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THE P-IDEAL DICHOTOMY, MARTIN'S AXIOM AND ENTANGLED SETS

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ABSTRACT

We build a model of the P-ideal dichotomy (PID) and Martin's axiom for ω_1 (MA_{ω_1}) in which there is a 2-entangled set of reals. In particular, it follows that the Open Graph Axiom or Baumgartner's axiom for ω_1-dense sets are not consequences of $\mathsf{PID} + \mathsf{MA}_{\omega_1}$. We review Neeman's iteration method using two type side conditions and provide an alternative proof for the preservation of properness.

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1. Introduction

The P-ideal dichotomy (PID) is one of the most important and strongest consequences of the Proper Forcing Axiom (PFA).[1] It was introduced in [48] by the second author and many applications of this dichotomy have been found since then. For example, PID implies the Suslin Hypothesis, that every gap in $\wp(\omega)/\mathsf{fin}$ is ccc-indestructible ([3], [48]), the bounding number is at most ω_2 ([51]), the Singular Cardinal Hypothesis ([57]) and every complete weakly distributive algebra $\mathbb{B}$ with the countable chain condition supports a strictly positive continuous submeasure ([7]). Another interesting aspect of the P-ideal dichotomy is that it is strong enough to imply the failure of several square principles. In [48] the second author proved that PID implies that $\square_\kappa$ fails for every uncountable cardinal κ. This was later improved by Raghavan in [39], where he proved that PID implies the failure of $\square_{\kappa,\omega}$ for all uncountable κ, as well as the failure of $\square_{\kappa,<\mathfrak{b}}$ for all κ such that

$$\mathsf{cof}(\kappa) > \omega_1.$$

It has been observed that under PID, several mathematical statements (not necessarily from set theory) become equivalent to an assertion regarding cardinal invariants. This program was initiated by Raghavan and the second author in [40] (see also [51]). In [40] the following general project was introduced:

Problem 1: Let φ be a consequence of PFA, Find a cardinal invariant $\mathfrak{j}$ such that φ and $\mathfrak{j} > \omega_1$ are equivalent under PID.

It is a remarkable result of the second author that PID is consistent with the Continuum Hypothesis (CH) (see [48]). The following quote is from [40]; "The problem asks if the influence of PFA on φ can be decomposed into a part which is consistent with CH and into another CH violating part that is precisely captured by the cardinal invariant $\mathfrak{j}$". We list some examples of this type:

Theorem 2 (Raghavan, Todorcevic [40]): *Under* PID*, the following statements are equivalent:*

(1) $\mathfrak{b} > \omega_1$.
(2) $\omega_1 \longrightarrow (\omega_1, \omega + 2)$.

[1] All the relevant undefined notions will be reviewed in the next sections.

Theorem 3 (Raghavan, Todorcevic [40]): *Under* PID, *the following statements are equivalent:*

(1) $\min\{\mathfrak{b}, \mathsf{cof}(\mathcal{F}_\sigma)\} > \omega_1$.
(2) *Every directed set of size at most* ω_1 *is Tukey equivalent to one of the following:* $1, \omega, \omega_1, \omega \times \omega_1$ *or* $[\omega_1]^{<\omega}$.

Theorem 4 (Borodulin-Nadzieja, Chodounský [12]): *Under* PID, *the following statements are equivalent:*

(1) $\mathfrak{b} > \omega_1$.
(2) *Every* ω_1*-tower is Hausdorff.*

Recall that a famous theorem of Cantor establishes that every two countable dense linear orders with no end-points are isomorphic. We may wonder about possible extensions of this result to uncountable cardinals. The straightforward generalization is false, but it may be true when restricted to subsets of reals in which all of its intervals have the same size. We say that $D \subseteq \mathbb{R}$ is **κ-dense** if $D \neq \emptyset$ has no end-points and for every $a, b \in D$ with $a < b$, the interval $(a, b) \cap D$ has size κ. The **Baumgartner Axiom for κ-dense sets** is the following assertion:

$\mathsf{BA}(\kappa)$: Every two κ-dense sets of reals are isomorphic.

Note that the theorem of Cantor mentioned above is simply $\mathsf{BA}(\omega)$. It is easy to see that $\mathsf{BA}(\mathfrak{c})$ is false (where $\mathfrak{c}$ is the cardinality of the continuum). Hence, $\mathsf{BA}(\omega_1)$ is consistently false. Nevertheless, the following is an impressive result of Baumgartner:

Theorem 5 (Baumgartner, [8][9]): PFA *implies* $\mathsf{BA}(\omega_1)$.

The reader may also consult [47], [52] or [53] for a proof. It is worth mentioning that the second author proved that $\mathsf{BA}(\mathfrak{b})$ is false (see [43] for a proof). It is currently unknown if $\mathsf{BA}(\mathfrak{p})$ is consistent. One of the major open problems in set theory is if $\mathsf{BA}(\omega_2)$ is consistent. A lot of progress on this problem has been done by Neeman, pointing to a positive solution. The reader may also consult the work of Moore and the second author ([36]) to learn more about $\mathsf{BA}(\omega_2)$. For more on the structure of uncountable linear orders, the reader may look at [50], [52], [33], [34], [27] and [26]. In [43] Steprāns and Watson studied topological versions of the Baumgartner axiom in $\mathbb{R}^n$.

Recall that a **complete set** in a graph is a set in which any two elements are connected, while an **independent set** is a set in which no two elements are connected. The **chromatic number** of a graph is the smallest size of a family of independent sets that covers the set of vertices. It is natural to wonder when a graph has countable chromatic number. Obviously this is impossible if there is an uncountable complete set. Although this is a sufficient condition, in general is far from necessary. Surprisingly, the existence of an uncountable complete subgraph may be the only obstruction for some "topologically nice" graphs. In his book *Partition Problems in Topology*, the author introduced the **Open Graph Axiom** (OGA), which is the following dichotomy:

OGA: Let X be a second countable space and $G \subseteq [X]^2$ an open graph. One of the following conditions holds:
(1) X contains an uncountable complete set.
(2) The chromatic number of G is at most countable.

The Open Graph Axiom is a remarkable dichotomy with many strong consequences. Just to name a few: all automorphisms of the Calkin algebra of a separable Hilbert space are inner (see [17]), the bounding number is exactly ω_2, if $\mathcal{G}$ is a (κ, λ)-gap in $\wp(\omega)/\mathsf{fin}$ with both κ and λ regular cardinals, then $\kappa = \lambda = \omega_1$, every uncountable Boolean algebra contains an uncountable set of pairwise disjoint elements, for every real valued function with an uncountable domain, there is an uncountable set in which it is monotone (see [47] and [53]). OGA has also very strong consequences on the quotients $\wp(\omega)/\mathcal{I}$ where $\mathcal{I}$ is an analytic ideal on ω (see [16]).

Theorem 6 ([47]): PFA *implies* OGA.

To learn more about the Open Graph Axiom, the reader may consult [47], [53], [51], [35], [32], [49], [54], [15] and [28] among many others.

Given the importance of both OGA and $\mathsf{BA}(\omega_1)$ and in light of the program described at the beginning, we may wonder if those principles are equivalent (under PID) to a cardinal inequality as described earlier. We will show that this is not the case for the usual cardinal invariants (like the ones described in [11]). More formally, we will prove the following:

Theorem 7 (LC): MA_{ω_1}+PID *do not imply* OGA *or* $\mathsf{BA}(\omega_1)$.

Above, by LC we denote a large cardinal hypothesis. The existence of a supercompact cardinal is enough for us. In this way, if there is a cardinal invariant related to $\mathsf{BA}(\omega_1)$ or OGA, it will not be possible to increase it with ccc forcings, which is the case for most of the cardinal invariants one finds in practice (of course, there might still be an interesting, non-artificial cardinal invariant with these properties).

In order to prove Theorem 7, we will show that $\mathsf{MA}_{\omega_1} + \mathsf{PID}$ is consistent with the existence of a 2-entangled set of reals (the definition of entangled set and its main properties will be reviewed in a later section). Since both OGA and $\mathsf{BA}(\omega_1)$ forbid the existence of 2-entangled sets of reals, clearly Theorem 7 will follow.

The paper is organized as follows: Section 2 contains the preliminaries. In Section 3, we present the basic notions and results regarding entangled sets of reals. In Section 4, we prove that for every partial order destroying a given 2-entangled set, there is a proper forcing that adds an uncountable antichain to the former. Abraham and Shelah proved that there is a ccc forcing with this property under the Continuum Hypothesis. Our forcing is not ccc, but it is proper and exists in any model, independently if CH holds or not. In Section 5, we prove that the usual side condition poset for forcing an instance of the P-ideal dichotomy preserves entangled sets. In Section 6, for every proper forcing $\mathbb{P}$, we introduce its "side condition hull", which is a proper forcing with side conditions in which $\mathbb{P}$ embeds. In Section 7, we review the technique of forcing with two type side conditions introduce by Neeman in [37]. Most of the section is devoted to studying this technique. Nevertheless, there are some new results, like a decomposition of the successor steps in the Neeman iteration, as well as a new proof of the preservation of properness. Part of this section is based on a graduate course the second author taught at the University of Toronto in 2019. In Section 8, we prove the preservation theorem for 2-entangled sets under Neeman iteration and finish the proof of Theorem 7. We list some open questions in Section 9.

2. Preliminaries and notation

Most of our definitions and notation are standard, but for the convenience of the reader, in this section we will review some notions that will be used through the paper.

Definition 8: Let X be a set and $A, B \subseteq X$. We say that B is an **almost subset** of A (which we denote as $B \subseteq^* A$) if $B \setminus A$ is finite.

We now recall the notions of ideal and P-ideal, which are fundamental concepts in infinite combinatorics.

Definition 9: Let X be a set and $\mathcal{I} \subseteq \wp(X)$.

(1) We say that $\mathcal{I}$ is an **ideal** if the following conditions hold:
   (a) $\emptyset \in \mathcal{I}$ and $X \notin \mathcal{I}$.
   (b) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.
   (c) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
   (d) $[X]^{<\omega} \subseteq \mathcal{I}$.

(2) Let $\mathcal{I}$ be an ideal. We say that $\mathcal{I}$ is a ***P*-ideal** if for every countable family $\mathcal{B} \subseteq \mathcal{I}$, there is $A \in \mathcal{I}$ such that $B \subseteq^* A$ for every $B \in \mathcal{B}$ (in this case, we say that A is a **pseudounion** of $\mathcal{B}$).

(3) $\mathcal{I}^\perp = \{S \subseteq X \mid \forall A \in \mathcal{I}(|A \cap S| < \omega)\}$.

(4) $\mathcal{I}^+ = \wp(X) \setminus \mathcal{I}$.

We will be mainly interested in the case where $\mathcal{I}$ is an ideal of countable sets (i.e., $\mathcal{I} \subseteq [X]^{\leq\omega}$). The ***P*-ideal dichotomy** (PID) is the following dichotomy:

PID: Let X be a set and and $\mathcal{I} \subseteq [X]^{\leq\omega}$ a P-ideal. One of the following conditions holds:
   (1) There is $Y \in [X]^{\omega_1}$ such that $[Y]^\omega \subseteq \mathcal{I}$.
   (2) There is $\{Z_n \mid n \in \omega\} \subseteq \mathcal{I}^\perp$ such that $X = \bigcup_{n\in\omega} Z_n$.

It was proved by the second author that PFA implies PID. To learn more about PID, the reader may consult [3], [48], [51], [52], [35], [48], [50], [25], [13], [30] and [39] among others.

Let κ be a cardinal. The **Martin axiom** (MA) **for** κ is the following statement:

MA_κ: Let $\mathbb{P}$ be a ccc partial order. If $\mathcal{D}$ is a family of open dense subsets of $\mathbb{P}$ and $|\mathcal{D}| \leq \kappa$, then there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

Martin's axiom (MA) is the statement that MA_κ holds for all $\kappa < \mathfrak{c}$ (by $\mathfrak{c}$ we denote the size of the continuum). It is easy to see that MA_ω is true and $\mathsf{MA}_\mathfrak{c}$ is false. To learn more about MA, the reader may consult [18], [24], [22], [10] and [53].

Let X be a set. We say that $T \subseteq X^{<\omega}$ is a **tree** if T is closed under taking initial segments. If $s, t \in X^{<\omega}$ by $s^\frown t$ we denote the **concatenation** of s and t. If $T \subseteq X^{<\omega}$ is a tree and $s \in T$, define $\mathsf{suc}_T(s) = \{x \in X \mid s^\frown\langle x\rangle \in T\}$. By $[T]$ we denote the set of **branches** of T, which is the set of all maximal paths through T. If $W \subseteq X^{<\omega}$, the **tree closure** of W is obtained by closing W under initial segments.

If $\mathbb{P}$ is a forcing and M is a countable elementary submodel of a large enough structure with $\mathbb{P} \in M$, we say that $p \in \mathbb{P}$ is an $(M, \mathbb{P})$**-generic condition** if for every $D \subseteq \mathbb{P}$ open dense with $D \in M$, the set $D \cap M$ is predense below p. The following equivalence of generic conditions is often useful and may be considered folklore:[2]

Lemma 10: *Let $\mathbb{P}$ be a forcing, $p \in \mathbb{P}$, θ a large enough regular cardinal and M an elementary submodel of $\mathsf{H}(\theta)$ with $\mathbb{P} \in M$. The following are equivalent:*

(1) *p is an $(M, \mathbb{P})$-generic condition.*
(2) *For every $E \subseteq \mathbb{P}$ with $E \in M$ and $q \leq p$, if $q \in E$, then there is $r \in E \cap M$ that is compatible with q.*

We will say that p is a **strong** $(M, \mathbb{P})$**-generic condition** if for every dense $D \subseteq \mathbb{P} \cap M$, we have that D is predense below p (in general, $D \notin M$). We say that $\mathbb{P}$ is (**strongly**) **proper** for M if every $q \in \mathbb{P} \cap M$ can be extended to a (strong) $(M, \mathbb{P})$-generic condition. A forcing is (strongly) proper if it is (strongly) proper for every countable elementary submodel of a large enough structure.

Let $\mathbb{P}$ be a partial order, M an elementary submodel of some $\mathsf{H}(\lambda)$ with $\mathbb{P} \in M$ and $G \subseteq \mathbb{P}$ a generic filter. Define

$$M[G] = \{\dot{a}[G] \mid \dot{a} \in M\}.$$

Since the forcing relation is definable, it follows that $M[G]$ is an elementary submodel of $\mathsf{H}(\lambda)[G]$ (for more details, see [41] and [1]). If G does not contain an $(M, \mathbb{P})$-generic condition, $M[G]$ will not be a forcing extension of M. Nevertheless, it is still a model and may be useful in some situations. We will often use the following result (for a proof, see [41, Chapter I, Claim 5.17 and Chapter III Theorem 2.11]):

[2] In the book [52] the condition (2) in the Lemma is taken as the definition of an $(M, \mathbb{P})$-generic condition.

Proposition 11: *Let λ be a regular cardinal and $\mathbb{P}$ a forcing such that $\mathbb{P}\in\mathsf{H}(\lambda)$. If $G\subseteq\mathbb{P}$ is a generic filter, then the following holds:*

(1) $\mathsf{H}^V(\lambda)[G]=\mathsf{H}^{V[G]}(\lambda)$.
(2) *If $M\preceq\mathsf{H}(\lambda)$ and $\mathbb{P}\in M$, then $M[G]\preceq\mathsf{H}^{V[G]}(\lambda)$.*

Let X be a set. We say that $\mathcal{C}\subseteq[X]^\omega$ is a **club** if it is cofinal and closed under countable directed unions. Let μ be a cardinal, we say that $\mathcal{S}\subseteq[X]^{<\mu}$ is **stationary** if for every $f:X^{<\omega}\longrightarrow X$, there is an element of $\mathcal{S}$ that is closed under f. It is worth noting that there is no real need to mention X at all. If $\mathcal{S}$ is a family of sets of size less than μ, then $\mathcal{S}$ is stationary if for every function $f:(\bigcup\mathcal{S})^{<\omega}\longrightarrow\bigcup\mathcal{S}$, there is $M\in\mathcal{S}$ that is closed under $\mathcal{S}$.

