Received: 2 July 2024 | Revised: 8 August 2024 | Accepted: 19 August 2024
DOI: 10.1002/mma.10463

RESEARCH ARTICLE

WILEY

Convex combinations of some convergent sequences

Stevo Stević[1,2]

[1]Mathematical Institute of the Serbian Academy of Sciences, Beograd, Serbia

[2]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

Correspondence
Stevo Stević, Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia.
Email: sscite1@gmail.com

Communicated by: W. Sprößig

We consider the convex combinations $c_n^\alpha := (1-\alpha)a_n + \alpha b_n$, $n \in \mathbb{N}$, $\alpha \in [0, 1]$, of a pair of sequences of real numbers $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that $a_n \le b_n$, $n \in \mathbb{N}$, converging to $\ln 2$, and study the location of the limit inside the intervals $[a_n, b_n]$, for every $n \in \mathbb{N}$ or for sufficiently large n. We also investigate the same problem for the case of two corresponding sequences converging to $\ln 3$. Among other results, we prove some, a bit, unexpected ones. Namely, for each $\alpha \in [0, 1]$, we determine the exact index $n_0 \in \mathbb{N}$ at which the sequence c_n^α changes the monotonicity, and we also determine the type of the monotonicity. A number of interesting remarks are also presented.

KEYWORDS
convex combinations of sequences, locations of limits, monotone sequences, pair of convergent sequences

MSC CLASSIFICATION
40A05, 52A41

1 | INTRODUCTION

By $\mathbb{N}$, we denote the set of positive natural numbers and by $\mathbb{R}$ the set of real numbers. Let $\mathbb{N}_k = \{n \in \mathbb{N} : n \ge k\}$, where $k \in \mathbb{N}$ is fixed. If $k, l \in \mathbb{N}$, where $k \le l$, then we use the notation $j = \overline{k, l}$ instead of writing the expression: $k \le j \le l$, $j \in \mathbb{N}$.

Sequences of real numbers have been studied for a long time. The sequences given by recursive relations were first studied analytically by De Moivre [1, 2], then by D. Bernoulli [3], Euler [4], Lagrange [5], Laplace [6], and several other mathematicians. Many results in this direction up to 1800 can be found in [7]. Their solvability was one of the first studied problems (see also [8–10]). For some recent results on the solvability, invariants, and their applications, see, for example, [11–23] and the references therein. On the other hand, there are some sequences which are given in some other ways. For example, they can be given in the form of some sums or explicitly as some functions of $\mathbb{N}$ (see, e.g., [4, 10, 24–35]).

Sometimes sequences occur in pairs, for example, a_n and b_n, $n \in \mathbb{N}$; they satisfy the condition

$$a_n \le b_n, \ \ n \in \mathbb{N}, \tag{1}$$

and it is proved by some methods that they both converge to the same limit, say, $c \in \mathbb{R}$.

It is a frequent situation that a_n is nondecreasing, whereas b_n is nonincreasing. In this situation, the sequences form a family of nonincreasing compact intervals $[a_n, b_n]$, that is, $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$, $n \in \mathbb{N}$. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$, we have that $\cap_{n\in\mathbb{N}} [a_n, b_n] = \{c\}$.

Such pairs of sequences have been studied for a long time. One of the most known pairs of the sequences is

$$a_n = \left(1 + \frac{1}{n}\right)^n \text{ and } b_n = \left(1 + \frac{1}{n}\right)^{n+1}, \ n \in \mathbb{N}, \tag{2}$$

which converge to e, a_n increasingly and b_n decreasingly.

An interesting question is to locate the limit inside the interval $[a_n, b_n]$ for each fixed n. For example, each of the intervals $[a_n, b_n]$ can be divided into k equal subintervals, and it should be investigated if there is one of them in which the limit c lies for every $n \in \mathbb{N}$ or for sufficiently large n. For example, Problem 171 in [33] asks: If a_n and b_n are defined in (2), in which quarter of the interval $[a_n, b_n]$ is e contained? This paper, among other things, is devoted to such a problem.

Our motivation stems from the following problem [34, Problem 3] by R.P. Ushakov published in [26].

Problem 1. Let

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, \tag{3}$$

and

$$b_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, \tag{4}$$

for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, divide the interval $[a_n, b_n]$ into eight equal intervals. Prove that $\ln 2$ lies in the second interval from the left for every $n \in \mathbb{N}$.

In this paper, we give two solutions to the problem, pose a more general problem, and present a solution to it; consider the convex combinations $c_n^\alpha := (1-\alpha)a_n + \alpha b_n$, $n \in \mathbb{N}$, $\alpha \in [0, 1]$ of the sequences in (3) and (4) and study the location of their limit inside the intervals $[a_n, b_n]$, for every $n \in \mathbb{N}$ or for sufficiently large n. For each $\alpha \in [0, 1]$, we determine the index $n_0 \in \mathbb{N}$ at which the sequence c_n^α changes the monotonicity and determine the type of the monotonicity. We also investigate the same problems for the case of two corresponding sequences converging to $\ln 3$.

2 | TWO SOLUTIONS TO PROBLEM 1

Here we give two proofs of Problem 1. We include all the details for the completeness, benefit of the reader, and as a motivation for some further investigations in this direction. The first proof is connected to some geometric interpretations of the sequences defined in (3) and (4). Similar methods are frequently connected with these and some related sequences (see, e.g., [25]), which suggested us to find a solution of the problem in this way. The second proof is standard and is the one proposer of the problem expected to be found [35].

2.1 | On Problem 1

It is well-known that the sequences (3) and (4) are convergent and that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \ln 2,$$

(see, e.g., [10, 32, 36]). It is also clear that $a_n < b_n$ for every $n \in \mathbb{N}$.

Now we present the first solution to Problem 1, which is based on some geometric considerations.

