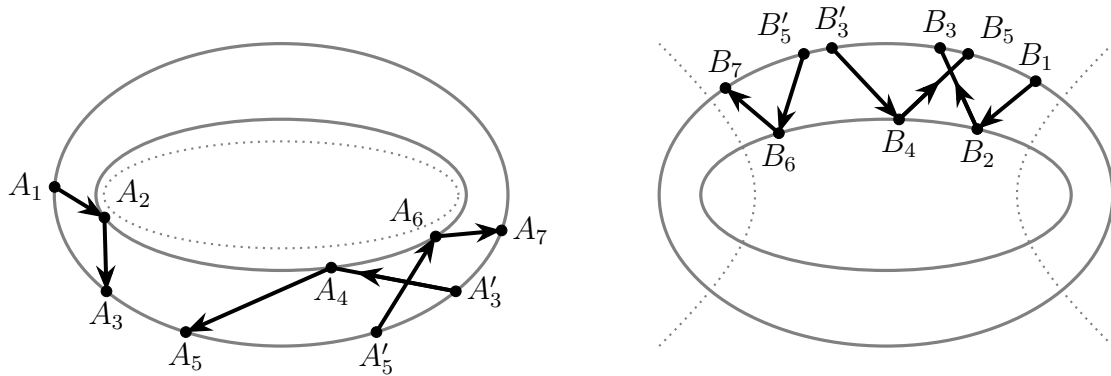


Figure 12: Billiard between two ellipses with magic flipping over the short axis when the particle hits the outer boundary. The dotted curves are caustics.

Theorem 4.5. *The magic billiard in the annulus between two confocal ellipses with flipping over the short axis on the outer boundary is roughly Liouville equivalent to the magic billiard within an ellipse with flipping over the long axis.*

Proof. In the phase space, any level set corresponding to the trajectories having an ellipse as caustic consists of a single Liouville torus. On the other hand, if the caustic is hyperbola, there are two tori: each corresponding to one connected component within the annulus between the two branches of the hyperbola.

The level sets corresponding to the caustic $\mathcal{E} = \mathcal{C}_0$ corresponds to the Fomenko atom **A**.

The level set with the caustic \mathcal{C}_a corresponds to the motion along two segments of the short axis which are within the annulus. Each segment is covered by one closed trajectory on that level set, thus both of them corresponds to the Fomenko atom **A**.

The level set with the caustic \mathcal{C}_b consists of the trajectories that contain the foci of the ellipse \mathcal{E} . One of those trajectories is closed and it is placed on the long axis, traversing the two segments. Any other trajectory on that level set is placed on one side of the long axis, thus there are two separatrices. We conclude that this level set corresponds to the the Fomenko atom **B**. \square

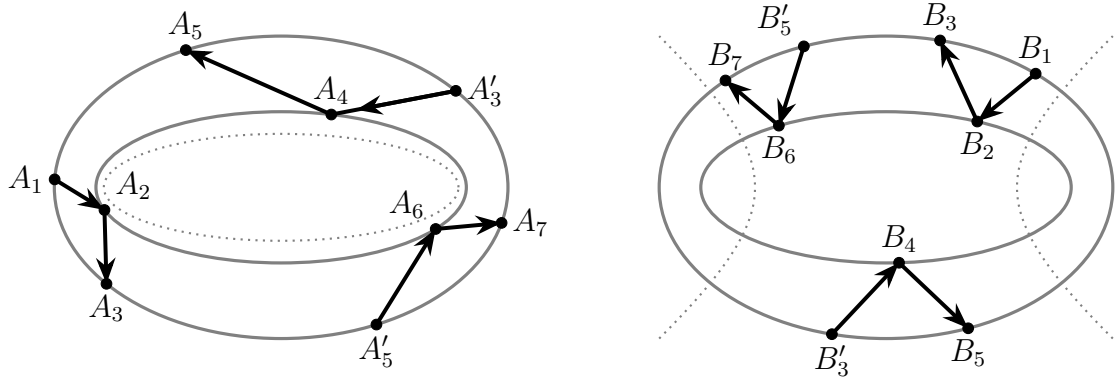


Figure 13: Billiard between two ellipses with slipping by half-ellipse along the outer boundary. The dotted curves are caustics.