3. Basic properties of entangled sets of reals

The notion of entangled sets of reals was introduced by Abraham and Shelah in [6] in order to prove that $\mathsf{BA}(\omega_1)$ does not follow by MA_{ω_1}. We will start by recalling this notion and some of its main properties. Let $a,b\in[\omega_1]^{<\omega}$. By $a<b$ we mean that $\max(a)<\min(b)$. We say that $\mathcal{B}=\{b_\alpha\mid\alpha\in\omega_1\}\subseteq[\omega_1]^{<\omega}$ is a **block-sequence** if $\alpha<\beta$ implies that $b_\alpha<b_\beta$. Given $a\in[\omega_1]^m$, whenever we take an enumeration $a=\{a(i)\mid i<m\}$, we implicitly assume that $a(i)<a(j)$ if $i<j$. By a **type** we mean a function

$$t:m\longrightarrow\{>,<\}$$

(where $m\in\omega$).

Definition 12: Let $E=\{e_\alpha\mid\alpha\in\omega_1\}\subseteq\mathbb{R}$, $m\in\omega$, $t:m\longrightarrow\{>,<\}$ a type and $a,b\in[\omega_1]^m$ disjoint.

(1) We say that (a,b) **realizes** t (over E) if for every $i<m$ the following holds:
$$e_{a(i)}\,t(i)\,e_{b(i)}.$$
(2) By $T(a,b)$ we denote the (unique) type realized (over E) by (a,b).

We will omit the phrase "over E" whenever E is clear by context.[3] We can now define the entangled sets:

[3] By convention, if a and b are not disjoint, their type is not defined.

Definition 13: Let $E = \{e_\alpha \mid \alpha \in \omega_1\} \subseteq \mathbb{R}$ and $m \in \omega$.

(1) E is m**-entangled** if for every block sequence $\mathcal{B} = \{b_\alpha \mid \alpha \in \omega_1\} \subseteq [\omega_1]^m$ and for every type $t : m \longrightarrow \{>, <\}$ there are $\alpha \neq \beta$ such that

$$T(b_\alpha, b_\beta) = t.$$

(2) E is **entangled** if it is n-entangled for every $n \in \omega$.

Entangled sets are very interesting objects with very strong combinatorial properties. In this article, we only defined entangled sets of size ω_1 (since those are relevant for our work) but it is worth pointing out that this notion extends to other cardinals and other linear orders; we refer the reader to [47] and [45] to learn more. Some theorems regarding entangled sets are the following:

(1) Every uncountable set of reals is 1-entangled.
(2) (Abraham, Shelah [6]) Adding ω_1-Cohen reals adds an entangled set.
(3) (Abraham, Shelah [6]) MA_{ω_1} implies that there are no entangled sets.
(4) (Abraham, Shelah [6]) For every $m \in \omega$, the statement "MA_{ω_1}+ There is an m-entangled set" is consistent.
(5) (Todorcevic [45]) If there is an entangled set, then there are two ccc partial orders whose product is not ccc.
(6) (Todorcevic [45]) If $\mathsf{cof}(\mathfrak{c}) = \omega_1$, then there is an entangled set.
(7) (Todorcevic [47] (page 55), see also [46]) Adding a single Cohen real or random real adds an entangled set.
(8) Using the proof of the theorem above, it can be shown that $\mathsf{cov}(\mathcal{M}) > \omega_1 + \upharpoonleft$ implies that there is an entangled set (recall that $\upharpoonleft$ is the following statement: "There is a family $\mathcal{S} = \{S_\alpha \mid \alpha \in \omega_1\} \subseteq [\omega_1]^\omega$ such that for every $A \in [\omega_1]^{\omega_1}$ there is $\alpha \in \omega_1$ such that $S_\alpha \subseteq A$").
(9) (Miyamoto, Yorioka [31]) For every $m \in \omega$, the statement

$$\text{“}\mathsf{PFA}^{\text{s-fin}}(\omega_1) + \text{ There is an } m\text{-entangled set”}$$

is consistent.[4]
(10) (Chodounský, Zapletal [13]) YPFA is consistent with the existence of an entangle set.[5]

[4] $\mathsf{PFA}^{\text{s-fin}}(\omega_1)$ is a weakening of the axiom $\mathsf{PFA}^{\text{fin}}(\omega_1)$ introduced by Aspero and Mota in [4]. The reader may consult [4] and [31] for the definitions of this axioms.

[5] YPFA is the forcing axiom for the class of Y*-proper* forcings. The reader may consult [13] for the definition of Y-properness.

The following proposition is very well-known, but we prove it here for the sake of completeness and because of the relevance to our Theorem 7. The part of $\mathsf{BA}(\omega_1)$ is due to Abraham and Shelah and the part of OGA is due to the second author.

Proposition 14: *If there is a 2-entangled set of reals, then both* $\mathsf{BA}(\omega_1)$ *and* OGA *fail.*

Proof. Let $E = \{e_\alpha \mid \alpha \in \omega_1\} \subseteq \mathbb{R}$ be a 2-entangled set. Let A, B be two disjoint uncountable subsets of ω_1. Define $E_A = \{e_\alpha \mid \alpha \in A\}$ and $E_B = \{e_\beta \mid \beta \in B\}$. We can find $X \subseteq E_A$ and $Y \subseteq E_B$ such that both are ω_1-dense. We claim that X and Y are not isomorphic (as linear orders). Let $f : X \longrightarrow Y$ be an injective function. We find a block-sequence $\mathcal{B} = \{b_\alpha \mid \alpha \in \omega_1\} \subseteq f$. Define the type $t : 2 \longrightarrow \{>, <\}$ given by $t(0)$ is $>$ and $t(1)$ is $<$. Since E is 2-entangled, we can find $\alpha \neq \beta$ such that $T(b_a, b_\beta) = t$. This means that $e_{b_\alpha(0)} > e_{b_\beta(0)}$ and $e_{b_\alpha(1)} < e_{b_\beta(1)}$ (where $b_\alpha = \{b_\alpha(0), b_\alpha(1)\}$ and $b_\alpha = \{b_\beta(0), b_\beta(1)\}$), both listed in increasing order. By definition, we know that

$$e_{b_\alpha(1)} = f(e_{b_\alpha(0)}) \quad \text{and} \quad e_{b_\beta(1)} = f(e_{b_\beta(0)}).$$

Hence, $e_{b_\alpha(0)} > e_{b_\beta(0)}$ but $f(e_{b_\alpha(0)}) < f(e_{b_\beta(0)})$ which implies that f is not an isomorphism (note that the argument in fact proves that there are no embeddings between two disjoint uncountable subsets of E). In this way we get the failure of $\mathsf{BA}(\omega_1)$.

We now turn our attention to the Open Graph Axiom. Let $f : E \longrightarrow E$ be an injective function without fixed points. For every $\alpha \in \omega_1$, define

$$b_\alpha = \{e_\alpha, f(e_\alpha)\}.$$

Let

$$X = \{(e_\alpha, f(e_\alpha)) \mid \alpha \in \omega_1\} \subseteq \mathbb{R}^2.$$

Define the graph $G \subseteq [X]^2$ where $(e_\alpha, f(e_\alpha))$ and $(e_\beta, f(e_\beta))$ are connected if and only if $f \restriction \{e_a, e_\beta\}$ is increasing. Let $W \subseteq X$ be uncountable; we claim that W is not complete or independent. Take $A \in [\omega_1]^{\omega_1}$ such that $\mathcal{B} = \{b_\alpha \mid \alpha \in A\}$ is a block-sequence such that $(e_{b_\alpha(0)}, e_{b_\alpha(1)}) \in W$ for every $\alpha \in W$. Since E is 2-entangled, we know every type is realized in $\mathcal{B}$, which implies that W is not complete or independent. This implies that OGA can not be true. ∎

For the rest of the section, we will prove some simple facts about entangled sets that will be helpful in future sections. We will often use implicitly the next simple observation:

LEMMA 15: *Let $\mathcal{A}$ be an uncountable subset of $[\omega_1]^m$.*

(1) *If $\{\min(a) \mid a \in \mathcal{A}\}$ is uncountable, then $\mathcal{A}$ contains an uncountable block-sequence.*
(2) *In particular, if M is a countable elementary submodel, $\mathcal{A} \in M$ and there is $a \in M$ such that $a \cap M = \emptyset$, then $\mathcal{A}$ contains an uncountable block-sequence.*

The following notions will be very useful:

Definition 16: Let $E = \{e_\alpha \mid \alpha \in \omega_1\} \subseteq \mathbb{R}$ and $m \in \omega$.

(1) Let $\mathcal{U} = \langle U_i \rangle_{i<m}$ and $b = \{b(i) \mid i < m\} \in [\omega_1]^m$. We say that $\mathcal{U}$ **covers** b if the following conditions hold:
   (a) $U_0, \ldots, U_{m-1}$ are disjoint rational intervals.
   (b) $e_{b(i)} \in U_i$ for every $i < m$.
(2) Let $\mathcal{B} = \{b_\alpha \mid \alpha \in \omega_1\} \subseteq [\omega_1]^m$ be a block-sequence. We say that $\mathcal{B}$ is ω_1**-dense** if for every $\mathcal{U} = \langle U_i \rangle_{i<m}$, if there is $\alpha \in \omega_1$ such that $\mathcal{U}$ covers b_α, then there are uncountable many $\gamma \in \omega_1$ such that $\mathcal{U}$ covers b_γ.
(3) Let $\mathcal{U} = \langle U_i \rangle_{i<m}$, $\mathcal{V} = \langle V_i \rangle_{i<m}$ and $a, b \in [\omega_1]^m$ disjoint. We say that $(\mathcal{U}, \mathcal{V})$ **freezes** (a, b) if the following conditions hold:
   (a) $U_i \cap V_j = \emptyset$ for every $i, j < m$.
   (b) $\mathcal{U}$ covers a.
   (c) $\mathcal{V}$ covers b.
   (d) For every $c, d \in [\omega_1]^{<\omega}$ if $\mathcal{U}$ covers c and V covers d, then
   $$T(a, b) = T(c, d)$$
   (note that this condition follows from points (a), (b) and (c) above, but we wrote it because it is useful to keep it in mind).
(4) Let $\mathcal{U} = \langle U_i \rangle_{i<m}$ be a sequence of rational open intervals and $b \in [\omega_1]^m$. If $T(a, b) = t$ holds for every a that is covered by $\mathcal{U}$ (where $t : m \longrightarrow \{>, <\}$), then we will denote this fact by $T(\mathcal{U}, b) = t$.

Note that every block-sequence contains one that is ω_1-dense. When working with entangled sets, it is often useful to use ω_1-dense block-sequences. We have the following:

Lemma 17: *Let* $E = \{e_\alpha \mid \alpha \in \omega_1\} \subseteq \mathbb{R}$ *and* $m \in \omega$. *The following are equivalent:*

1. E *is* m*-entangled.*
2. *For every block-sequence* $\mathcal{B} = \{b_\alpha \mid \alpha \in \omega_1\} \subseteq [\omega_1]^m$ *and for every type* $t : m \longrightarrow \{>, <\}$ *there are* $\alpha < \beta$ *such that* $T(b_\alpha, b_\beta) = t$.

Proof. The only difference between points (1) and (2) is that in item (2) we require that $\alpha < \beta$ and in (1) only that $\alpha \neq \beta$. Clearly item (2) implies item (1). Assume E is m-entangled, we will prove that it satisfies the extra requirement in point (2). Let $\mathcal{B} = \{b_\alpha \mid \alpha \in \omega_1\} \subseteq [\omega_1]^m$ be a block-sequence and $t : m \longrightarrow \{>, <\}$ a type. We may assume that $\mathcal{B}$ is ω_1-dense.

Since E is m-entangled, we can find $\alpha, \beta \in \omega_1$ (with $\alpha \neq \beta$) such that $T(b_\alpha, b_\beta) = t$. Now, let $\mathcal{U}$ and $\mathcal{V}$ be sequences of rational intervals freezing (b_α, b_β). Since $\mathcal{B}$ is ω_1-dense, we can find $\gamma \in \omega_1$ such $\gamma > \alpha$ and $\mathcal{V}$ covers b_γ. It follows that $T(b_\alpha, b_\gamma) = t$ and we are done. ∎

The following proposition is due to the second author and was published in [31] as Proposition 2.2.

Proposition 18: *Let* $m \in \omega$, $E = \{e_\alpha \mid \alpha \in \omega_1\} \subseteq \mathbb{R}$ *an* m*-entangled set and* M *a countable elementary submodel such that* $E \in M$. *Let* $\mathcal{W} \subseteq [\omega_1]^m$ *with the following properties:*

1. $\mathcal{W} \in M$.
2. *There is* $b \in \mathcal{W}$ *such that* $b \cap M = \emptyset$.

For every type $t : m \longrightarrow \{>, <\}$ *there is* $a \in M \cap \mathcal{W}$ *such that* $T(a, b) = m$.

We will use the following notions in the next section:

Definition 19: Let E be a 2-entangled set and $\mathbb{P}$ a partial order.

1. We say that $\mathbb{P}$ **destroys** E if $\mathbb{P} \Vdash$"E is not 2-entangled".
2. We say that $\mathbb{P}$ **preserves** E if $\mathbb{P} \Vdash$"E is 2-entangled".

Obviously, a forcing collapsing ω_1 will destroy all 2-entangled sets. Furthermore, since OGA can be forced with a proper forcing, it follows that every 2-entangled set can be destroyed with a proper forcing. Moreover, if V is a model of CH, then the relevant instances of OGA can be forced using a ccc partial order (see [47]) so under the Continuum Hypothesis, every 2-entangled set can be destroyed with a ccc partial order.

It is easy to see that the property of preserving E is preserved under finite support iteration of ccc partial orders (see [6]). Regarding proper forcing, we have the following equivalence:

Proposition 20: *Let $\mathbb{P}$ be a proper forcing and $E = \{e_\alpha \mid \alpha \in \omega_1\}$ a 2-entangled set. The following are equivalent:*

1. *$\mathbb{P}$ preserves E.*
2. *Let λ be a large enough regular cardinal, $\dot{\mathcal{B}}$ a $\mathbb{P}$-name for a subset of $[\omega_1]^2$, M a countable elementary submodel of $\mathsf{H}(\lambda)$ such that $\mathbb{P}, E, \dot{\mathcal{B}} \in M$. If $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic, $t : 2 \longrightarrow \{>, <\}$ is a type, $b \in [\omega_1]^2$ is such that $p \Vdash$ "$b \in \dot{\mathcal{B}}$" and $b \cap M = \emptyset$, then there are $q \in \mathbb{P} \cap M$ and $a \in [\omega_1]^2 \cap M$ such that $q \Vdash$ "$a \in \dot{\mathcal{B}}$", p and q are compatible and $T(a, b) = t$.*

Proof. We will first prove that (1) implies (2). Let $G \subseteq \mathbb{P}$ be a generic filter with $p \in G$. We go to $V[G]$. Since p is an $(M, \mathbb{P})$-generic condition, we know that $M[G]$ is a forcing extension of M and it is a countable elementary submodel of $\mathsf{H}^{V[G]}(\lambda)$ (see Proposition 11). Since $\dot{\mathcal{B}}[G] \in M[G]$, E is 2-entangled in $V[G]$ (since $\mathbb{P}$ preserves E), $b \in \dot{\mathcal{B}}[G]$ and $b \cap M[G] = \emptyset$ (since M and $M[G]$ have the same ordinals), by Proposition 18, there is $a \in M[G] \cap \dot{\mathcal{B}}[G]$ such that $T(a, b) = t$. Since $M[G]$ is a forcing extension of M, there is $q \in M \cap G$ such that $q \Vdash$ "$a \in \dot{\mathcal{B}}$". Since both p and q are in the generic filter, they are compatible.