2.2 | First solution to Problem 1

Note that

$$\ln 2 = \int_0^1 \frac{dx}{x+1} = \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{dx}{x+1}. \tag{5}$$

The Hermite–Hadamard inequalities say that

$$f\left(\frac{a+b}{2}\right)(b-a) \le \int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}(b-a) \tag{6}$$

for each convex continuous function $f : [a, b] \to \mathbb{R}$, where the equality is achieved for the linear function ([31, 37, 38]).

Since the function

$$f_0(x) = \frac{1}{x+1} \tag{7}$$

is convex on the interval $[0,1]$, by the second inequality in (6), we have

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{dx}{x+1} \leq \frac{f_0(\frac{k}{n}) + f_0(\frac{k+1}{n})}{2n} = \frac{\frac{n}{n+k} + \frac{n}{n+k+1}}{2n} \tag{8}$$

for $k = \overline{0, n-1}$.

From (5) and (8), we obtain

$$\ln 2 \leq \sum_{k=0}^{n-1} \frac{\frac{n}{n+k} + \frac{n}{n+k+1}}{2n} = \frac{1}{2}\left(\frac{1}{n} + \frac{2}{n+1} + \frac{2}{n+2} + \cdots + \frac{2}{2n-1} + \frac{1}{2n}\right) = \frac{1}{4n} + a_n. \tag{9}$$

From that, we have that $\ln 2$ lies in the first subinterval if we divide the interval $[a_n, b_n]$ into four equal subintervals, for each $n \in \mathbb{N}$.

Using the first inequality in (6), we have

$$\frac{2}{2n+2k+1} = \frac{f_0(\frac{1}{2}(\frac{k}{n} + \frac{k+1}{n}))}{n} \leq \int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{dx}{x+1},$$

for $k = \overline{0, n-1}$, from which it follows that

$$\sum_{k=0}^{n-1} \frac{2}{2n+2k+1} \leq \int_0^1 f_0(t)dt = \ln 2. \tag{10}$$

Further, we have

$$\begin{aligned}\sum_{k=0}^{n-1} \frac{2}{2n+2k+1} - a_n &= \sum_{k=0}^{n-1} \left(\frac{2}{2n+2k+1} - \frac{2}{2n+2k+2}\right)\\ &= \sum_{k=0}^{n-1} \frac{2}{(2n+2k+1)(2n+2k+2)}\\ &\geq n\frac{2}{(4n-1)4n} > \frac{1}{8n},\end{aligned} \tag{11}$$

for $n \in \mathbb{N}$.

From (10) and (11), we obtain

$$\frac{1}{8n} + a_n < \ln 2, \tag{12}$$

for every $n \in \mathbb{N}$, from which together with (9) we have that $\ln 2$ lies in the second subinterval if we divide the interval $[a_n, b_n]$ into eight equal subintervals, for each $n \in \mathbb{N}$. □

Remark 1. Inequalities (9) and (12) are, in fact, strict, from which it follows that

$$\ln 2 \in \left(a_n + \frac{1}{8n}, a_n + \frac{1}{4n}\right), \text{ for } n \in \mathbb{N};$$

that is, $\ln 2$ lies inside the open interval $(a_n + \frac{1}{8n}, a_n + \frac{1}{4n})$, for every $n \in \mathbb{N}$.

Remark 2. If an integrable function f is nonincreasing on the interval $[0,1]$, then we have

$$\frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right) \leq \int_0^1 f(t)dt,$$

for each $n \in \mathbb{N}$.

From this and since (7) is a decreasing function, we have

$$a_n = \frac{1}{n}\left(\frac{n}{n+1} + \frac{n}{n+2} + \cdots + \frac{n}{2n}\right) \leq \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f_0(t)dt = \int_0^1 f_0(t)dt = \ln 2,$$

for each $n \in \mathbb{N}$, from which we cannot conclude in which of the two intervals $\left[a_n, a_n + \frac{1}{8n}\right]$ and $[a_n + \frac{1}{8n}, a_n + \frac{1}{4n}]$, $\ln 2$ lies, by using this argument.

Remark 3. Inequality (12) can be obtained without using the Hadamard–Hermite inequality. Namely, we can use the following inequality:

$$Area(T_k) \leq \int_{\frac{k-1}{n}}^{\frac{k}{n}} f_0(t)dt, \tag{13}$$

where T_k, $k = \overline{1,n}$, is the trapezoid with the lateral sides consisting of the interval $[\frac{k-1}{n}, \frac{k}{n}]$ and the interval obtained as the intersection of the tangent line at the point $(\frac{k}{n}, f_0(\frac{k}{n}))$ with the lines $x = \frac{k-1}{n}$ and $x = \frac{k}{n}$. Inequality (13) holds due to the convexity of the function f_0 (if f_0 is a differentiable function; see, e.g., [39, p.246, Proposition 6]).

The tangent line is

$$y(x) = f_0\left(\frac{k}{n}\right) + f_0'\left(\frac{k}{n}\right)\left(x - \frac{k}{n}\right) = \frac{n}{n+k} - \frac{n^2}{(n+k)^2}\left(x - \frac{k}{n}\right).$$

From this, we have

$$y\left(\frac{k-1}{n}\right) = \frac{n}{n+k} + \frac{n}{(n+k)^2},$$

and consequently,

$$Area(T_k) = \frac{1}{2n}\left(y\left(\frac{k-1}{n}\right) + y\left(\frac{k}{n}\right)\right) = \frac{1}{n+k} + \frac{1}{2(n+k)^2}, \tag{14}$$

for $k = \overline{1,n}$.

Using (14) in (13) and summing such obtained inequalities for $k = \overline{1,n}$, it follows that

$$a_n + \sum_{k=1}^{n} \frac{1}{2(n+k)^2} \leq \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f_0(t)dt = \ln 2, \tag{15}$$

for $n \in \mathbb{N}$.