Example 4.6. Two trajectories when φ on the outer boundary is slipping by half-ellipse are shown in Figure 13.

Theorem 4.7. *The Liouville equivalence class of the magic billiard in an elliptic annulus with slipping by half-ellipse along the outer boundary is given by the Fomenko graph in Figure 14.*

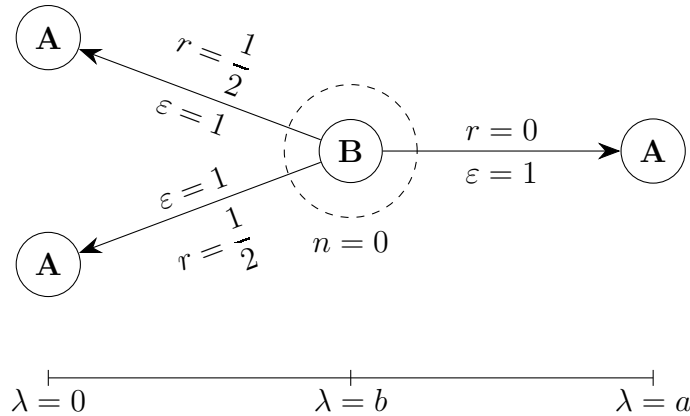


Figure 14: Theorem 4.7: Fomenko graph for the magic billiard in the annulus with flipping over the long axis.

Proof. Any level set corresponding to the trajectories having a fixed ellipse as caustic consists of two Liouville tori: each torus contains the trajectories the winding in one direction around the annulus. On the other hand, if the caustic is hyperbola, there is only one torus.

The level set corresponding to the caustic $\mathcal{E} = \mathcal{C}_0$ contains only two closed trajectories: each is winding in one direction along the boundary. Thus, we have two Fomenko atoms of type **A** there.

The level set with the caustic \mathcal{C}_a corresponds to the motion along two segments of the short axis which are within the annulus. There is only one trajectory on that level set, so it corresponds to the Fomenko atom **A**.

The level set with the caustic \mathcal{C}_b consists of the trajectories that contain the foci of the ellipse \mathcal{E} . One of those trajectories is closed and it is placed on the long axis, traversing the two segments. There are two separatrices on that level set: one separatrix contains the trajectories

where the motion is above the long axis on the left and below the axis on the right as $t \rightarrow +\infty$, and vice versa for $t \rightarrow -\infty$. The opposite holds for the other separatrix. We can conclude that the corresponding Fomenko atom is **B**. \square

Remark 4.8. The example of billiard within elliptic annulus with slipping along the outer ellipse was analyzed in [FVZ2021]. There, the configuration space was defined differently: by gluing two identical annuli along inner and the outer boundaries, so the resulting Fomenko graph there is different.

Remark 4.9. The billiards within an annulus with flipping over one of the axes on the outer boundary correspond to the topological billiards $\Delta_\beta(2B_2)_{xx}$ and $\Delta_\beta(2B_1)_{yy}$ from [Fok2015].

Remark 4.10. Various classes of glued sets and billiard systems on them were considered in [Kud2015].

5 Conclusions and discussion

We note that the magic billiards, as introduced in Section 2 represents a very broad class, where various subclasses may be of interest for exciting future research.

In particular, integrable cases for magic billiards within ellipse are not exhausted by the list considered in this paper. It would be interesting to explore a more general class of magic billiards $(\mathcal{E}, \varphi, \varphi^*)$, including those which preserve caustic. One natural generalization includes studying domains with piece-wise smooth boundaries.

We note that billiard ordered games introduced in [DR2004] and studied further in [DGR2022a] can also be generalised using magic reflections.

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