We will now prove that (2) implies (1). Let $r \in \mathbb{P}$, $\dot{\mathcal{B}}$ a $\mathbb{P}$-name for an uncountable block sequence of $[\omega_1]^2$ and a type $t : 2 \longrightarrow \{>, <\}$. We need to extend r to a condition forcing that t is realized in $\dot{\mathcal{B}}$. Let λ be a large enough regular cardinal, M a countable elementary submodel of $\mathsf{H}(\lambda)$ such that $\mathbb{P}, E, \dot{\mathcal{B}}, r \in M$. Since $\mathbb{P}$ is a proper forcing, we can find $p_1 \leq r$ such that p_1 is $(M, \mathbb{P})$-generic. We now find a further extension $p \leq p_1$ and $b \in [\omega_1]^2$ such that $p \Vdash$ "$b \in \dot{\mathcal{B}}$" and $b \cap M = \emptyset$. By point (2), we know that there are $q \in \mathbb{P} \cap M$ and $a \in [\omega_1]^2 \cap M$ such that $q \Vdash$ "$a \in \dot{\mathcal{B}}$", p and q are compatible and $T(a, b) = t$. A common extension of both p and q is the condition we are looking for. ∎

4. Destroying "bad" partial orders with side conditions

We mentioned before that Abraham and Shelah proved that the existence of a 2-entangled set is consistent with MA_{ω_1}. The key result for their argument is the following:

THEOREM 21 (Abraham, Shelah [6]): *Assume the Continuum Hypothesis and let E be a 2-entangled set. If $\mathbb{P}$ is a ccc partial order that destroys E, then there is a partial order $\mathbb{Q}$ with the following properties:*

(1) $\mathbb{Q}$ *is ccc.*
(2) $\mathbb{Q}$ *preserves E.*
(3) $\mathbb{Q}$ *adds an uncountable antichain to $\mathbb{P}$.*

With the knowledge of this Theorem, it is now easy to build a model of MA_{ω_1} where there is a 2-entangled set. We start with a model of GCH and we choose E a 2-entangled set (in [6] it was forced by adding ω_1-Cohen reals, but we now know that CH already implies that there is a 2-entangled set, see [45]). We perform a finite support iteration of length ω_2 and we use a suitable bookkeeping device that will be handing us ccc partial orders in order to force MA_{ω_1}. However, at every step of the iteration, if the partial order given to us by the bookkeeping device is a ccc partial order that destroys E, instead of forcing with it, we will add an uncountable antichain to it using the proposition above (see [6] for more details). The reader may consult [2] and [42] for a deeper discussion on constructing models of Martin's axiom.

The aim of this section is to prove a result similar to Theorem 21 but with some key differences: our forcing $\mathbb{Q}$ will be proper instead of ccc, however, its existence does not depend on the Continuum Hypothesis. Moreover, we use the method of "models as side conditions", which is a very powerful method developed by the second author in order to build proper partial orders (see [47], [52] and [35] to learn more about this method). The situation resembles the one with the Open Graph Axiom. It is known that OGA can be forced with a ccc partial order under CH (plus a diamond principle, see [47]) or with a proper forcing using side conditions (see [52]). While working with OGA, it is often useful to keep in mind these two different approaches, we expect that the situation will be similar with entangled sets.

It is worth pointing out that our forcing shares some similarities with the one introduced by Miyamoto and Yorioka in [31]. Our forcing is simpler, but this is because here we are dealing with ccc partial orders, while the authors of [31] are working with s-finitely proper forcings.

For the rest of this section, we fix $E = \{e_\alpha \mid \alpha \in \omega_1\} \subseteq \mathbb{R}$, $\mathbb{Q}$ a partial order, $\dot{\mathcal{B}}$, $(\kappa, <_W)$, a type $t : 2 \longrightarrow \{>, <\}$ and $h : ([\omega_1]^2)^3 \longrightarrow 2$ with the following properties:

(1) E is 2-entangled.
(2) $\mathbb{Q}$ is a ccc partial order that destroys E.
(3) Moreover, $\dot{\mathcal{B}} = \{\dot{b}_\alpha \mid \alpha \in \omega_1\}$ is a $\mathbb{Q}$-name for an ω_1-dense block-sequence such that if $\alpha, \beta \in \omega_1$, then

$$\mathbb{Q} \Vdash \text{“}T(\dot{b}_\alpha, \dot{b}_\beta) \neq t\text{”}.$$

(4) $\kappa > (2^{|\mathbb{Q}|})^+$ is a large enough regular cardinal and $<_w$ is a well-order of $\mathsf{H}(\kappa)$.
(5) The function $h : ([\omega_1]^2)^3 \longrightarrow 2$ is defined as follows: given $\overline{s}, \overline{z} \in ([\omega_1]^2)^3$ define $h(\overline{s}, \overline{z}) = 0$ if and only if the following conditions hold:
   (a) $\overline{s}$ and $\overline{z}$ are block-sequences.
   (b) There are $a \in \overline{s}$ and $b \in \overline{z}$ such that $T(a, b) = t$.

Note that we are only assuming that E is 2-entangled, we do not need it to be entangled. For this section, given $M \in \mathsf{H}(\kappa)$ with $\mathbb{Q}$, E, $\dot{\mathcal{B}} \in M$, we write $M \preceq \mathsf{H}(\kappa)$ to denote that $(M, \in, <_W)$ is an elementary submodel of $(\mathsf{H}(\kappa), \in, <_W)$.

Definition 22: Let $M \preceq \mathsf{H}(\kappa)$ be countable, $m \in \omega$ and $\mathcal{D} = \{d_i \mid i < m\} \subseteq [\omega_1]^2$ be a block-sequence. We say that $(M, \mathcal{D})$ is **separated by models** if there is a sequence $\langle N_i \rangle_{i<m}$ of countable elementary submodels of $\mathsf{H}(\kappa)$ such that:

(1) $M = N_0$.
(2) $N_i \in N_{i+1}$ whenever $i + 1 < m$.
(3) $d_i \subseteq N_{i+1} \setminus N_i$ (where $N_m = V$ by convention).

Proposition 18 has the following extension:

Proposition 23: *Let $M \preceq \mathsf{H}(\kappa)$ be countable, $m \in \omega$ and $\mathcal{D} = \{d_i \mid i < m\} \subseteq [\omega_1]^2$ be a block-sequence such that $(M, \mathcal{D})$ is separated by models. Let $S \subseteq ([\omega_1]^2)^{<m}$ be a tree with the following properties:*

(1) $S \in M$.
(2) $\langle d_0, \ldots, d_{m-1} \rangle \in [S]$.

Let $l_0, \ldots, l_{m-1} : 2 \longrightarrow \{>, <\}$ be types. There is $\langle a_0, \ldots, a_{m-1} \rangle \in [S] \cap M$ such that

$$T(a_i, d_i) = l_i \quad \textit{for every } i < m.$$

Proof. We will prove the proposition by induction over m. The case $m = 1$ follows by Proposition 18. We now assume that the proposition is true for m, we will prove that it is also true for $m + 1$.

In this way, we have

$$\mathcal{D} = \{d_0, \dots, d_m\}$$

separated by the models $M = N_0, N_1, \dots, N_m$. Let $\mathcal{D}' = \{d_0, \dots, d_{m-1}\}$ which obviously is separated by the models $M = N_0,\ N_1, \dots, N_{m-1}$. Since $S \in N_m$ and $\langle d_0, \dots, d_{m-1}\rangle \in N_m$, it follows that

$$L = \mathsf{suc}_S(\langle d_0, \dots, d_{m-1}\rangle) \in N_m.$$

We also know that $d_m \in L$ and

$$d_m \cap N_m = \emptyset.$$

By Proposition 18, there is $e \in L \cap N_m$ such that $T(e, d_m) = l_m$. Let $\mathcal{U}$ and $\mathcal{V}$ be sequences of rational disjoint intervals such that $(\mathcal{U}, \mathcal{V})$ freezes (e, d_m). Now, we define $\widetilde{S}$ as the set of all $\overline{x} = \langle x_0, \dots, x_{m-1}\rangle \in S$ such that:

$$\text{There is } y \in \mathsf{suc}_S(\overline{x}) \quad \text{such that } \mathcal{U} \text{ covers } y.$$

Note that $\widetilde{S} \in M$ and $\langle d_0, \dots, d_{m-1}\rangle$ is a branch of $\widetilde{S}$. By the inductive hypothesis, there is $\overline{a} = \langle a_0, \dots, a_{m-1}\rangle \in [\widetilde{S}] \cap M$ such that $T(a_i, d_i) = l_i$ for $i \leq m-1$. Since $\overline{a} \in [\widetilde{S}]$, we know that there is $y \in \mathsf{suc}_S(\overline{a})$ such that $\mathcal{U}$ covers y. It follows that $\overline{a}^\frown y \in S$ and $T(y, d_m) = l_m$ (since $(\mathcal{U}, \mathcal{V})$ freezes (e, d_m)). ∎

We now introduce the following:

Definition 24:

(1) Let $X \in \mathsf{H}(\kappa)$, by $\mathcal{SK}(X)$ we denote the Skolem closure of X (where the set of Skolem functions is defined using the well-order $<_w$).
(2) If $M \preceq \mathsf{H}(\kappa)$ is countable, by M^+ we denote $\mathcal{SK}(M \cup \{M\})$.

Note that if $M \preceq \mathsf{H}(\kappa)$, then $M^+ \preceq \mathsf{H}(\kappa)$. The idea of using successors of models in side conditions was first used by Kuzeljevic and the second author (see [25]) in order to prove that PID is consistent with the existence of an almost Suslin tree (an Aronszajn with no stationary antichains). This idea will be very fruitful for us in this section.

We can now define our forcing:

Definition 25: By $\mathbb{P}_E(\mathbb{Q})$ we denote the set of all $p = (\mathcal{M}_p, f_p)$ that satisfy the following conditions:

(1) $\mathcal{M}_p = \{M_0, \dots, M_n\}$ has the following properties:
   (a) $M_i \in M_{i+1}$ for all $i < n$.
   (b) $M_i \preceq \mathsf{H}(\kappa)$.
   (c) If $i < n$, then $M_i \in M_i^+ \in M_i^{++} \in M_{i+1}$.

(2) $f_p : \mathcal{M}_p \longrightarrow ([\omega_1]^2)^3$ is such that if $f_p(M_i) = (a, b, c)$, then the following holds:
   (a) $a \subseteq M_i^+ \setminus M_i$, $b \subseteq M_i^{++} \setminus M_i^+$ and $c \subseteq M_{i+1} \setminus M_i^{++}$ (where $M_{n+1} = V$, for convenience).
   (b) There is $q^i \in \mathbb{Q}$ such that $q^i \Vdash$ "$a, b, c \in \dot{\mathcal{B}}$" (in this case, q^i is called a **witness** for $f_p(M_i)$).
   (c) $\mathrm{im}(f_p)$ is 0-monochromatic with respect to h (where $\mathrm{im}(f_p)$ denotes the image of f_p).

If $p = (\mathcal{M}_p, f_p)$ and $q = (\mathcal{M}_q, f_q)$ are conditions in $\mathbb{P}_E(\mathbb{Q})$, define $p \leq q$ if $f_q \subseteq f_p$ (which implies that $\mathcal{M}_q \subseteq \mathcal{M}_p$).

During this section, we will write $\mathbb{P}(\mathbb{Q})$ instead of $\mathbb{P}_E(\mathbb{Q})$. Let

$$p = (\mathcal{M}_p, f_p) \in \mathbb{P}(\mathbb{Q}),$$

whenever we write $\mathcal{M}_p = \{M_0, \dots, M_n\}$ we are implicitly assuming that $M_i \in M_{i+1}$ for all $i < n$.

Let $p = (\mathcal{M}_p, f_p)$ be a condition of $\mathbb{P}(\mathbb{Q})$ and M_i, $M_j \in \mathcal{M}_p$ with $i \neq j$. By definition, $h(f_p(M_i), f_q(M_j)) = 0$. This means that there are $x \in f_p(M_i)$ and $y \in f_q(M_j)$ such that $T(x, y) = t$. It follows that if q^i is a witness for $f_p(M_i)$ and q^j is a witness for $f_p(M_j)$, then q^i and q^j are incompatible in $\mathbb{Q}$.[6]

Definition 26: Let θ be a large enough regular cardinal such that $\mathsf{H}(\kappa) \in \mathsf{H}(\theta)$. We say that N is a **big model** if the following conditions hold:

(1) $N \in \mathsf{H}(\theta)$ is a countable elementary submodel.
(2) $\mathsf{H}(\kappa)$, $<_w$, E, $\mathbb{Q}$, $\dot{\mathcal{B}}$, $\mathbb{P}(\mathbb{Q}) \in N$.

We will need the following notion:

[6] At this point, the reader may wonder why f_p takes values in $([\omega_1]^2)^3$ and not just in $([\omega_1]^2)^2$. The reason for this will be clear in Proposition 33.

Definition 27: Let $p = (\mathcal{M}_p, f_p)$ and $q = (\mathcal{M}_q, f_q)$. Let $\mathcal{M}_p = \{M_0, \dots, M_n\}$ and $\mathcal{M}_q = \{N_0, \dots, N_m\}$. We say that q is an **initial segment** of p (denoted by $q \sqsubseteq p$) if the following conditions hold:[(1)]

(1) $N_i = M_i$ for $i \leq m$.
(2) $f_p \restriction \mathcal{M}_q = f_q$.

It follows by definition that if $q \sqsubseteq p$, then $p \leq q$.

Definition 28: Let $p = (\mathcal{M}_p, f_p) \in \mathbb{P}(\mathbb{Q})$ with $\mathcal{M}_p = \{M_0, \dots, M_n\}$. Let

$$\mathcal{U} = \langle (U_i^0, V_i^0), (U_i^1, V_i^1), (U_i^2, V_i^2) \rangle_{i \leq n}.$$

We say that $\mathcal{U}$ **covers** p if the following conditions hold:

(1) Each U_i^j and V_i^j are rational open intervals.
(2) $\{U_i^j \mid i \leq n \wedge j < 3\} \cup \{V_i^j \mid i \leq n \wedge j < 3\}$ is pairwise disjoint.
(3) If $f_p(M_i) = (a_i^0, a_i^1, a_i^2)$, then (U_i^j, V_i^j) covers a_i^j for every $i \leq n$ and $j < 3$.