Now note that

$$\sum_{k=1}^{n} \frac{1}{2(n+k)^2} \geq n\frac{1}{2(2n)^2} = \frac{1}{8n}. \tag{16}$$

Combining (15) and (16), we get

$$a_n + \frac{1}{8n} \leq \ln 2,$$

from which together with inequality (9) it follows that $\ln 2 \in [a_n + \frac{1}{8n}, a_n + \frac{1}{4n}]$, for every $n \in \mathbb{N}$, as claimed.

Remark 4. Inequalities (9) and (13) are, in fact, strict (and so is (15)), from which it also follows that $\ln 2 \in \left(a_n + \frac{1}{8n}, a_n + \frac{1}{4n}\right)$, for every $n \in \mathbb{N}$.

Remark 5. Inequality (16) is not strict. The equality is achieved for $n = 1$.

Remark 6. Since

$$\begin{aligned}\frac{2}{2n+2k-1} - \frac{1}{n+k} - \frac{1}{2(n+k)^2} &= \frac{1}{(2n+2k-1)(n+k)} - \frac{1}{2(n+k)^2}\\ &= \frac{1}{2(n+k)^2(2n+2k-1)} > 0,\end{aligned}$$

for $1 \le k \le n$, $n \in \mathbb{N}$, we have that the approximation of the integral used in the first proof is better than in the second one.

In fact, the last inequality holds in a much more general situation. Namely, the following result holds.

Proposition 1. *Let f be a convex and continuously differentiable function on the interval $[a,b]$. Then, the following inequality holds*

$$Area(\widetilde{T}) \ge Area(T), \tag{17}$$

where $\widetilde{T}$ is the trapezoid with the lateral sides consisting of the interval $[a,b]$ and the interval obtained as the intersection of the tangent line at the point $(\frac{a+b}{2}, f(\frac{a+b}{2}))$ with the lines $x=a$ and $x=b$, whereas T is the trapezoid with the lateral sides consisting of the interval $[a,b]$ and the interval obtained as the intersection of the tangent line at the point $(b, f(b))$ with the lines $x=a$ and $x=b$.

Proof. First, note that

$$Area(\widetilde{T}) = (b-a) f\left(\frac{a+b}{2}\right).$$

The tangent line of f at the point $(b, f(b))$ is

$$y = f(b) + f'(b)(x-b).$$

Hence, we have

$$Area(T) = (b-a)\frac{1}{2}\left(f(b) + f(b) + f'(b)(a-b)\right).$$

From these two relations and the Lagrange mean value theorem, we have

$$\begin{aligned} Area(\widetilde{T}) - Area(T) &= (b-a)\left(f\left(\frac{a+b}{2}\right) - f(b) - \frac{1}{2}f'(b)(a-b)\right) \\ &= (b-a)\left(f'(\zeta)\frac{a-b}{2} - \frac{1}{2}f'(b)(a-b)\right) \\ &= \frac{(b-a)^2}{2}(f'(b) - f'(\zeta)), \end{aligned} \tag{18}$$

for some $\zeta \in (\frac{a+b}{2}, b)$. Since the function f is convex, f' is a nondecreasing function (see, e.g., [39, p.245]), from which together with (18), inequality in (17) follows. □

Remark 7. If in Proposition 1 we assume that the function f is two times differentiable on the interval $[a,b]$, then we can apply the Lagrange mean value theorem to the function f' on the interval (ζ, b), and from (18), we obtain

$$Area(\widetilde{T}) - Area(T) = \frac{(b-a)^2(b-\zeta)f''(\zeta_1)}{2}, \tag{19}$$

for some $\zeta_1 \in (\zeta, b)$. Since the function f is convex, it must be $f''(t) \ge 0$, $t \in [a,b]$, (see, e.g., [39, p.245]), from which together with (19) inequality in (17) follows, in this case.

2.3 | Second solution to Problem 1

The complexity of the above presented solution to Problem 1 suggested us to try to find an easier solution to the problem. There is a simpler solution which is based on studying the monotonicity of sequences ([35]), which is one of the basic methods for their investigations (see, e.g., [12, 25, 29–32, 40, 41]). Namely, note that the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are monotone; the first one is increasing, whereas the second one is decreasing. Recall that they both converge to the same limit. These facts suggest investigating the monotonicity of the sequences

$$c_n = \frac{1}{8n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, \tag{20}$$

$$d_n = \frac{1}{4n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, \tag{21}$$

for $n \in \mathbb{N}$.

We have

$$c_{n+1} - c_n = \frac{1}{8(n+1)} + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{8n} - \frac{1}{n+1} = \frac{2n-1}{8n(n+1)(2n+1)} > 0,$$

for each $n \in \mathbb{N}$; that is, the sequence c_n is (strictly) increasing.

On the other hand, we have

$$d_{n+1} - d_n = \frac{1}{4(n+1)} + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{4n} - \frac{1}{n+1} = \frac{-1}{4n(n+1)(2n+1)} < 0,$$

for each $n \in \mathbb{N}$; that is, the sequence d_n is (strictly) decreasing.

Since we obviously have

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = \ln 2,$$

we get $c_n < \ln 2 < d_n$ for each $n \in \mathbb{N}$. Therefore, if we divide the interval $[a_n, b_n]$ into eight equal subintervals, $\ln 2$ really lies in the second one from the left for each $n \in \mathbb{N}$. □

3 | A NATURAL GENERALIZATION OF PROBLEM 1 AND ITS SOLUTION

Having solved Problem 1, the following problem naturally occurs.

Problem 2. Let the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be defined in (3) and (4), respectively, and let $k \in \mathbb{N}_3$. For each $n \in \mathbb{N}$ divide the interval $[a_n, b_n]$ into 2^k equal subintervals. Find the subinterval in which $\ln 2$ lies for every $n \in \mathbb{N}$ or for sufficiently large n.

If we want to develop the first proof of Problem 1, we have some technical difficulties, because of which it is better to find a less involved solution to Problem 1. The second solution to Problem 1 gives us a simple method for solving Problem 2. We prove the following result.

Proposition 2. *Let the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be defined in (3) and (4), respectively, and let $k \in \mathbb{N}_3$. For each $n \in \mathbb{N}$, divide the interval $[a_n, b_n]$ into 2^k equal subintervals. Then $\ln 2$ lies in the 2^{k-2}-th subinterval from the left for sufficiently large n.*

Proof. Let

$$\widetilde{d}_n = \frac{2^{k-2}-1}{2^k n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, \quad n \in \mathbb{N}.$$

Then, we have

$$\widetilde{d}_{n+1} - \widetilde{d}_n = \frac{2^{k-2}-1}{2^k(n+1)} + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{2^{k-2}-1}{2^k n} - \frac{1}{n+1} = \frac{2n+1-2^{k-2}}{2^k n(n+1)(2n+1)} > 0$$

for $n \geq 2^{k-3}$.

Thus, the sequence $\widetilde{d}_n$ is increasing for $n \geq 2^{k-3}$. Since obviously $\widetilde{d}_n \to \ln 2$ as $n \to \infty$, we get $\widetilde{d}_n < \ln 2$ for $n \geq 2^{k-3}$. From this and the first or the second proof of Problem 1, we have

$$\widetilde{d}_n < \ln 2 < \frac{1}{4n} + a_n, \tag{22}$$

for $n \geq 2^{k-3}$, as claimed. □

Remark 8. Recall that the second inequality in (22) holds for every $n \in \mathbb{N}$.

Remark 9. Note that we have managed to find the exact value of the natural number, say, n_0 from which the inequalities in (22) hold.

4 | CONVEX COMBINATIONS OF THE SEQUENCES (3) AND (4)

Note that the sequence $(\widetilde{d}_n)_{n\in\mathbb{N}}$ in Proposition 2 is, in fact, a convex combination of the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$. This naturally suggests an investigation of the other convex combinations of the sequences, that is, of the sequences

$$c_n^\alpha := (1-\alpha)a_n + \alpha b_n,\ n\in\mathbb{N}, \tag{23}$$

where $\alpha \in (0,1)$.

Proposition 3. *Consider the sequence $(c_n^\alpha)_{n\in\mathbb{N}}$ defined in (23), where $\alpha\in[0,1]$. Then, the following statements hold.*

(a) If $\alpha \in [1/4, 1]$, then the sequence is strictly decreasing.
(b) If $\alpha \in [0, 1/6)$, then the sequence is strictly increasing.
(c) If $\alpha \in [\alpha_k, \alpha_{k+1})$ for some fixed $k \in \mathbb{N}$, where

$$\alpha_k = \frac{k}{2(2k+1)},\ k\in\mathbb{N},$$

then the sequence is strictly increasing for $n \geq k+1$ and nonincreasing for $1 \leq n \leq k$.

Proof. First, note that

$$c_n^\alpha = \frac{\alpha}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n},\ n\in\mathbb{N}. \tag{24}$$

We have

$$c_{n+1}^\alpha - c_n^\alpha = \frac{\alpha}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{\alpha}{n} - \frac{1}{n+1} = \frac{(1-4\alpha)n - 2\alpha}{2n(n+1)(2n+1)}, \tag{25}$$

for $n\in\mathbb{N}$.

(a) If $\alpha\in[1/4,1]$, then from (25), we have

$$c_{n+1}^\alpha - c_n^\alpha \leq \frac{-2\alpha}{2n(n+1)(2n+1)} < 0$$

for $n\in\mathbb{N}$, from which the claim immediately follows.

(b) If $\alpha\in[0,1/6)$, then from (25), we have

$$c_{n+1}^\alpha - c_n^\alpha \geq \frac{1-6\alpha}{2n(n+1)(2n+1)} > 0,$$

for $n\in\mathbb{N}$, from which the claim immediately follows.

(c) If $\alpha\in[\alpha_k,\alpha_{k+1})$, then we have $\alpha_{k+1}\in[1/6,1/4)$, so from (25), it follows that

$$\begin{aligned} c_{n+1}^\alpha - c_n^\alpha &= \frac{(1-4\alpha)n-2\alpha}{2n(n+1)(2n+1)} \geq \frac{(1-4\alpha)(k+1)-2\alpha}{2n(n+1)(2n+1)} \\ &= \frac{k+1-2(2k+3)\alpha}{2n(n+1)(2n+1)} > \frac{k+1-2(2k+3)\alpha_{k+1}}{2n(n+1)(2n+1)} = 0, \end{aligned} \tag{26}$$

for $n \geq k+1$.

We also have

$$c_{n+1}^{\alpha} - c_n^{\alpha} = \frac{(1-4\alpha)n - 2\alpha}{2n(n+1)(2n+1)} \le \frac{(1-4\alpha)k - 2\alpha}{2n(n+1)(2n+1)}$$
$$= \frac{k - 2(2k+1)\alpha}{2n(n+1)(2n+1)} \le \frac{k - 2(2k+1)\alpha_k}{2n(n+1)(2n+1)} = 0, \tag{27}$$

for $n \le k$. From (26) and (27), the claim follows. □

Remark 10. Note that

$$\left[\frac{1}{6}, \frac{1}{4}\right) = \bigsqcup_{k\in\mathbb{N}} [\alpha_k, \alpha_{k+1}),$$

where the symbol $\bigsqcup$ stands for a union of disjoint sets.