The following lemma is trivial, we just write it to keep it in mind:

Lemma 29: *Let $p = (\mathcal{M}_p, f_p)$ and $q = (\mathcal{M}_q, f_q)$ be conditions in $\mathbb{P}(\mathbb{Q})$ such that $|\mathcal{M}_p| = |\mathcal{M}_q| = n$. Let*

$$f_p(M_i) = (a_i^0, a_i^1, a_i^2) \quad \text{and} \quad f_q(N_i) = (c_i^0, c_i^1, c_i^2)$$

(where $\mathcal{M}_p = \{M_1, \dots, M_n\}$ and $\mathcal{M}_p = \{N_1, \dots, N_n\}$). Let $\mathcal{U}$ be covering both p and q. If $(i,j) \neq (k,l)$, then

$$T(a_i^j, c_k^l) = T(a_i^j, a_k^l) = T(c_i^j, c_k^l).$$

The following is the expected proposition one usually finds when working with models as side conditions:

Proposition 30: *Let $\overline{M}$ be a big model, $M = \overline{M} \cap \mathsf{H}(\kappa)$ and $\widetilde{p} = (\mathcal{M}_{\widetilde{p}}, f_{\widetilde{p}}) \in \mathbb{P}(\mathbb{Q})$. If $M \in \mathcal{M}_{\widetilde{p}}$, then $\widetilde{p}$ is an $(\overline{M}, \mathbb{P}(\mathbb{Q}))$-generic condition.*

Proof. Let $D \in \overline{M}$ be an open dense subset of $\mathbb{P}(\mathbb{Q})$ and $p = (\mathcal{M}_p, f_p) \leq \widetilde{p}$. We need to prove that p is compatible with an element of $D \cap \overline{M}$. Without loss of generality, we may assume that $p \in D$.

We need to introduce some items that will aid us in proving the result. Define

$$p_M = (\mathcal{M}_p \cap M, f_p \restriction M).$$

It is easy to see that $p_M \in \mathbb{P}(\mathbb{Q}) \cap M$ and is an initial segment of p (in particular, $p \leq p_M$). Let $\mathcal{M}_p \setminus M = \{N_0, \ldots, N_m\}$ (where $N_0 = M$) and $f_p(N_i) = (a_i, c_i, d_i)$. Choose $\mathcal{U}$ that covers p.

Now, define L as the set of all $(x_0, y_0, z_0, \ldots, x_m, y_m, z_m) \in ([\omega_1]^2)^{<\omega}$ such that there is $q \in \mathbb{P}(\mathbb{Q})$ with the following properties:

(1) $q \in D$.
(2) $p_M \sqsubseteq q$.
(3) $\mathcal{M}_q \setminus \mathcal{M}_{p_M}$ has size $m+1$. Say $\mathcal{M}_q \setminus \mathcal{M}_{p_M} = \{K_0, \ldots, K_m\}$.
(4) $f_q(K_i) = (x_i, y_i, z_i)$.
(5) $\mathcal{U}$ covers q.

Note that $L \in \overline{M}$ by elementarity. Moreover, since $L \subseteq ([\omega_1]^2)^{<\omega}$ it follows that $L \in \mathsf{H}(\kappa)$, so $L \in M$. Let S be the tree closure of L. Clearly S is in M as well and $(a_0, c_0, d_0, \ldots, a_m, c_m, d_m) \in [S]$. By Proposition 23, we know that there is $s = (x_0, \ldots, z_m) \in M \cap [S]$ such that:[7]

$$T(x_i, a_i) = T(y_i, c_i) = T(z_i, d_i) = t,$$

for every $i \leq m$. By the definition of L and elementarity, we may find $q \in M \cap D$ witnessing that $s \in L$. By Lemma 29, we get that p and q are compatible. ∎

Let $l : 2 \longrightarrow \{>, <\}$ be a type, define $-l : 2 \longrightarrow \{>, <\}$ such that

$$l(i) \neq -l(i) \quad \text{for all } i < 2.$$

PROPOSITION 31: *Let $\overline{M}$ be a big model, $M = \overline{M} \cap \mathsf{H}(\kappa)$ and $p \in M \cap \mathbb{P}(\mathbb{Q})$. There is $r \leq p$ such that $M \in \mathcal{M}_r$.*

Proof. Let N be the largest model in $\mathcal{M}_p$ and $f_p(N) = (a, c, d)$. Choose $\mathcal{U}$ covering p and (U_0, V_0), (U_1, V_1), (U_2, V_2) in $\mathcal{U}$ such that (U_0, V_0) covers a, (U_1, V_1) covers c and (U_2, V_2) covers d.

Let L be the set of all $(x, y, z) \in ([\omega_1]^2)^3$ such that there is $q \in \mathbb{Q}$ with the following properties:

(1) $q \Vdash$"$x, y, z \subseteq \dot{\mathcal{B}}$".
(2) (U_0, V_0) covers x, (U_1, V_1) covers y and (U_2, V_2) covers z.

[7] Here, we are making the three values equal to t. We are doing it like that because we can, but in order to get a condition, it would have been enough that only one value is equal to t.

Let S be the tree closure of L. Clearly $S \in N$ and $(a,c,d) \in [S] = L$. By Proposition 23 we know there is $(x,y,z) \in L \cap N$ such that:[8]

$$T(x,a) = T(y,c) = T(z,d) = -t,$$

so

$$T(a,x) = T(c,y) = T(d,z) = t.$$

By elementarity, we can find $q \in N$ such that $q \Vdash$ "$x,y,z \subseteq \dot{\mathcal{B}}$". Now, let $(\overline{U}_0, \overline{V}_0)$, $(\overline{U}_1, \overline{V}_1)$, $(\overline{U}_2, \overline{V}_2)$ be rational open intervals such that:

(1) $\overline{U}_i \subseteq U_i$ and $\overline{V}_i \subseteq V_i$ for $i < 3$.
(2) $x(0) \in \overline{U}_0$, $x(1) \in \overline{V}_0$ while $a(0) \notin \overline{U}_0$, $a(1) \notin \overline{V}_0$.
(3) $y(0) \in \overline{U}_1$, $y(1) \in \overline{V}_1$ while $c(0) \notin \overline{U}_1$, $c(1) \notin \overline{V}_1$.
(4) $z(0) \in \overline{U}_2$, $z(1) \in \overline{V}_2$ while $d(0) \notin \overline{U}_2$, $d(1) \notin \overline{V}_2$.

Since $\dot{\mathcal{B}}$ is forced to be ω_1-dense, we know that q forces that there are uncountable many elements in $\dot{\mathcal{B}}$ that are separated by $(\overline{U}_0, \overline{V}_0)$, $(\overline{U}_1, \overline{V}_1)$ and $(\overline{U}_2, \overline{V}_2)$. In this way, we can find $q_1 \leq q$ and $\{\widetilde{x}, \widetilde{y}, \widetilde{z}\} \subseteq [\omega_1]^{<\omega}$ block-sequence such that:

(1) $(\overline{U}_0, \overline{V}_0)$ separates $\widetilde{x}$.
(2) $(\overline{U}_1, \overline{V}_1)$ separates $\widetilde{y}$.
(3) $(\overline{U}_2, \overline{V}_2)$ separates $\widetilde{z}$.
(4) $\widetilde{x} \subseteq M^+ \setminus M$, $\widetilde{y} \subseteq M^{++} \setminus M^+$ and $\widetilde{z} \cap M^{++} = \emptyset$.
(5) $q \Vdash$ "$\widetilde{x}, \widetilde{y}, \widetilde{z} \in \dot{\mathcal{B}}$".

Now, define $r = (\mathcal{M}_r, f_r)$ where $\mathcal{M}_r = \mathcal{M}_p \cup \{M\}$, $f_p \subseteq f_r$ and $f_r(M) = (\widetilde{x}, \widetilde{y}, \widetilde{z})$. Clearly $r \leq p$ and $M \in \mathcal{M}_r$. ∎

Now we get the following:

Corollary 32: $\mathbb{P}(\mathbb{Q})$ *is a proper forcing and* $\mathbb{P}(\mathbb{Q}) \Vdash$ "$\mathbb{Q}$ *is not ccc*".

Proof. By combining Proposition 30 and Proposition 31 we conclude that $\mathbb{P}(\mathbb{Q})$ is proper. We will now show that it adds an uncountable antichain to $\mathbb{Q}$.

Let $G \subseteq \mathbb{P}(\mathbb{Q})$ be a generic filter. We go to $V[G]$. Here, define

$$\mathcal{D}_{\text{gen}} = \{f_p(M) \mid p \in G\}.$$

Clearly $\mathcal{D}_{\text{gen}}$ is a block-sequence and by Proposition 31 it follows that $\mathcal{D}_{\text{gen}}$ is uncountable. For every $a = (x,y,z) \in \mathcal{D}_{\text{gen}}$ we choose $q_a \in \mathbb{Q}$ such that $q_a \Vdash$ "$a \subseteq \dot{\mathcal{B}}$". It follows that $\{q_a \mid a \in \mathcal{D}_{\text{gen}}\}$ is an uncountable antichain. ∎

[8] Once again, it was enough that only one of those is equal to t.

It remains to prove that $\mathbb{P}(\mathbb{Q})$ does not destroy the 2-entangledness of E.

Proposition 33: $\mathbb{P}(\mathbb{Q})$ *preserves* E.

Proof. Let $\overline{p} \in \mathbb{P}(\mathbb{Q})$ and $\dot{\mathcal{A}}$ be such that $\overline{p}$ forces that $\dot{\mathcal{A}}$ is an ω_1-dense block-sequence of pairs. Let $l : 2 \longrightarrow \{>, <\}$ be a type. We need to prove that we can extend $\overline{p}$ to a condition that forces that l is realized in $\dot{\mathcal{A}}$. The argument is very similar to the one used in Proposition 30.

Let $\overline{M}$ be a big model with $\overline{p}, \dot{\mathcal{A}} \in \overline{M}$ and $M = \overline{M} \cap \mathsf{H}(\kappa)$. By Proposition 31, we can find $p \in \mathbb{P}(\mathbb{Q})$ such that:

(1) $p \leq \overline{p}$.
(2) $M \in \mathcal{M}_p$.
(3) There is $w \in [\omega_1]^2$ such that:
   (a) $p \Vdash$ "$w \in \dot{\mathcal{A}}$".
   (b) $w \cap M = \emptyset$.
   (c) w is contained in the last model of $\mathcal{M}_p$.

Let $\mathcal{M}_p \setminus M = \{N_0, \ldots, N_m\}$ (where $N_0 = M$) and $f_p(N_i) = (a_i, c_i, d_i)$. Let $p_M = (\mathcal{M}_p \cap M, f_p \restriction M)$ and $\mathcal{U}$ covering p. Define

$$\delta_i = N_i \cap \omega_1, \quad \delta_i^+ = N_i^+ \cap \omega_1 \quad \text{and} \quad \delta_i^{++} = N_i^{++} \cap \omega_1.$$

We also define

$$I_i = [\delta_i, \delta_i^+), \quad I_i^+ = [\delta_i^+, \delta_i^{++}) \quad \text{and} \quad I_i^{++} = [\delta_i^{++}, \delta_{i+1}).$$

Note that $\mathcal{P} = \{I_i, I_i^+, I_i^{++} \mid i < m\}$ is a partition of $[\delta_0, \delta_m)$ and $w \subseteq [\delta_0, \delta_m)$. There are two cases to consider:

Case 34: *w is contained in one of the intervals in $\mathcal{P}$.*

For concreteness, we assume that $w \subseteq I_0^+$ (every other case is practically the same). Define L as the set of all

$$(x_0, u, z_0, x_1, y_1, z_1, \ldots, x_m, y_m, z_m) \in ([\omega_1]^2)^{<\omega}$$

such that there is $q \in \mathbb{P}(\mathbb{Q})$ with the following properties:

(1) $p_M \sqsubseteq q$.
(2) $\mathcal{M}_q \setminus \mathcal{M}_{p_M}$ has size $m+1$. Say $\mathcal{M}_q \setminus \mathcal{M}_{p_M} = \{K_0, \ldots, K_m\}$.
(3) There is y_0 such that $f_q(K_0) = (x_0, y_0, z_0)$.
(4) $f_q(K_i) = (x_i, y_i, z_i)$ for $i \neq 0$.
(5) $\mathcal{U}$ covers q.
(6) $u \subseteq [K_0^+ \cap \omega_1, K_0^{++} \cap \omega_1)$.
(7) $q \Vdash$ "$u \in \dot{\mathcal{A}}$".

Clearly $L \in \overline{M}$ by elementarity. Moreover, since $L \subseteq ([\omega_1]^2)^{<\omega}$ it follows that $L \in \mathsf{H}(\kappa)$, so $L \in M$. Let S be the tree closure of L, which is in M as well. Note that $(a_0, w, d_0, \ldots, a_m, c_m, d_m) \in [S]$. By Proposition 23, we know that there is $s = (x_0, u, z_0, \ldots, x_m, y_m, z_m) \in M \cap [S]$ such that:[9]

$$T(x_0, a_0) = T(z_0, d_0) = t,$$
$$T(u, w) = l,$$
$$T(x_i, a_i) = T(y_i, c_i) = T(z_i, d_i) = t \quad \text{for } i \neq 0.$$

By the definition of L and elementarity, we may find $q \in M \cap D$ witnessing that $s \in L$. By Lemma 29, we get that p and q are compatible. We are done in this case.

Case 35: *w is not contained in one of the intervals in $\mathcal{P}$, but there is $i < m$ such that $w \subseteq I_i \cup I_i^+ \cup I_i^{++}$.*

Again for concreteness, we assume that $i = 0$, $w(0) \in I_0$ and $w(1) \in I_0^{++}$ (every other case is essentially the same). Define

$$w^0 = \{w(0), w(0)+1\} \quad \text{and} \quad w^1 = \{w(1), w(1)+1\}.$$

In this case, we define L as the set of all

$$(u^0, y_0, u^1, x_1, y_1, z_1, \ldots, x_m, y_m, z_m) \in ([\omega_1]^2)^{<\omega}$$

such that there is $q \in \mathbb{P}(\mathbb{Q})$ with the following properties:

(1) $p_M \sqsubseteq q$.
(2) $\mathcal{M}_q \setminus \mathcal{M}_{p_M}$ has size $m+1$. Say $\mathcal{M}_q \setminus \mathcal{M}_{p_M} = \{K_0, \ldots, K_m\}$.
(3) There are x_0, z_0 such that $f_q(K_0) = (x_0, y_0, z_0)$.
(4) $f_q(K_i) = (x_i, y_i, z_i)$ for $i \neq 0$.
(5) $\mathcal{U}$ covers q.
(6) $u^0 \subseteq [K_0 \cap \omega_1, K_0^+ \cap \omega_1)$ and $u^1 \subseteq [K_0^{++} \cap \omega_1, K_1 \cap \omega_1)$.
(7) If $u = \{u^0(0), u^1(1)\}$, then $q \Vdash$ "$u \in \dot{\mathcal{A}}$".

Once again, L is in $\overline{M}$ by elementarity. Moreover, since $L \subseteq ([\omega_1]^2)^{<\omega}$, it follows that $L \in \mathsf{H}(\kappa)$, so $L \in M$. Let S be the tree closure of L, which is in M as well. Note that $(w^0, c_0, w^1, \ldots, a_m, c_m, d_m) \in [S]$. By Proposition 23,

[9] In this way, it might be impossible to achieve $T(y_0, b_0) = t$, but in any other place it is possible.

we know that there is $s = (u^0, y_0, u^1, x_1, y_1, z_1, \ldots, x_m, y_m, z_m) \in M \cap [S]$ such that[10]

$$\begin{aligned} T(u^0, w^0) &= l, \\ T(u^1, w^1) &= l, \\ T(y_0, c_0) &= t, \\ T(x_i, a_i) &= T(y_i, c_i) = T(z_i, d_i) = t \quad \text{for } i \neq 0. \end{aligned}$$

By the definition of L and elementarity, we may find $q \in M \cap D$ witnessing that $s \in L$. Using Lemma 29, we get that p and q are compatible. We are done in this case.