Remark 11. Note that if $\alpha = 1/6$, then the sequence $c_n^{1/6}$ increases for $n \in \mathbb{N}_2$ and that from (25) with $n = 1$, we have

$$c_2^{1/6} = c_1^{1/6}.$$

Remark 12. The above analysis shows that the monotonicity of the family of sequences $(c_n^{\alpha})_{n\in\mathbb{N}}$ changes at $\alpha = 1/4$. On the other hand, in the dyadic decompositions of the interval $[a_n, b_n]$, $n \in \mathbb{N}$, the point $c_n^{1/4}$ is the end point of the 2^{k-2}th and $2^{k-2}+1$th subintervals from the left for each $k \in \mathbb{N}_3$. These are the main reasons why $\ln 2$ eventually lies in the 2^{k-2}th subinterval from the left.

5 | ON TWO SEQUENCES CONVERGING TO ln3

Here we consider the following two sequences

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}, \tag{28}$$

and

$$b_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}, \tag{29}$$

for $n \in \mathbb{N}$.

It is well-known that the sequences (28) and (29) are convergent and that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \ln 3, \tag{30}$$

(see, e.g., [10, 32]). It is also clear that

$$a_n < b_n$$

for every $n \in \mathbb{N}$.

Besides, $(a_n)_{n\in\mathbb{N}}$ is strictly increasing, whereas $(b_n)_{n\in\mathbb{N}}$ is strictly decreasing. Indeed, we have

$$a_{n+1} - a_n = \frac{1}{3n+1} + \frac{1}{3n+2} + \frac{1}{3n+3} - \frac{1}{n+1} = \frac{2}{(3n+1)(3n+3)} + \frac{1}{(3n+2)(3n+3)} > 0,$$

for each $n \in \mathbb{N}$, from which it follows that the sequence $(a_n)_{n\in\mathbb{N}}$ is strictly increasing, and we have

$$b_{n+1} - b_n = \frac{1}{3n+1} + \frac{1}{3n+2} + \frac{1}{3n+3} - \frac{1}{n}$$
$$= \left(\frac{1}{3n+1} - \frac{1}{3n}\right) + \left(\frac{1}{3n+2} - \frac{1}{3n}\right) + \left(\frac{1}{3n+3} - \frac{1}{3n}\right)$$
$$= -\frac{1}{3n(3n+1)} - \frac{2}{3n(3n+2)} - \frac{3}{3n(3n+3)} < 0,$$

for every $n \in \mathbb{N}$, from which it follows that the sequence $(b_n)_{n\in\mathbb{N}}$ is strictly decreasing.

Now, as in the case of the sequences in (3) and (4), consider the convex combinations of the sequences

$$c_n^\alpha := (1-\alpha)a_n + \alpha b_n, \ \ n \in \mathbb{N}, \tag{31}$$

where $\alpha \in [0,1]$. From (30), we have that the sequence converges to $\ln 3$ as $n \to +\infty$.

Now we investigate the eventual monotonicity of the sequence in terms of the parameter α. Note that

$$c_n^\alpha = \frac{\alpha}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}, \ \ n \in \mathbb{N}.$$

Proposition 4. *Consider the sequence $(c_n^\alpha)_{n\in\mathbb{N}}$ defined in (31), where $\alpha \in [0,1]$. Then, the following statements hold.*

(a) If $\alpha \in [1/3, 1]$, then the sequence is strictly decreasing.
(b) If $\alpha \in [0, 7/30)$, then the sequence is strictly increasing.
(c) If $\alpha \in [\alpha_k, \alpha_{k+1})$ for some fixed $k \in \mathbb{N}$, where

$$\alpha_k = \frac{9k^2+5k}{27k^2+27k+6}, \ \ k \in \mathbb{N},$$

then the sequence is strictly increasing for $n \geq k+1$ and nonincreasing for $1 \leq n \leq k$.

Proof. We have

$$\begin{aligned} c_{n+1}^\alpha - c_n^\alpha &= \frac{\alpha}{n+1} + \frac{1}{3n+1} + \frac{1}{3n+2} + \frac{1}{3n+3} - \frac{\alpha}{n} - \frac{1}{n+1} \\ &= \frac{2}{3(n+1)(3n+1)} + \frac{1}{3(n+1)(3n+2)} - \frac{\alpha}{n(n+1)} \\ &= \frac{9n^2+5n-\alpha(27n^2+27n+6)}{3n(n+1)(3n+1)(3n+2)} \end{aligned} \tag{32}$$

$$= \frac{9(1-3\alpha)n^2 + (5-27\alpha)n - 6\alpha}{3n(n+1)(3n+1)(3n+2)}, \tag{33}$$

for $n \in \mathbb{N}$.

(a) If $\alpha \in [1/3, 1]$, then from (33), we have

$$c_{n+1}^\alpha - c_n^\alpha \leq \frac{-6\alpha}{3n(n+1)(3n+1)(3n+2)} < 0$$

for $n \in \mathbb{N}$, from which the claim immediately follows.

(b) Let

$$f_\alpha(t) = 9(1-3\alpha)t^2 + (5-27\alpha)t - 6\alpha. \tag{34}$$

Then

$$f'_\alpha(t) = 18(1-3\alpha)t + 5 - 27\alpha. \tag{35}$$

If $\alpha \in [0, 1/3)$, then the function f'_α is increasing, and consequently, we have that

$$f'_\alpha(t) \geq f'_\alpha(1) = 23 - 81\alpha,$$

for $t \geq 1$.

Hence, if $\alpha \in [0, 23/81)$, the function $f_\alpha(t)$ is increasing for $t \geq 1$, from which it follows that

$$f_\alpha(t) \geq f_\alpha(1) = 14 - 60\alpha > 0,$$

if $\alpha \in [0, 7/30)$.

Now note that

$$[0, 7/30) \subset [0, 23/81).$$

So, if $\alpha \in [0, 7/30)$, then from (33), we have

$$c_{n+1}^{\alpha} - c_n^{\alpha} \geq \frac{14 - 60\alpha}{3n(n+1)(3n+1)(3n+2)} > 0,$$

for $n \in \mathbb{N}$, from which the claim immediately follows.

(c) Let $f_\alpha(t)$ be defined in (34) and $\alpha \in [\alpha_k, \alpha_{k+1})$. Then (35) holds, and since $\alpha \in [7/30, 1/3)$, we have that $f'_\alpha(t)$ is increasing. On the other hand,

$$f_\alpha(t) = 9t^2 + 5t - \alpha(27t^2 + 27t + 6)$$

from which it follows that it is decreasing in α for each $t \geq 0$.

Note that $f_{\alpha_k}(k) = 0$. Since f_α is a quadratic polynomial and its discriminant

$$\Delta = (5 - 27\alpha)^2 + 216\alpha(1 - 3\alpha) > 0$$

for $\alpha \in [0, 1/3)$, we have that f_α has two real roots $t_j(\alpha) = t_j$, $j = 1, 2$,

$$t_1 = \frac{-(5 - 27\alpha) + \sqrt{\Delta}}{18(1 - 3\alpha)}$$

and

$$t_2 = \frac{-(5 - 27\alpha) - \sqrt{\Delta}}{18(1 - 3\alpha)}.$$

Since

$$\sqrt{\Delta} > |5 - 27\alpha| > 0,$$

for $\alpha \in [0, 1/3)$, the root t_1 is positive, whereas the root t_2 is negative. We also have

$$\min_{t \in \mathbb{R}} f_\alpha(t) = f_\alpha\left(\frac{27\alpha - 5}{18(1 - 3\alpha)}\right)$$

and

$$\frac{27\alpha - 5}{18(1 - 3\alpha)} > 0,$$

for $\alpha \in [7/30, 1/3)$.

Hence, we have

$$f_\alpha(t) < 0, \text{ for } t \in (t_2, t_1), \tag{36}$$

$$f_\alpha(t) > 0, \text{ for } t > t_1, \tag{37}$$

and the sequence $f_\alpha(n)$ is increasing on the set $\mathbb{N}_{[t_1]+1}$.

Especially, we have

$$f_{\alpha_k}(t) < 0, \text{ for } t \in (t_2, k), \tag{38}$$

$$f_{\alpha_k}(t) > 0, \text{ for } t > k, \tag{39}$$

and the sequence $f_{\alpha_k}(n)$ is increasing on the set $\mathbb{N}_{k+1}$.

Let

$$g(t) = \frac{9t^2 + 5t}{27t^2 + 27t + 6} = \frac{1}{3}\left(1 - \frac{4t + 2}{9t^2 + 9t + 2}\right).$$

Then

$$g'(t) = \frac{36t^2 + 36t + 10}{3(9t^2 + 9t + 2)^2} > 0, \ t \in \mathbb{R},$$

from which it follows that the sequence $\alpha_k = g(k)$ is increasing, and since it converges to 1/3 as $k \to \infty$, we have

$$\left[\frac{7}{30}, \frac{1}{3}\right) = \bigsqcup_{k\in\mathbb{N}} [\alpha_k, \alpha_{k+1}).$$

If $\alpha \in [\alpha_k, \alpha_{k+1})$, then $\alpha_{k+1} \in [7/30, 1/3)$ and

$$k = t_1(\alpha_k) \le t_1(\alpha) < t_1(\alpha_{k+1}) = k + 1. \tag{40}$$

Hence, from the relations (32) and (40), the monotonicity of the function $f_\alpha(n)$ on the set $\mathbb{N}_{[t_1(\alpha)]+1}$, and the monotonicity of f_α in variable α, we have

$$\begin{aligned} c_{n+1}^\alpha - c_n^\alpha &= \frac{9n^2 + 5n - \alpha(27n^2 + 27n + 6)}{3n(n+1)(3n+1)(3n+2)} \\ &\ge \frac{9(k+1)^2 + 5(k+1) - \alpha(27(k+1)^2 + 27(k+1) + 6)}{3n(n+1)(3n+1)(3n+2)} \\ &> \frac{9(k+1)^2 + 5(k+1) - \alpha_{k+1}(27(k+1)^2 + 27(k+1) + 6)}{3n(n+1)(3n+1)(3n+2)} = 0, \end{aligned} \tag{41}$$

for $n \ge k + 1$.

By using (32), (36), and (40), we have

$$\begin{aligned} c_{n+1}^\alpha - c_n^\alpha &= \frac{9n^2 + 5n - \alpha(27n^2 + 27n + 6)}{3n(n+1)(3n+1)(3n+2)} \\ &\le \frac{9t_1(\alpha)^2 + 5t_1(\alpha) - \alpha(27t_1(\alpha)^2 + 27t_1(\alpha) + 6)}{3n(n+1)(3n+1)(3n+2)} = 0, \end{aligned} \tag{42}$$

for $1 \le n \le k$.

Finally, from the inequalities in (41) and (42), the claim immediately follows. □

Now we address to Problem 2 with respect to the sequences defined in (28) and (29). We make dyadic divisions of the interval $[0, 1]$. First, if we divide the interval $[0, 1]$ in two equal halves, then the number 1/3 belongs to the first one from the left, and we have $0 < \frac{1}{3} < \frac{1}{2} < 1$. If we divide the interval $[0, 1]$ in four equal subintervals, then 1/3 belongs to the second one from the left and we have $0 < \frac{1}{4} < \frac{1}{3} < \frac{1}{2} < 1$. Further, if we divide the interval $[0, 1]$ in eight equal subintervals, then 1/3 belongs to the third one from the left and we have $0 < \frac{1}{4} < \frac{1}{3} < \frac{3}{8} < \frac{1}{2} < 1$. If we divide the interval $[0, 1]$ in 16 equal subintervals, then 1/3 belongs to the sixth subinterval from the left and we have $0 < \frac{1}{4} < \frac{5}{16} < \frac{1}{3} < \frac{3}{8} < \frac{1}{2} < 1$.