Case 36: *w is not contained in one of the intervals in $\mathcal{P}$ and there is no $i < m$ such that $w \subseteq I_i \cup I_i^+ \cup I_i^{++}$.*

Very similar to the previous case. ∎

For the convenience of the reader, we summarize the results of this section in the following theorem:

Theorem 37: *Let E be a 2-entangled set and $\mathbb{Q}$ a ccc forcing that destroys E. There is a forcing $\mathbb{P}_E(\mathbb{Q})$ such that:*

1. $\mathbb{P}_E(\mathbb{Q})$ *is proper.*
2. $\mathbb{P}_E(\mathbb{Q})$ *preserves E.*
3. $\mathbb{P}_E(\mathbb{Q})$ *adds an uncountable antichain to $\mathbb{Q}$.*

5. The P-ideal dichotomy and entangled sets

In the last section we developed the tools needed to force MA_{ω_1} while preserving a 2-entangled set using a proper forcing. In this section, we will obtain the analogous results for the P-ideal dichotomy. There are two usual ways for forcing PID, one that does not add reals (see [48] and [3]) and one with models as side conditions (see [52] and [35]). We will use the latter approach (which was historically the first one). We will now recall (without proofs) how this is done (the reader may consult [52] for the missing proofs).

[10] Here we might not be able to achieve $T(x_0, a_0) = t$ or $T(z_0, d_0) = t$, but we can get $T(y_0, c_0)$, so we do what we must, because we can.

For this section fix S an uncountable set, $\mathcal{I} \subseteq [S]^{\leq\omega}$ a P-ideal such that the second alternative of the P-ideal dichotomy fails, or in other words:

$$S \text{ can not be decomposed into countably many sets of } \mathcal{I}^{\perp}.$$

We need a proper forcing that adds an uncountable set such that all its countable sets are in $\mathcal{I}$. Let κ be a large enough regular cardinal such that $[S]^{\leq\omega} \in \mathsf{H}(\kappa)$ and let $<_w$ be a well order of $\mathsf{H}(\kappa)$. For this section, given $M \in \mathsf{H}(\kappa)$ with $S, \mathcal{I} \in M$, we write $M \preceq \mathsf{H}(\kappa)$ to denote that $(M, \in, <_W)$ is an elementary submodel of $(\mathsf{H}(\kappa), \in, <_W)$. Moreover, for $M \preceq \mathsf{H}(\kappa)$, let $B_M \in \mathcal{I}$ be the $<_W$-least pseudounion of $\mathcal{I} \cap M$.

Definition 38: Define $\mathbb{P}(\mathcal{I})$ as the set of all $p = (\mathcal{M}_p, f_p)$ such that:[11]

1. $\mathcal{M}_p = \{M_0, \ldots, M_n\}$ where $M_i \preceq \mathsf{H}(\kappa)$ for all $i \leq n$.
2. $M_i \in M_{i+1}$.
3. $f_p : \mathcal{M}_p \longrightarrow S$.
4. $f_p(M_i) \in M_{i+1} \setminus M_i$ (where $M_{n+1} = V$ for convenience).
5. $f_p(M_i) \notin \bigcup(M_i \cap \mathcal{I}^{\perp})$.

Given $p = (\mathcal{M}_p, f_p)$ and $q = (\mathcal{M}_q, f_q)$ conditions in $\mathbb{P}(\mathcal{I})$, define $p \leq q$ if the following conditions hold:

1. $f_q \subseteq f_p$ (so $\mathcal{M}_q \subseteq \mathcal{M}_p$).
2. If $M \in \mathcal{M}_q$ and $N \in \mathcal{M}_p \setminus \mathcal{M}_q$ with $N \in M$, then:

$$f_p(N) \in B_M.$$

We need the following notion for this section:

Definition 39: Let $\theta > (2^{\kappa})^{+}$ be a large enough regular cardinal such that $\mathsf{H}(\kappa) \in \mathsf{H}(\theta)$. We say that N is a **big model** if the following conditions hold:

1. $N \in \mathsf{H}(\theta)$ is a countable elementary submodel.
2. $\mathsf{H}(\kappa)$, $<_w$, S, $\mathcal{I}$, $\mathbb{P}(\mathcal{I}) \in N$.

We have the following:

[11] The forcing in [52] is slightly different from the one presented here. In the book, the forcing omits the component f_p (or rather, $f_p(M)$ is always the least element in S that is not in $\bigcup(M_i \cap \mathcal{I}^{\perp})$). At least for the purpose of this paper, the difference between the two partial orders is inconsequential.

THEOREM 40 ([52]): *Let $\overline{M}$ be a big model and $M = \overline{M} \cap \mathsf{H}(\kappa)$.*

(1) *If $p \in \mathbb{P}(\mathcal{I})$ and $M \in \mathcal{M}_p$, then p is an $(\overline{M}, \mathbb{P}(\mathcal{I}))$-generic condition.*
(2) *For every $q \in M \cap \mathbb{P}(\mathcal{I})$ there is $p \leq q$ such that $M \in \mathcal{M}_p$.*
(3) *$\mathbb{P}(\mathcal{I})$ is a proper forcing.*
(4) *$\mathbb{P}(\mathcal{I})$ adds an uncountable set such that all of its countable sets are in $\mathcal{I}$.*

Let X be a subset of S. Note that $X \in (\mathcal{I}^{\perp})^{+}$ if and only if X has infinite intersection with a member of $\mathcal{I}$. We now prove the following:

PROPOSITION 41: *Let $E = \{e_\alpha \mid \alpha \in \omega_1\} \subseteq \mathbb{R}$ be a 2-entangled set, $M \preceq \mathsf{H}(\kappa)$ with $E \in M$ and $L \subseteq [\omega_1]^2 \times S$ with $L \in M$. Let (d, x) such that:*

(1) *$(d, x) \in L$.*
(2) *$d \cap M = \emptyset$.*
(3) *$x \notin \bigcup(M \cap \mathcal{I}^{\perp})$.*

For every type $t : 2 \longrightarrow \{>, <\}$ there is $\mathcal{V}$ a sequence of rational intervals such that:

(1) *$T(\mathcal{V}, d) = t$.*
(2) *The set $\{y \in S \mid \exists c(((c, y) \in L) \wedge (\mathcal{V}$ covers $c))\}$ is in $(\mathcal{I}^{\perp})^{+}$.*

Proof. Let $\mathcal{U} = (U_0, U_1)$ be a sequence of rational open intervals that covers d. By shrinking L if needed, we may assume that if $(a, y) \in L$, then $\mathcal{U}$ covers a. Let

$$Z = \{a \in [\omega_1]^2 \mid \exists y((a, y) \in L)\}.$$

Given $a \in Z$, we define

$$Y(a) = \{w \in S \mid \exists b((b, w) \in L \wedge T(b, a) = t)\}.$$

Note that if $a \in M$, then $Y(a) \in M$. We will now prove the following:

CLAIM 42: $Y(d) \in (\mathcal{I}^{\perp})^{+}$.

Assume this is not the case. Let $A = \{b \in Z \mid Y(b) \in \mathcal{I}^{\perp}\}$, note that $A \in M$, $d \in A$ and $M \cap d = \emptyset$. By Proposition 18, we can find $a \in M \cap A$ such that

$$T(a, d) = -t$$

(so $T(d, a) = t$). Since $T(d, a) = t$, it follows that $x \in Y(a)$. Now, note that $Y(a) \in M$ (since $a \in M$) and $Y(a) \in \mathcal{I}^{\perp}$ (since $a \in A$), but this is a contradiction because $x \notin \bigcup(M \cap \mathcal{I}^{\perp})$. This finishes the proof of the claim.

We now know that $Y(d) \in (\mathcal{I}^{\perp})^{+}$. Now, let $B = \{a \in [\omega_1]^2 \mid Y(a) \in (\mathcal{I}^{\perp})^{+}\}$. Obviously, $d \in B$ and $B \in M$. Once more we apply Proposition 18 and obtain $a \in B \cap M$ such that $T(a,d) = t$. We now define $\mathcal{V} = (V_0, V_1)$ such that:

(1) V_0 and V_1 are two rational open disjoint intervals.
(2) $\mathcal{V}$ covers a.
(3) $e_{d(0)}, e_{d(1)} \notin V_0 \cup V_1$.
(4) If $i < 2$, the following holds:[12]
   (a) If $t(i) = <$, then $(\inf(U_i), e_{a(i)}) \subseteq V_i$.
   (b) If $t(i) = >$, then $(e_{a(i)}, \sup(U_i)) \subseteq V_i$.

Note that $T(\mathcal{V}, d) = t$. In order to finish the proof, we must argue that the set

$$H = \{y \in S \mid \exists c(((c,y) \in L) \wedge (\mathcal{V} \text{ covers } c))\}$$

is in $(\mathcal{I}^{\perp})^{+}$. For this, it is enough to prove that $Y(a) \subseteq H$ (recall that $a \in B$). Let $y \in Y(a)$, by definition, we know there is b such that:

(1) $(b,y) \in L$.
(2) $T(b,a) = t$.

In this way, it will be enough to prove that $\mathcal{V}$ covers b. Let $i < 2$, we proceed by cases:

Case 43: $t(i) =<$.

Since $T(b,a) = t$ it follows that $e_{b(i)} < e_{a(i)}$, so $e_{b(i)} \in (\inf(U_i), e_{a(i)}) \subseteq V_i$.

Case 44: $t(i) =>$.

Since $T(b,a) = t$ it follows that $e_{b(i)} > e_{a(i)}$, so $e_{b(i)} \in (e_{a(i)}, \sup(U_i)) \subseteq V_i$.
This finishes the proof. ∎

We need the following notion:

Definition 45: Let $M \preceq \mathsf{H}(\kappa)$ be countable, $m \in \omega$ and

$$\overline{s} = \langle (d_i, x_i)_{i<m} \rangle \in ([\omega_1]^2 \times S)^{<\omega}.$$

We say that $(M, \overline{s})$ is **separated by models** if there is a sequence $\langle N_i \rangle_{i<m}$ of countable elementary submodels of $\mathsf{H}(\kappa)$ such that:

(1) $M = N_0$.
(2) $N_i \in N_{i+1}$ whenever $i + 1 < m$.
(3) $d_i \subseteq N_{i+1} \setminus N_i$ (where $N_m = V$ by convention).
(4) $x_i \notin \bigcup (N_i \cap \mathcal{I}^{\perp})$ for all $i < m$.

[12] Recall that $\mathcal{U}$ covers both a and d.

The next result is the "tree-version" of Proposition 41:

Proposition 46: *Let $E = \{e_\alpha \mid \alpha \in \omega_1\} \subseteq \mathbb{R}$ be a 2-entangled set, $M \preceq \mathsf{H}(\kappa)$ a countable submodel with $E \in M$. Let $m \in \omega$ and $\overline{s} = \langle (d_i, x_i) \rangle_{i<m}$ such that $(M, \overline{s})$ is separated by models. Let $Z \subseteq ([\omega_1]^2 \times S)^{<m}$ be a tree such that:*

1. *$Z \in M$.*
2. *$\overline{s} \in [Z]$.*

For every $\langle t_i \rangle_{i<m}$ sequence of types, there is $\langle \mathcal{V}_i \rangle_{i<m}$ a sequence of rational disjoint open intervals and $R \subseteq Z$ a subtree with the following properties:

1. *$T(\mathcal{V}_i, d_i) = t_i$.*
2. *$R \in M$.*
3. *For every $w \in R$ of height less than m, the set*

$$\{y \in S \mid \exists c(((c, y) \in \mathsf{suc}_R(w)) \wedge (\mathcal{V} \text{ covers } c))\}$$

   is in $(\mathcal{I}^\perp)^+$.

Proof. We proceed by induction over m. Proposition 41 takes care of the case $m = 0$. Assume the proposition is true for m, we will prove that it is also true for $m+1$. Let $\overline{s} = \langle (d_i, x_i) \rangle_{i<m+1}$ such that $(M, \overline{s})$ is separated by models and

$$Z \subseteq ([\omega_1]^2 \times S)^{<m+1}$$

with the properties above. First, we find a sequence of models $\langle N_0, \ldots, N_m \rangle$ with the following properties:

1. $N_0 = M$.
2. $N_i \in N_{i+1}$ for $i < m$.
3. $d_i \subseteq N_{i+1} \setminus N_i$ for $i \leq m$ (where $N_{i+1} = V$ for convenience).

Define $\overline{w} = \langle (d_i, x_i) \rangle_{i<m}$ (so $\overline{s} = \overline{w}^\frown (d_m, x_m)$) and $L = \mathsf{suc}_Z(\overline{w})$. Note that $L \in N_m$ and $(d_m, x_m) \in L$. By Proposition 41, we can find $\mathcal{V}_m$ a sequence of rational open intervals such that:

1. $T(\mathcal{V}_m, d_m) = t_m$.
2. The set $\{y \in S \mid \exists c(((c, y) \in L) \wedge (\mathcal{V}_m \text{ covers } c))\}$ is in $(\mathcal{I}^\perp)^+$.

Now, let J be the set of all $\overline{u} = \langle (c_i, y_i) \rangle_{i<m}$ that satisfy the following properties:

1. $\overline{u} \in Z$.
2. The set $\{y \in S \mid \exists c(((c, y) \in \mathsf{suc}_Z(\overline{u})) \wedge (\mathcal{V}_m \text{ covers } c))\}$ is in $(\mathcal{I}^\perp)^+$.

Let $\widetilde{Z}$ be the tree closure of J. Note that $\widetilde{Z} \subseteq ([\omega_1]^2 \times S)^{<m}$, $\widetilde{Z} \in M$ and $\overline{w}$ is a branch of $\widetilde{Z}$. By the inductive hypothesis, there are $\widetilde{R}$ and $\mathcal{V}_0, \ldots, \mathcal{V}_{m+1}$ sequences of disjoint open rational intervals such that:

(1) $\widetilde{R} \in M$ and is a subtree of $\widetilde{Z}$.
(2) $T(\mathcal{V}_i, d_i) = t_i$ for $i \leq m-1$.
(3) For every $l \in \widetilde{R}$ of height less than m, the set $\{y \in S \mid \exists c(((c, y) \in \mathsf{suc}_{\widetilde{R}}(l)) \wedge (\mathcal{V} \text{ covers } c))\}$ is in $(\mathcal{I}^{\perp})^{+}$.

We can now easily add a new level to $\widetilde{R}$ and find the desired tree. ∎

With these results, we can now prove the main result of this section:

Theorem 47: *Let S be an uncountable set, $\mathcal{I} \subseteq [S]^{<\omega_1}$ a P-ideal for which the second alternative of the P-ideal dichotomy does not hold and*

$$E = \{e_\alpha \mid \alpha \in \omega_1\} \subseteq \mathbb{R}$$

a 2-entangled set. The forcing $\mathbb{P}(\mathcal{I})$ preserves E.

Proof. Let $\overline{p} \in \mathbb{P}(\mathcal{I})$ and $\dot{\mathcal{A}}$ be a $\mathbb{P}(\mathcal{I})$-name such that $\overline{p}$ forces that $\dot{\mathcal{A}}$ is an ω_1-dense block sequence of pairs. Let $l : 2 \longrightarrow \{>, <\}$ be a type. We need to prove that we can extend $\overline{p}$ to a condition that forces that l is realized in $\dot{\mathcal{A}}$.