Continuing the procedure, we get

$$0 < \frac{1}{4} < \frac{5}{16} < \frac{21}{64} < \cdots < \frac{1}{3} < \cdots < \frac{43}{128} < \frac{11}{32} < \frac{3}{8} < \frac{1}{2} < 1.$$

Now we describe the exact position of the dyadic subinterval in which 1/3 lies, if we divide the interval $[0, 1]$ into 2^n equal subinterval.

The dyadic decimal expression for 1/3 is $1/3 = 0.010101\cdots$. Indeed, note that

$$0.010101\cdots = \sum_{j=1}^{\infty} \frac{1}{2^{2j}} = \frac{1}{4}\left(1 - \frac{1}{4}\right)^{-1} = \frac{1}{3},$$

where we have used the formula for the infinite sum of a geometric progression.

Let

$$a_1 = \frac{1}{2} \text{ and } a_2 = \frac{1}{4}, \tag{43}$$

$$a_{n+2} = \frac{a_{n+1} + a_n}{2}, \ n \in \mathbb{N}, \tag{44}$$

and

$$a_n = \frac{b_n}{2^n}, \tag{45}$$

for $n \in \mathbb{N}$.

Then, from (43), we get

$$b_1 = b_2 = 1, \tag{46}$$

whereas from (44) and (45), we have

$$b_{n+2} = b_{n+1} + 2b_n, \tag{47}$$

for every $n \in \mathbb{N}$.

By solving the initial value problems (46) and (47), we get

$$b_n = \frac{2^n - (-1)^n}{3}, \tag{48}$$

for $n \in \mathbb{N}$.

This shows that for each $n \in \mathbb{N}$, $1/3$ belongs to the interval with the end points a_n and a_{n+1} and that

$$0 < a_{2m} < a_{2m+2} \leq \frac{1}{3} \leq a_{2m+1} < a_{2m-1} < 1,$$

for every $m \in \mathbb{N}$.

From this, it follows that if we divide the interval $[0, 1]$ into 2^n equal subintervals, then $1/3$ belongs to the subinterval which is located on the position

$$\frac{2^n - (-1)^n}{3} + \frac{1 + (-1)^n}{2}$$

from the left.

Moreover, $1/3$ is not an end point of any of these intervals. Indeed, if it were, then we would have $\frac{1}{3} = \frac{p}{2^k}$, for some $p, k \in \mathbb{N}$, $1 \leq p \leq 2^k - 1$. We may assume that $\gcd(p, 2^k) = 1$; otherwise, we will cancel the factor of p containing the power of 2. So, we would get $3p = 2^k$, which is not possible since from the left-hand side it is an odd number, but on the right-hand side is an even number, which is a contradiction. Hence, $1/3$ belongs to each open interval with the end points a_n and a_{n+1}; that is, we have

$$0 < a_{2m} < a_{2m+2} < \frac{1}{3} < a_{2m+1} < a_{2m-1} < 1,$$

for every $m \in \mathbb{N}$.

The above consideration shows that Problem 2 in the case of the sequences defined in (28) and (29) has a different solution, so it is less connected to them than for the sequences in (3) and (4). Regarding the problem, we have the following result, whose proof follows from Proposition 4.

Corollary 1. *Let the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be defined in (28) and (29), respectively, and let $k \in \mathbb{N}_3$. For each $n \in \mathbb{N}$, divide the interval $[a_n, b_n]$ into 3^k equal subintervals. Then $\ln 3$ lies in the 3^{k-1}-th subinterval from the left for sufficiently large n.*

6 | CONCLUSION

Here we give two solutions to a problem of locating the limit of two sequences a_n and b_n inside each of the intervals $[a_n, b_n]$, and motivated by it, we start investigating convex combinations of two pairs of classical sequences in detail. For each value of the convexity parameter, we managed to determine the exact value of the index at which the corresponding convex combination changes the monotonicity and determine the type of the monotonicity, that is, if it is increasing, decreasing, or eventually increasing or decreasing. The problems studied in the paper could be a starting point for further investigations in the topic.

AUTHOR CONTRIBUTIONS

The paper is based on some ideas and analyses by the author (Stevo Stević).

ACKNOWLEDGEMENTS

There are no funders to report for this submission.

CONFLICT OF INTEREST STATEMENT

This work does not have any conflict of interest.