Let $\overline{M}$ be a big model with $E, \overline{p}, \dot{\mathcal{A}} \in \overline{M}$ and $M = \overline{M} \cap \mathsf{H}(\kappa)$. By Theorem 40, we can find $p \in \mathbb{P}(\mathcal{I})$ such that:

(1) $p \leq \overline{p}$.
(2) $M \in \mathcal{M}_p$.
(3) There is $w \in [\omega_1]^2$ such that:
   (a) $p \Vdash$ "$w \in \dot{\mathcal{A}}$".
   (b) $w \cap M = \emptyset$.
   (c) w is contained in the last model of $\mathcal{M}_p$.

Let $\mathcal{M}_p \setminus M = \{N_0, \ldots, N_m\}$ (where $N_0 = M$) and $f_p(N_i) = x_i$. Let

$$p_M = (\mathcal{M}_p \cap M, f_p \restriction M)$$

and $\mathcal{U}$ covering p. Define

$$\delta_i = N_i \cap \omega_1 \quad \text{and} \quad I_i = [\delta_i, \delta_{i+1}).$$

Note that $\mathcal{P} = \{I_i \mid i < m\}$ is a partition of $[\delta_0, \delta_m)$ and $w \subseteq [\delta_0, \delta_m)$. The proof is now very similar to the one of Theorem 33 but using Proposition 46 and the proof of Theorem 40. ∎

6. The side condition hull

The method of using models as side conditions is extremely powerful. For this reason, one may wonder if everything that can be achieved by a proper forcing can also be achieved using a forcing with models as side conditions. We will see in this section that this is indeed the case, since any proper forcing can be embedded in a forcing with models as side conditions. The results of this section will be used to prove the properness of the Neeman iteration and the preservation of 2-entangled sets.

The $\in$**-collapse forcing** is defined as the set of all finite chains of countable submodels of $\mathsf{H}(\theta)$ ordered by inclusion. This is a very interesting forcing on its own, it is strongly proper and it collapses the size of $\mathsf{H}(\theta)$ to ω_1. The reader can learn more about this interesting forcing in [52, Chapter 7]. In [25], Kuzeljevic and the second author studied a variant using matrices of models (see also [44], [4] and [5] for more on forcing with matrices of models). Moreover, the $\in$-collapse may be parametrized using a stationary subset of $[\mathsf{H}(\theta)]^{\omega}$ (see [52] for further discussion and results). We will now also parametrize with a sufficiently proper forcing.

Definition 48: Let $\mathbb{P}$ be a forcing, θ a large enough regular cardinal and $\mathcal{S}\subseteq[\mathsf{H}(\theta)]^{\omega}$ a stationary set. We define the **side condition hull** of $\mathbb{P}$ with respect to $\mathcal{S}$ (which we denote $\mathbb{S}_{\in}(\mathbb{P},\mathcal{S})$) as the set of all pairs (p,a) with the following properties:

1. $p=\{M_0,\dots,M_n\}\subseteq\mathcal{S}$ is an $\in$-chain of countable elementary submodels with $\mathbb{P}\in M_0$.
2. $a\in\mathbb{P}$ and is an $(M_i,\mathbb{P})$-generic condition for every $i\leq n$.

Let (p,a), $(q,b)\in\mathbb{S}_{\in}(\mathbb{P},\mathcal{S})$. Define $(p,a)\leq(q,b)$ if the following conditions hold:

1. $q\subseteq p$.
2. $a\leq b$ (as conditions in $\mathbb{P}$).

We will use the following notion, which was introduced by Shelah.

Definition 49: Let $\mathbb{P}$ be a partial order and $\mathcal{S}$ a family of countable sets. We say that $\mathbb{P}$ is $\mathcal{S}$**-proper** if for every large enough λ and M a countable elementary submodel of $\mathsf{H}(\lambda)$ with $\mathbb{P}\in M$, if $M\cap(\bigcup\mathcal{S})\in\mathcal{S}$, then every condition in $\mathbb{P}\cap M$ can be extended to an $(M,\mathbb{P})$-generic condition.

Of course, this notion is most interesting when $\mathcal{S}$ is at least a stationary set. We can now prove the following:

PROPOSITION 50: *Let $\mathbb{P}$ be a forcing, θ a large enough regular cardinal, $\mathcal{S} \subseteq [\mathsf{H}(\theta)]^\omega$ a stationary set such that $\mathbb{P}$ is $\mathcal{S}$-proper and $(\overline{p}, \overline{a}) \in \mathbb{S}_\in(\mathbb{P}, \mathcal{S})$. Let $\overline{M}$ be a countable elementary submodel of a large enough structure such that $\mathbb{S}_\in(\mathbb{P}, \mathcal{S}) \in \overline{M}$ and $M = \overline{M} \cap \mathsf{H}(\theta) \in \mathcal{S}$.*

1. *If $M \in \overline{p}$, then $(\overline{p}, \overline{a})$ is an $(M, \mathbb{S}_\in(\mathbb{P}, \mathcal{S}))$-generic condition.*
2. *If $(\overline{p}, \overline{a}) \in M$, then there is $(q, b) \leq (\overline{p}, \overline{a})$ such that $M \in q$.*
3. *The side condition hull $\mathbb{S}_\in(\mathbb{P}, \mathcal{S})$ is $\mathcal{S}$-proper.*
4. *If $\mathcal{S}$ is a club in $[\mathsf{H}(\theta)]^\omega$, then $\mathbb{S}_\in(\mathbb{P}, \mathcal{S})$ is proper.*

Proof. It is clear that points (3) and (4) follow from points (1) and (2). We will start proving the first point; assume $M \in \overline{p}$, we must prove that $(\overline{p}, \overline{a})$ is an $(M, \mathbb{S}_\in(\mathbb{P}, \mathcal{S}))$-generic condition. Let $(p, a) \leq (\overline{p}, \overline{a})$ and $D \in \overline{M}$ an open dense subset of $\mathbb{S}_\in(\mathbb{P}, \mathcal{S})$. We need to prove that (p, a) is compatible with an element of $\overline{M} \cap D$. We may assume that $(p, a) \in D$.

Let $p_M = p \cap M$, it is clear that $p_M \in M$. Define $E \subseteq \mathbb{P}$ as the set of all $x \in \mathbb{P}$ such that there is q for which the following conditions hold:

1. $(q, x) \in D$.
2. p_M is an initial segment of q.

It is clear that $E \in \overline{M}$ and $E \in \mathsf{H}(\theta)$, so $E \in \overline{M} \cap \mathsf{H}(\theta) = M$. Note that $a \in E$. Since a is an $(M, \mathbb{P})$-generic condition and it is in E, it follows by Lemma 10 that there is $b \in E \cap M$ such that a and b are compatible. Let $c \in \mathbb{P}$ be a common extension. Since $b \in E \cap M$, we can find $q \in M$ such that $(q, b) \in M \cap D$ and p_M is an initial segment of q. Define $r = q \cup p$; it is easy to see that $(r, c) \in \mathbb{S}_\in(\mathbb{P}, \mathcal{S})$ and extends both (p, a) and (q, b).

We will now prove point (2), so assume that $(\overline{p}, \overline{a}) \in M$. Let $q = \overline{p} \cup \{M\}$ and since $a \in M$ and $\mathbb{P}$ is proper for M, we know there is $b \in \mathbb{P}$ an $(M, \mathbb{P})$-generic condition extending a. It is clear that $(q, b) \leq (\overline{p}, a)$. ∎

The next task is to prove that forcing with $\mathbb{S}_\in(\mathbb{P}, \mathcal{S})$ adds $(V, \mathbb{P})$-generic filters. Recall the following notion:

Definition 51: Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders. We say that $\pi : \mathbb{Q} \longrightarrow \mathbb{P}$ is a **projection** if the following conditions hold:

1. If $q_1 \leq q_2$, then $\pi(q_1) \leq \pi(q_2)$.

(2) For every $q \in \mathbb{Q}$ and $p \in \mathbb{P}$, if $p \leq \pi(q)$, then there is $q_1 \leq q$ such that $\pi(q_1) \leq p$.

It is not hard to prove that if there is a projection from $\mathbb{Q}$ to $\mathbb{P}$, then forcing with $\mathbb{Q}$ adds generic filters for $\mathbb{P}$ (see [1]). We will now prove the following:

Lemma 52: *Let $\mathbb{P}$ be a forcing, θ a large enough regular cardinal, $\mathcal{S} \subseteq [\mathsf{H}(\theta)]^\omega$ a stationary set such that $\mathbb{P}$ is $\mathcal{S}$-proper. There is a projection from $\mathbb{S}_\in(\mathbb{P}, \mathcal{S})$ to $\mathbb{P}$.*

Proof. Define $\pi : \mathbb{S}_\in(\mathbb{P}, \mathcal{S}) \longrightarrow \mathbb{P}$ by $\pi(p, a) = a$. It is clear that π is a projection. ∎

Now, we will prove a preservation theorem for 2-entangled sets:

Proposition 53: *Let $E = \{e_\alpha \mid \alpha \in \omega_1\} \subseteq \mathbb{R}$ be a 2-entangled set, $\mathbb{P}$ be a forcing, θ a large enough regular cardinal, $\mathcal{S} \subseteq [\mathsf{H}(\theta)]^\omega$ a stationary set such that $\mathbb{P}$ is $\mathcal{S}$-proper. If $\mathbb{P}$ preserves E, then $\mathbb{S}_\in(\mathbb{P}, \mathcal{S})$ preserves E.*

Proof. Let $(p_1, a_1) \in \mathbb{S}_\in(\mathbb{P}, \mathcal{S})$ and $\dot{\mathcal{B}}$ an $\mathbb{S}_\in(\mathbb{P}, \mathcal{S})$-name for an uncountable block sequence of pairs of ω_1. Let $t : 2 \longrightarrow \{>, <\}$ be a type. We need to prove that (p_1, a_1) can be extended to a condition that forces that t is realized in $\dot{\mathcal{B}}$.

Let λ be a large enough regular cardinal. Since $\mathcal{S}$ is stationary, we can find a countable $\overline{M} \preceq \mathsf{H}(\lambda)$ such that the following holds:

(1) $M = \overline{M} \cap \mathsf{H}(\theta)$ is in $\mathcal{S}$.
(2) $(p_1, a_1), E, \dot{\mathcal{B}} \in \overline{M}$.

Now, by Proposition 50, we can find a condition $(p_2, a_2) \leq (p_1, a_1)$ such that $M \in p_2$. We can now find a further extension $(p, a) \leq (p_2, a_2)$ and $b \in [\omega_1]^2$ such that $(p, a) \Vdash$ "$b \in \dot{\mathcal{B}}$" and $b \cap M = \emptyset$. Let $\mathcal{U}$ be a sequence of disjoint rational intervals that cover b and $p_M = p \cap M$.

Define $\dot{W}$ as the set of all (u, x) such that there is a q with the following properties:

(1) $u \in [\omega_1]^2$ and $x \in \mathbb{P}$.
(2) $p_M \subseteq q$.
(3) $(q, x) \Vdash$ "$u \in \dot{\mathcal{B}}$".
(4) $\mathcal{U}$ covers u.

It is clear that $\dot{W} \in M$ and it is a $\mathbb{P}$-name for a subset of pairs of ω_1. It is also easy to see that $(p, a) \in \dot{W}$, which means that $a \Vdash_{\mathbb{P}}$ "$b \in \dot{W}$". Now, let $G \subseteq \mathbb{P}$ be a generic filter with $a \in G$. We go to the extension $V[G]$.

Since a is an $(M, \mathbb{P})$-generic condition and $a \in G$, we know that $M[G]$ is a forcing extension of M. In this way, we get that $M[G] \cap b = \emptyset$. Since $V[G]$ is a forcing extension by $\mathbb{P}$, we know that E is still a 2-entangled set and $b \in \dot{W}[G]$. By Proposition 18, there is $u \in \dot{W}[G] \cap M[G]$ such that $T(u, b) = t$. Let $x \in G$ such that $(u, x) \in \dot{W}$. Since $a, x \in G$, there is $y \in G$ such that $y \leq a, x$.

We now go back to V. Since $(u, x) \in \dot{W}$, there must be a q such that $p_M \subseteq q$ and $(q, x) \Vdash$"$u \in \dot{\mathcal{B}}$". Furthermore, we may assume that $q \in M$. Let $r = q \cup p$, it is easy to see that (r, y) is in $\mathbb{S}_{\in}(\mathbb{P}, \mathcal{S})$, it extends (p, a) and $(r, y) \Vdash$"$u, b \in \dot{\mathcal{B}}$". ∎

Properties that satisfy the conclusion of the Proposition above and are preserved by two step iterations have a good opportunity of being preserved under Neeman's iteration, which we will review in the next section.

7. Two type side conditions

Let $E \subseteq \mathbb{R}$ be a 2-entangled set. By our work in the previous sections, we know that we can force any instance of the P-ideal dichotomy with a proper forcing while preserving E. We also know that if a ccc forcing $\mathbb{P}$ destroys E, then we can add an uncountable antichain to $\mathbb{P}$ with a proper forcing that preserves E. What we are missing now is an iteration theorem. Just like in [25], we find it more convenient to use the iteration method introduced by Neeman in [37] rather than the usual countable support iteration. For the convenience of the reader, we will review the work of Neeman.

For this section, fix θ an inaccessible cardinal and $<_w$ a well-order of $\mathsf{H}(\theta)$. For now, if $M \in \mathsf{H}(\theta)$, we will write $M \preceq \mathsf{H}(\theta)$ if $(M, \in, <_w)$ is an elementary submodel of $(\mathsf{H}(\theta), \in, <_w)$. We now fix the following items:

$$\mathcal{S} \subseteq \{M \in [\mathsf{H}(\theta)]^{\omega} \mid M \preceq \mathsf{H}(\theta)\},$$
$$\mathcal{T} = \{\mathsf{H}(\lambda) \mid \mathsf{H}(\lambda) \preceq \mathsf{H}(\theta) \wedge \mathsf{cof}(\lambda) > \omega\}.$$

Moreover, we demand the following:

(1) $\mathcal{S}$ is stationary in $[\mathsf{H}(\theta)]^{\omega}$ and $\mathcal{T}$ is stationary in $[\mathsf{H}(\theta)]^{<\theta}$.
(2) $\mathcal{S} \cup \mathcal{T}$ is closed under intersections (note that $\mathcal{T}$ is closed under intersections since given any two elements of $\mathcal{T}$, one is contained in the other).

Recall the following definition:

Definition 54: A cardinal κ is **countably inaccessible** if it is regular and $\lambda^\omega < \kappa$ for every $\lambda < \kappa$.

In order to meet the requirements above, it is enough that θ is countably inaccessible (see Proposition 2.5 and Proposition 2.12 of [20]).

Every element of $\mathcal{S}$ is countable while all the elements of $\mathcal{T}$ are uncountable. Following the terminology of [37], we call the elements of $\mathcal{S} \cup \mathcal{T}$ **nodes,** the elements of $\mathcal{S}$ are called **small models** or **small nodes** and the elements of $\mathcal{T}$ are called **transitive models** or **transitive nodes**. In this paper, we will be using the following convention:

M, N, L will always be small models,
W, X, Y, Z will always be transitive models,
A, B, C, D will be elements of $\mathcal{S} \cup \mathcal{T}$ whose type is unknown or irrelevant

We have the following simple remarks:

(1) $\mathcal{S}$ is closed under intersections (this is because $\mathcal{S} \cup \mathcal{T}$ is closed under intersections).
(2) If $M \in \mathcal{S}$ and $X \in \mathcal{T}$, then $M \cap X \in \mathcal{S}$ (this is because $M \cap X$ is countable).
(3) The elements of $\mathcal{T}$ are closed under taking countable subsets. In particular, if $M \in \mathcal{S}$ and $X \in \mathcal{T}$, then $M \cap X \in X$.

We need the following notions:

Definition 55: Let $p \subseteq \mathcal{S} \cup \mathcal{T}$.

(1) We say that p is a **chain** if for every $A, B \in p$ either $A = B$, or $A \in B$ or $B \in A$.
(2) We say that p is a **path** if it is of the form $p = \{A_0, \ldots, A_n\}$ where $A_i \in A_{i+1}$ for all $i < n$.

Obviously every chain is a path. Moreover, any path consisting only of small models or only of transitive models is a chain. However, by using both small and transitive models, we can build a path that is not a chain. Whenever we write a path $p = \{A_0, \ldots, A_n\}$, we are implicitly assuming that we enumerate it in such a way that $A_i \in A_{i+1}$ for all $i < n$.

Definition 56: Let $p \subseteq \mathcal{S} \cup \mathcal{T}$ be a path and $A, B \in p$.

(1) Define $A <_p B$ if $A \neq B$ and there are $\{C_0, \ldots, C_n\} \subseteq p$ such that $A = C_0$, $B = C_n$ and $C_i \in C_{i+1}$ for all $i < n$ (note that $n \neq 0$ since A is different from B).
(2) Define $A \leq_p B$ if $A = B$ or $A <_p B$.
(3) Define the interval $(A, B)_p = \{C \in p \mid A <_p C <_p B\}$. The expressions $[A, B]_p$, $(A, B]_p$ and $[A, B)_p$ have the expected meaning.
(4) Define $A_{<p} = \{C \in p \mid C <_p A\}$.

Let $p = \{A_0, \ldots, A_n\}$ be a path. Following the convention mentioned before the definition, it follows that if $i, j \leq n$, then $A_i <_p A_j$ if and only if $i < j$. We also have that $(A_i, A_j)_p = \{A_k \mid i < k < j\}$. Similarly for $[A_i, A_j]_p$, $(A_i, A_j]_p$ and $[A_i, A_j)_p$.

Note that if $X \in p$ is a transitive model, then $X_{<p} = X \cap p$. However, if $M \in p$ is small model, then $M \cap p$ and $M_{<p}$ may be different (but note that $M \cap p \subseteq M_{<p}$). By the remarks above, it follows that if an interval has only small nodes, then it will be a chain.

Definition 57: Let $p \subseteq \mathcal{S} \cup \mathcal{T}$ (not necessarily a path). Define:

(1) $\mathcal{S}(p) = \mathcal{S} \cap p$.
(2) $\mathcal{T}(p) = \mathcal{T} \cap p$.

As mentioned in the previous chapter, the $\in$-collapse forcing plays a fundamental role while working with the usual (or "one type") models as side conditions. The analogue of the $\in$-collapse for two type side conditions is the following forcing introduced by Neeman:

Definition 58: Define $\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$ as the set of all $p \subseteq \mathcal{S} \cup \mathcal{T}$ such that:

(1) p is a path.
(2) p is closed under intersections.

Given $p, q \in \mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$, define $p \leq q$ if $q \subseteq p$.

For convenience, we will simply write $\mathbb{P}_{\in}$ instead of $\mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$ where there is no risk of confusion. It follows by the axiom of foundation that if $A, B \in p$ (for p a condition in $\mathbb{P}_{\in}$), then $A \cap B \leq_p A, B$. Checking if a path is closed under intersections might be a little tedious, but fortunately, the following result simplifies some of the work:

LEMMA 59 ([37]): *Let $p \subseteq \mathcal{S} \cup \mathcal{T}$ be a path. The following are equivalent:*

(1) *$p \in \mathbb{P}_{\in}$ (i.e., p is closed under intersections).*
(2) *For every $M \in \mathcal{S}(p)$ and $X \in \mathcal{T}(p)$, if $X \in M$, then $M \cap X \in p$.*

We need one more definition:

Definition 60: Let $p \in \mathbb{P}_{\in}$, $M \in \mathcal{S}(p)$ and $X \in \mathcal{T}(p)$ with $X \in M$. The **residue gap** of p induced by M and X is defined as $[M \cap X, X)_p$.

Understanding the structure of the residue gaps is fundamental in order to work with $\mathbb{P}_{\in}$. We quote the following result:

LEMMA 61 ([37]): *Let $p \in \mathbb{P}_{\in}$, $M \in \mathcal{S}(p)$ and $X, Y \in \mathcal{T}(p)$ with $X, Y \in M$ and $X \neq Y$.*

(1) *The residue gaps $[M \cap X, X)_p$ and $[M \cap Y, Y)_p$ are disjoint.*
(2) *$[M \cap X, X)_p$ and M are disjoint.*
(3) *$p_{<M} = (p \cap M) \cup \bigcup_{Z \in \mathcal{T}(p) \cap M} [M \cap Z, Z)_p$ (and this is a disjoint union).*

Proving strong properness for transitive models is easy.

PROPOSITION 62 ([37]): *Let $p \in \mathbb{P}_{\in}$ and $X \in \mathcal{T}(p)$. If $q \in \mathbb{P}_{\in}$ has the following properties:*

(1) *$q \in X$.*
(2) *$q \leq p \cap X$.*

Then $p \cup q \in \mathbb{P}_{\in}$ (and obviously it is a common extension of p and q).

It is also straight-forward to prove the following:

LEMMA 63 ([37]): *Let $q \in \mathbb{P}_{\in}$ and $X \in \mathcal{T}$. If $q \in X$, then $q \cup \{X\} \in \mathbb{P}_{\in}$.*

From this results we get the following:

PROPOSITION 64 (Strong properness for transitive models [37]): *Let $\lambda > \theta$ be a large enough regular cardinal such that $\mathsf{H}(\theta), \mathbb{P}_{\in} \in \mathsf{H}(\lambda)$ and $K \preceq \mathsf{H}(\lambda)$ such that $\mathsf{H}(\theta), \mathbb{P}_{\in} \in K$ and $X = \mathsf{H}(\theta) \cap K \in \mathcal{T}$. The following holds:*

(1) *If $p \in \mathbb{P}_{\in}$ is such that $X \in p$, then p is a strong $(K, \mathbb{P}_{\in})$-generic condition.*
(2) *$\mathbb{P}_{\in}$ is strongly proper for K.*

Proving properness for countable models is much harder. The difficulty is that (unlike in the transitive case) if $p, q \in \mathbb{P}_\in$ and $M \in \mathcal{S}(p)$ such that $q \leq p \cap M$ and $q \in M$, then $q \cup p$ may not be a condition. The good news is that it can be extended to one:

Proposition 65: *Let $p, q \in \mathbb{P}_\in$ and $M \in \mathcal{S}(p)$ such that $q \leq p \cap M$ and $q \in M$. There is a condition $q \wedge p \in \mathbb{P}_\in$ such that:*

1. *$q \cup p$ is a path.*
2. *$q \wedge p$ is obtained by closing $q \cup p$ under intersections.*
3. *$q \wedge p$ is the largest common extension of both p and q.*
4. *$\mathcal{T}(q \wedge p) = \mathcal{T}(p) \cup \mathcal{T}(q)$.*
5. *$(q \wedge p) \cap M = q$.*
6. *Every node in $(q \wedge p) \setminus M$ is in p or it is of the form $N \cap X$ where $X \in \mathcal{T}(q)$ and $N \in \mathcal{S}(p) \cap M$.*

Above we mention that $q \wedge p$ is obtained by closing $q \cup p$ under intersections. However, it is worth pointing out that there is a nice and concrete construction of $q \wedge p$ from $q \cup p$ (see [37]). In fact, this explicit construction is what allows to prove the proposition just mentioned. From these results, it is possible to conclude the following:

Theorem 66 (Properness for countable models [37]): *$\mathbb{P}_\in$ is $\mathcal{S}$-proper. In particular, if $\mathcal{S}$ is a club, then $\mathbb{P}_\in$ is proper.*

Furthermore, we have the following:

Proposition 67 ([37]): *If $X \in \mathcal{T}$, then for every $p \in \mathbb{P}_\in$ there is $q \leq p$ such that $X \in q$.*

The chain condition of $\mathbb{P}_\in$ was not mentioned in [37]. The following was proved by Holy, Lücke and Njegomir. It was also independently proved by the second author while teaching his forcing course at the University of Toronto:

Proposition 68 ([20]): *$\mathbb{P}_\in$ has the θ-chain condition.*

Proof. Let $A \subseteq \mathbb{P}_\in$ be a set of size θ, we need to find two compatible elements in A. Since θ is an inaccessible cardinal, we know that $\mathsf{H}(\theta)$ has size θ (see [23]). In this way, we may enumerate $A = \{p_X \mid X \in \mathcal{T}\}$. By Proposition 67, for every transitive node X, we may find a condition $q_X \leq p_X$ such that $X \in q_X$.

Define $F : \mathcal{T} \longrightarrow \mathsf{H}(\theta)$ where

$$F(X) = X \cap q_X.$$

Clearly F is a choice function. Since $\mathcal{T}$ is stationary, we can find a stationary subset $\mathcal{T}_1 \subseteq \mathcal{T}$ such that F is constant on $\mathcal{T}_1$ (see [22]). We can now find $W \subseteq \mathcal{T}_1$ of size θ such that if $X, Y \in W$ and $X \in Y$, then $q_X \in Y$. It follows that if $X, Y \in W$ and $X \in Y$, then $q_X \leq q_Y \cap Y$ and $q_X \in Y$, so by Proposition 62, we know that q_X and q_Y are compatible. ∎

The following summarizes the effect of $\mathbb{P}_{\in}$ on the cardinals of V:

Proposition 69:

(1) $\mathbb{P}_{\in}$ *preserves* ω_1.
(2) *If* $\omega_1 < \kappa < \theta$, *then* $\mathbb{P}_{\in}$ *collapses* κ *to* ω_1.
(3) $\mathbb{P}_{\in}$ *has the* θ*-chain condition, so it preserves all cardinals that are larger than or equal to* θ.
(4) $\mathbb{P}_{\in} \Vdash$ "$\omega_2 = \theta$".

With the above results we can get a very clear picture of the generic object added by $\mathbb{P}_{\in}$. Let $G \subseteq \mathbb{P}_{\in}$ be a generic filter. In $V[G]$ we define the **generic path** $\mathcal{P}_{\text{gen}} = \bigcup G$. This is a path of length ω_2 that covers $\mathsf{H}(\theta)^V$. The transitive models now have size ω_1 and between any two of them there is an $\in$-chain of countable models of length ω_1.

Given any set A, by $\wp(A)$ we denote the power set of A.

Lemma 70: *Let* $p \in \mathbb{P}_{\in}$, $Y \in \mathcal{T}(p)$ *and* $M \in \mathcal{S}(p)$ *with* $Y <_p M$, $\mathbb{Q} \in M \cap Y$ *a partial order and* $\dot{\mathbb{P}} \in M \cap Y$ *a* $\mathbb{Q}$*-name for a partial order. Let* $G \subseteq \mathbb{Q}$ *be a generic filter and* $a \in \dot{\mathbb{P}}[G]$. *In* $V[G]$, *the following statements are equivalent:*

(1) a *is an* $((M \cap Y)[G], \dot{\mathbb{P}}[G])$*-generic condition.*
(2) a *is an* $(M[G], \dot{\mathbb{P}}[G])$*-generic condition.*

Proof. First, note that since Y is an elemental submodel of $\mathsf{H}(\theta)$ (and θ is inaccessible), it follows that $\wp(A) \in Y$ whenever $A \in Y$. Now, we will prove the following:

Claim 71: *In* $V[G]$ *the following holds:*

$$\wp(\mathbb{P}) \cap M[G] = \wp(\mathbb{P}) \cap (M \cap Y)[G].$$

We may assume that $\mathbb{P}$ is of the form $(\alpha, \leq_{\mathbb{P}})$ where α is an ordinal. By the remark above, every nice name (see [23, Chapter VII]) of a subset of α is in Y, the claim follows.

The conclusion of the lemma follows by the above claim since the definition of generic condition depends only on the subsets of $\dot{\mathbb{P}}[G]$ that are in the model. ∎

We can now explain the iteration technique introduce by Neeman. From now on, fix a function $J : \theta \longrightarrow \mathsf{H}(\theta)$, which we will use as a bookkeeping device. We will require that the elements of $\mathcal{S} \cup \mathcal{T}$ are also elemental with respect to J, by this we mean that if $A \in \mathcal{S} \cup \mathcal{T}$, then $(A, \in, \leq_w, J \restriction (A \cap \theta))$ is an elementary submodel of $(\mathsf{H}(\theta), \in, \leq_w, J)$. Note that this implies that each $A \in \mathcal{S} \cup \mathcal{T}$ is closed under J. Occasionally, we will write $J(\mathsf{H}(\lambda))$ instead $J(\lambda)$.[13] Clearly $\mathcal{T}$ and $\mathcal{T} \cup \{\mathsf{H}(\theta)\}$ are well-ordered by the membership relation. In this way, we can make recursive constructions and inductive proofs over them. Expressions like "Y is limit" or "Y is the successor of X" will refer to this order. By $Y = X^+$ we denote that Y is the successor of X in $\mathcal{T}$. The following definition is done by recursion over $\mathcal{T} \cup \{\mathsf{H}(\theta)\}$:

Definition 72: Define $\mathbb{P} = \mathbb{P}(J)$ as the set of all (p, f_p) with the following properties:

(1) $p \in \mathbb{P}_{\in}^{\mathcal{S},\mathcal{T}}$.
(2) Given $X \in \mathcal{T}$, $G_X \subseteq \mathbb{P} \cap X$ a generic filter and $Y = X^+$, define (in $V[G_X]$) the sets

$$\mathcal{S}_X[G_X] = \{M[G_X] \mid (M \in \mathcal{S}) \wedge (X \in M) \wedge (\{M \cap X\}, \emptyset) \in G_X\},$$
$$\mathcal{S}_{(X,Y)}[G_X] = \{M[G_X] \mid (M \in \mathcal{S}) \wedge (X \in M \in Y) \wedge (\{M \cap X\}, \emptyset) \in G_X\}.$$

(3) f_p is a function with domain contained in

$$\{X \in \mathcal{T}(p) \mid 1_{\mathbb{P}\cap X} \Vdash \text{"}J(X) \text{ is a } \mathcal{S}_{(X,Y)}[G_X]\text{-proper forcing"}\}$$

(4) If $X \in \text{dom}(f_p)$, then $(p \cap X, f_p \restriction X) \Vdash \text{"}f_p(X) \in J(X)\text{"})$.
(5) If $X \in \text{dom}(f_p)$, $M \in \mathcal{S}(p)$ and $X \in M$, then

$$(p \cap X, f_p \restriction X) \Vdash \text{"}f_p(X) \text{ is a } (M[\dot{G}_X], J(X)[\dot{G}_X])\text{-generic condition"}$$
$$(\text{where } \dot{G}_X \text{ is the name for the generic filter of } \mathbb{P} \cap X).$$

[13] In this way, if $A \in \mathcal{T}$, by $J(A)$ we denote $J(\lambda)$ where $A = \mathsf{H}(\lambda)$.

Let (p, f_p), $(q, f_q) \in \mathbb{P}$. Define $(p, f_p) \leq (q, f_q)$ if the following hold:

(1) $q \subseteq p$.
(2) $\mathrm{dom}(f_q) \subseteq \mathrm{dom}(f_p)$.
(3) If $X \in \mathrm{dom}(f_q)$, then $(p \cap X, f_p \restriction X) \Vdash$ "$f_p(X) \leq f_q(X)$".