ORCID

Stevo Stević https://orcid.org/0000-0002-7202-9764

REFERENCES

1. A. de Moivre, *De Fractionibus Algebraicis Radicalitate immunibus ad Fractiones Simpliciores reducendis, deque summandis Terminis quarumdam Serierum aequali Intervallo a se distantibus*, Philos. Trans. **32** (1722), 162–178. (in Latin).
2. A. de Moivre, *Miscellanea analytica de seriebus et quadraturis*, 1730. (in Latin).
3. D. Bernoulli, *Observationes de seriebus quae formantur ex additione vel substractione quacunque terminorum se mutuo consequentium, ubi praesertim earundem insignis usus pro inveniendis radicum omnium aequationum algebraicarum ostenditur*, Commentarii Acad. Petropol. **III** (1732), no. 1728, 85–100. (in Latin).
4. L. Eulero, *Introductio in Analysin Infinitorum, Tomus Primus*, 1748. (in Latin).
5. J.-L. Lagrange, *OEuvres*, t. II, Gauthier-Villars, Paris, 1868. (in French).
6. P. S. Laplace, *Recherches sur l'intégration des équations différentielles aux différences finies et sur leur usage dans la théorie des hasards*. Mémoires de l' Académie Royale des Sciences de Paris, 1773, t. VII, (1776) (Laplace OEuvres, VIII, 69-197, 1891). (in French).
7. S. F. Lacroix, *Traité des differénces et des séries*, J. B. M. Duprat, Paris, 1800. (in French).
8. C. Jordan, *Calculus of finite differences*, Chelsea Publishing Company, New York, 1965.
9. H. Levy and F. Lessman, *Finite difference equations*, The Macmillan Company, New York, NY, USA, 1961.
10. D. S. Mitrinović and J. D. Kečkić, *Metodi Izračunavanja Konačnih Zbirova*, Beograd, Naučna Knjiga, 1984. (in Serbian).
11. I. Bajo and E. Liz, *Global behaviour of a second-order nonlinear difference equation*, J. Differ. Equat. Appl. **17** (2011), no. 10, 1471–1486.
12. K. Berenhaut and S. Stević, *The behaviour of the positive solutions of the difference equation* $x_n = A + (x_{n-2}/x_{n-1})^p$, J. Difference Equ. Appl. **12** (2006), no. 9, 909–918.
13. L. Berg and S. Stević, *On the asymptotics of the difference equation* $y_n(1 + y_{n-1} \cdots y_{n-k+1}) = y_{n-k}$, J. Differ. Equ. Appl. **17** (2011), no. 4, 577–586.
14. G. Papaschinopoulos and C. J. Schinas, *Invariants for systems of two nonlinear difference equations*, Differ. Equ. Dyn. Syst. **7** (1999), 181–196.
15. G. Papaschinopoulos and C. J. Schinas, *Invariants and oscillation for systems of two nonlinear difference equations*, Nonlinear Anal. Theory Methods Appl. **46** (2001), 967–978.
16. G. Papaschinopoulos, C. J. Schinas, and G. Stefanidou, *On a k-order system of Lyness-type difference equations*, Adv. Difference Equ. **2007** (2007), 31272.
17. G. Papaschinopoulos and G. Stefanidou, *Asymptotic behavior of the solutions of a class of rational difference equations*, Inter. J. Difference Equ. **5** (2010), no. 2, 233–249.
18. C. Schinas, *Invariants for difference equations and systems of difference equations of rational form*, J. Math. Anal. Appl. **216** (1997), 164–179.
19. S. Stević, *Solutions of a max-type system of difference equations*, Appl. Math. Comput. **218** (2012), 9825–9830.
20. S. Stević, *On the system of difference equations* $x_n = c_n y_{n-3}/(a_n + b_n y_{n-1} x_{n-2} y_{n-3})$, $y_n = \gamma_n x_{n-3}/(\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3})$, Appl. Math. Comput. **219** (2013), 4755–4764.
21. S. Stević, J. Diblik, B. Iričanin, and Z. Šmarda, *On a solvable system of rational difference equations*, J. Differ. Equ. Appl. **20** (2014), no. 5-6, 811–825.
22. S. Stević, J. Diblik, B. Iričanin, and Z. Šmarda, *Solvability of nonlinear difference equations of fourth order*, Electr. J. Differ. Equ. **2014** (2014), 264.
23. S. Stević, B. Iričanin, and Z. Šmarda, *On a product-type system of difference equations of second order solvable in closed form*, J. Inequal. Appl. **2015** (2015), 327.
24. D. Adamović, *Collected works*, University of Belgrade, Belgrade, 2009.
25. M. I. Bashmakov, B. M. Bekker, and V. M. Gol'hovoi, *Zadachi po Matematike. Algebra i Analiz*, Moskva, Nauka, 1982. (in Russian).
26. A. Y. Dorogovtsev Ed., *Matematika Segodnya '90, Edited by* A. Y. Dorogovtsev, 1990. (in Russian).
27. L. Eulero, *De progressionibvs harmonicis observationes*, Commentarii Academiae Scientiarum Petropolitanae **7** (1740), 150–161. (in Latin).
28. P. P. Korovkin, *Neravenstva, Popularnie lektsii po matematike*, 1983. (in Russian).
29. V. A. Krechmar, *A problem book in algebra*, Mir Publishers, Moscow, 1974.

30. D. S. Mitrinović, *Problem 108*, Mat. Vesnik **19** (1967), no. 4, 338. (in French).
31. D. S. Mitrinović, *Analytic inequalities*, Springer-Verlag, Berlin, Heidelberg New York, 1970.
32. D. S. Mitrinović and D. D. Adamović, *Nizovi i Redovi*, Naučna Knjiga, Beograd Serbia, 1980. (in Serbian).
33. G. Polya and G. Szegö, *Aufgaben und Lehrsätze aus der analysis*, Verlag von Julius Springer Berlin, 1925. (in German).
34. R. P. Ushakov, *Zadacha 3, Matematika Segodnya '90*, 1990. (in Russian).
35. R. P. Ushakov, *Reshenie Zadachi 3 iz Matematika Segodnya '90, Matematika Segodnya '94*, 1994. (in Russian).
36. N. Nielsen, *Handbuch der Theorie der Gammafunktion*, B. G. Teübner, Leipzig, 1906. (in German).
37. J. Hadamard, *Étude sur les propriétés des fonctions entiéres et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl. **58** (1893), 171–215. (in French).
38. Ch. Hermite, *Sur deux limites d'une intégrale définie*, Mathesis **3** (1883), 82. (in French).
39. V. A. Zorich, *Mathematical analysis*, vol. **I**, Springer, Berlin, Heidelberg, 2004.
40. S. Stević, *On the recursive sequence* $x_{n+1} = g(x_n, x_{n-1})/(A + x_n)$, Appl. Math. Lett. **15** (2002), 305–308.
41. S. Stević, *On the recursive sequence* $x_{n+1} = \alpha_n + (x_{n-1}/x_n)$ *II*, Dyn. Contin. Discrete Impuls. Syst. **10a** (2003), no. 6, 911–917.

How to cite this article: S. Stević, *Convex combinations of some convergent sequences*. Math. Meth. Appl. Sci. (2024), 1–14, DOI 10.1002/mma.10463.