It is clear that (in the extension) $\mathcal{S}_{(X,Y)}[G_X]$ is a subset of $\mathcal{S}_X[G_X]$. Note that

$$\mathcal{S}_{(X,Y)}[G_X] \subseteq [Y[G_X]]^{\omega}.$$

Note that thanks to Lemma 70, in order to satisfy Definition 72(5), it is enough to check the condition for those $M \in \mathcal{S}(p)$ such that $X \in M$ and $(X, M)_p \cap \mathcal{T} = \emptyset$. Although this is a very simple remark, it is indeed very useful.

It is always possible to add transitive nodes:

Lemma 73 ([37]): *Let $(p, f) \in \mathbb{P}$ and $X \in \mathcal{T}$. There is $q \in \mathbb{P}_{\in}$ such that:*

(1) $X \in q$.
(2) $q \leq p$.
(3) *If $A \in q \setminus p$ then one of the following conditions hold:*
   (a) *A is transitive.*
   (b) *There is $N \in \mathcal{S}(p)$ and $W \in \mathcal{T}(q)$ such that $A = N \cap W$.*
(4) $(q, f) \in \mathbb{P}$, *so* $(q, f) \leq (p, f)$.

With this, we can prove the following:

Proposition 74: *$\mathbb{P}$ has the θ-chain condition.*

Proof. This is almost the same argument as the one for Proposition 68. Let $A \subseteq \mathbb{P}$ be a set of size θ, we need to find two compatible elements in A. Take an enumeration $A = \{(p_X, f_X) \mid X \in \mathcal{T}\}$. By Lemma 73, for every $X \in \mathcal{T}$, we may find $(q_X, g_X) \leq (p_X, f_X)$ such that $X \in q_X$.

Define $F : \mathcal{T} \longrightarrow \mathsf{H}(\theta)$ where $F(X) = X \cap q_X$. Clearly F is a choice function. Since $\mathcal{T}$ is stationary, we can find a stationary subset $\mathcal{T}_1 \subseteq \mathcal{T}$ such that F is constant on $\mathcal{T}_1$. We can now find $W \subseteq \mathcal{T}_1$ of size θ such that if $X, Y \in W$ and $X \in Y$, then $q_X \in Y$. It follows that if $X, Y \in W$ and $X \in Y$, then $q_X \leq q_y \cap Y$ and $q_X \in Y$, so by Proposition 62, we know that q_X and q_Y are compatible in $\mathbb{P}_{\in}$. Furthermore, by Lemma 70, we conclude that (q_X, g_X) and (q_Y, g_Y) are compatible. ∎

In some sense the models in $\mathcal{T}$ play a similar role to the ordinals in the usual finite support iteration. An instance of this analogy is the following:

Lemma 75 ([37]): *If $X \in \mathcal{T}$, then $\mathbb{P} \cap X$ is a regular suborder of $\mathbb{P}$.*

For convenience, we will say a node $X \in \mathcal{T}$ is **not trivial** if

$$1_{\mathbb{P}\cap X} \Vdash \text{“}J(X) \text{ is a } \mathcal{S}_{(X,X^+)}[G_X]\text{-proper forcing”}.$$

We can always add non-trivial nodes to the domain:

Lemma 76 ([37]): *Let $(p,f) \in \mathbb{P}$ and $X \in \mathcal{T}(p)$ that is not trivial. There is a function g with the following property:*

(1) $\mathrm{dom}(g) = \mathrm{dom}(f) \cup \{X\}$.
(2) *$(p,g) \in \mathbb{P}$ and $(p,g) \leq (p,f)$.*

By combining the two lemmas, we get the following:

Lemma 77 ([37]): *Let $(p,f) \in \mathbb{P}$ and $X \in \mathcal{T}$ not trivial. There is $(q,g) \in \mathbb{P}$ such that:*

(1) $X \in q$.
(2) $(q,g) \leq (p,f)$.
(3) $\mathrm{dom}(g) = \mathrm{dom}(f) \cup \{X\}$.
(4) *If $A \in q \setminus p$ then one of the following conditions hold:*
  (a) *A is transitive.*
  (b) *There is $N \in \mathcal{S}(p)$ and $W \in \mathcal{T}(q)$ such that $A = N \cap W$.*

The following is an important step in order to prove that $\mathbb{P}$ is proper:

Proposition 78 ([37]): *Let $M \in \mathcal{S}$ and $(p,f) \in M \cap \mathbb{P}$. There is $(q,g) \in \mathbb{P}$ with the following properties:*

(1) $(q,g) \leq (p,f)$.
(2) $M \in q$.
(3) $\mathrm{dom}(g) = \mathrm{dom}(p)$.

Now we want to prove the $\mathcal{S}$-properness of $\mathbb{P}$. Our proof is different from the one in [37]. The main difference is that we will use the results of the side condition hull obtained earlier.

We will need the following:

Lemma 79: *Let $X \in \mathcal{T}$, $G_X \subseteq \mathbb{P} \cap X$ a generic filter and $M \in \mathcal{S}$ such that $X \in M$. The following two statements are equivalent:*

(1) $M[G_X] \in \mathcal{S}_X[G_X]$.
(2) $(\{M\}, \emptyset)$ *is compatible (in* $\mathbb{P}$*) with every element of* G_X.

Proof. We will first prove that (2) implies (1). Since G_X is a generic filter, in order for $(\{M \cap X\}, \emptyset)$ to be in G_X, it is enough to prove that every element of G_X is compatible with $(\{M \cap X\}, \emptyset)$, which clearly is a consequence of (2).

We will now prove that (1) implies (2). Let $(p, f) \in G_X$. Since $(\{M \cap X\}, \emptyset) \in G_X$, we know that there is $(q, g) \in G_X$ such that $(q, g) \leq (p, f)$ and $M \cap X \in q$. Define $r = q \cup \{X, M\}$, we claim that $(r, g) \in \mathbb{P}$.

It is clear that r is a path. We will now prove that r is closed under intersections. It is enough to prove that if $A \in q$, then $M \cap A \in r$. Since $M \cap X \in q$, we have that $(M \cap X) \cap A$ is in q. Since $A \in X$, we get that

$$(M \cap X) \cap A = M \cap (X \cap A) = M \cap A,$$

so we are done. Finally, let $L \in \mathrm{dom}(g)$ such that $L \in M$, we need to prove that $g(L)$ is generic for M, but this is true since it is generic for $M \cap X$. ∎

We will now get the following:

Lemma 80: *Let $X, Y \in \mathcal{T}$ such that $Y = X^+$. Let $G_X \subseteq \mathbb{P} \cap X$ be a generic filter and $M \in \mathcal{S}$ with $X \in M$. The following are equivalent:*

(1) $M[G_X] \in \mathcal{S}_X[G_X]$.
(2) $(\{M\}, \emptyset)$ *is compatible with every element of* G_X.
(3) $(M \cap Y)[G_X] \in \mathcal{S}_{(X,Y)}[G_X]$.

Proof. We already know from Lemma 79 that $M[G] \in \mathcal{S}_X[G_X]$ if and only if $(\{M\}, \emptyset)$ is compatible with every element of G_X. Now, we have the following:

$$\begin{aligned} M[G_X] \in \mathcal{S}_X[G_X] &\iff (\{M \cap X\}, \emptyset) \in G_X \\ &\iff (\{M \cap (X \cap Y)\}, \emptyset) \in G_X \\ &\iff (\{(M \cap Y) \cap X\}, \emptyset) \in G_X \\ &\iff (M \cap Y)[G_X] \in \mathcal{S}_{(X,Y)}[G_X]. \end{aligned}$$ ∎

Now we will prove the following:

Proposition 81: *Let $X \in \mathcal{T}$.*

1. $\mathbb{P} \cap X \Vdash$ "$\mathcal{S}_X[G]$ *is stationary in* $[\mathsf{H}(\theta)]^\omega$".
2. *If* $Y = X^+$, *then* $\mathbb{P} \cap X \Vdash$ "$\mathcal{S}_{(X,Y)}[G]$ *is stationary in* $[Y[G]]^\omega$".

Proof. We start with point (1). Since θ is inaccessible, we have that

$$\mathbb{P} \cap X \Vdash \text{"}\mathsf{H}^{V[G]}(\theta) = \mathsf{H}^V(\theta)[G]\text{"}$$

(see Proposition 11). Let $(p, f) \in \mathbb{P} \cap X$ and $\dot{K}$ a $\mathbb{P} \cap X$-name such that

$$(p, f) \Vdash \text{"}\dot{K} : [\mathsf{H}(\theta)]^{<\omega} \longrightarrow \mathsf{H}(\theta)\text{"}.$$

Since $\mathcal{S}$ is stationary, we can find $\overline{M}$ a countable elementary submodel of a large enough structure such that:

1. $\mathbb{P}, X, (p, f), \dot{K} \in \overline{M}$.
2. $M = \overline{M} \cap \mathsf{H}(\theta)$ is in $\mathcal{S}$.

Note that $X, (p, f) \in M$. By Proposition 78 and Lemma 73, we can find $(q, g) \in \mathbb{P}$ such that $(q, g) \leq (p, f)$, $M, X \in q$ and $\text{dom}(g) = \text{dom}(p)$. Let $\overline{p} = q \cap X$ and $\overline{f} = g \restriction X$. Since $M, X \in q$, it follows that $M \cap X \in \overline{p}$, so $(\overline{p}, \overline{f}) \Vdash$"$M[G] \in \mathcal{S}_X[G]$".

We claim that $(\overline{p}, \overline{f})$ forces that $M[G]$ is closed under $\dot{K}$. Let $\dot{E}_1, \ldots, \dot{E}_n \in M$. We want to prove that $(\overline{p}, \overline{f}) \Vdash$"$\dot{K}(\dot{E}_1, \ldots, \dot{E}_n) \in M[G]$". To see this, note that if $\dot{R}$ is a nice name for $\dot{K}(\dot{E}_1, \ldots, \dot{E}_n)$, then $\dot{R} \in \overline{M}$. This is because $\dot{K}, \dot{E}_1, \ldots, \dot{E}_n \in \overline{M}$. Furthermore, since $\dot{R}$ is a name for an element of $\mathsf{H}(\theta)[G]$, then $\dot{R} \in \mathsf{H}(\theta)$ (see Proposition 11; note that although $\dot{K}$ might not be in $\mathsf{H}(\theta)$, it is nevertheless true that $\dot{R}$ is). It follows that $\dot{R} \in \overline{M} \cap \mathsf{H}(\theta) = M$. This implies that $\dot{R}[G]$ will be in $M[G]$.

The proof of the second point in the proposition is essentially the same. ∎

We now recall the following well-known definition:

Definition 82: Let $\mathbb{R}$ and $\mathbb{Q}$ be two partial orders. We say that $i : \mathbb{R} \longrightarrow \mathbb{Q}$ is a **dense embedding** if the following conditions hold for every $p_1, p_2 \in \mathbb{R}$:

1. If $p_1 \leq p_2$, then $i(p_1) \leq i(p_2)$.
2. If p_1 and p_2 are incompatible, then $i(p_1)$ and $i(p_2)$ are incompatible (or equivalently, if $i(p_1)$ and $i(p_2)$ are compatible, then p_1 and p_2 are compatible).
3. $i[\mathbb{R}]$ is a dense subset of $\mathbb{Q}$.

If there is a dense embedding $i:\mathbb{R}\longrightarrow\mathbb{Q}$, then $\mathbb{R}$ and $\mathbb{Q}$ yield the same generic extensions. To learn more about dense embeddings, the reader may consult [24]. We can now obtain a "factorization" theorem for the successors steps:

Proposition 83: *Let $X, Y \in \mathcal{T}$ with $Y = X^+$.*

(1) *If X is not trivial, then $\mathbb{P} \cap Y$ and $(\mathbb{P} \cap X) * \mathbb{S}_\in(J(X), \mathcal{S}_{(X,Y)}[G_X])$ are forcing equivalent (where $\dot{G}_X$ is the canonical name for the $\mathbb{P} \cap X$ generic filter).*
(2) *If X is trivial, then $\mathbb{P} \cap Y$ and $(\mathbb{P} \cap X) * \mathbb{S}_\in(1, \mathcal{S}_{(X,Y)}[G_X])$ are forcing equivalent (where $\dot{G}_X$ is the canonical name for the $\mathbb{P}\cap X$ generic filter and 1 is the trivial forcing).*

Proof. We will assume that X is not trivial, since the other case is similar, yet simpler. First, define D as the set of all $((r, h), (\{N_0[\dot{G}_X], \dots, N_n[\dot{G}_X]\}), \dot{a})$ with the following properties:

(1) $(r, h) \in \mathbb{P} \cap X$.
(2) $N_0, \dots, N_n \in \mathcal{S}$.
(3) $X \in N_0 \in \cdots \in N_n \in Y$.
(4) $N_i \cap X \in r$ for all $i \leq n$.
(5) $(r, h) \Vdash$"$\dot{a}$ is $(N_i[\dot{G}_X], J(X))$-generic" for all $i \leq n$.

Clearly $D \subseteq \mathbb{P} \cap X * \mathbb{S}_\in(J(X), \mathcal{S}_{(X,Y)}[G_X])$. We now have the following:

Claim 84: *D is a dense subset of $\mathbb{P} \cap X * \mathbb{S}_\in(J(X), \mathcal{S}_{(X,Y)}[G_X])$.*

We will prove the claim. Let $((p, f), (\dot{F}, \dot{a}))$ be an element of $(\mathbb{P}\cap X)*\mathbb{S}_\in(J(X), \mathcal{S}_{(X,Y)}[G_X])$. By definition, we know that (p, f) forces that $\dot{F}$ is a finite chain of $\mathcal{S}_{(X,Y)}[G_X]$. In this way, we can find $(p_1, f_1) \leq (p, f)$ and $\{N_0, \dots, N_n\}$ such that $(p_1, f_1) \Vdash$"$\dot{F} = \{N_0[\dot{G}_X], \dots, N_n[\dot{G}_X]\}$". Furthermore, since $(p_1, f_1) \Vdash$"$N_i[\dot{G}_X] \in \mathcal{S}_{(X,Y)}[G_X]$" (for every $i \leq n$), we can find $(r, h) \leq (p_1, f_1)$ such that $N_i \cap X \in r$ for all $i \leq n$. This finishes the proof of the claim.

Now, define $E = \{(p, f) \in \mathbb{P} \cap Y \mid X \in \mathrm{dom}(f)\}$. By Lemma 77, we know that E is a dense subset of $\mathbb{P}\cap Y$. Since E is forcing equivalent to $\mathbb{P}\cap Y$ and D is forcing equivalent to $(\mathbb{P} \cap X) * \mathbb{S}_\in(J(X), \mathcal{S}_{(X,Y)}[G_X])$, it is enough to prove that E and D are forcing equivalent. In order to do so, we define a function $i : E \longrightarrow D$ given by

$$i(p, f) = ((p \cap X, f \restriction X), (\{M[\dot{G}] \mid X \in M \in p\}), f(X)).$$

We claim that i is a dense embedding. It is clear that if $(p,f)\leq(q,g)$, then $i(p,f)\leq i(q,g)$. Now, assume that $i(p,f)$ and $i(q,g)$ are compatible, we must prove that (p,f) and (q,g) are compatible. Let $((r,h),(\{N_0[\dot{G}_X],\ldots,N_n[\dot{G}_X]\}),\dot{a})$ be a common extension of $i(p,f)$ and $i(q,g)$. It follows that

$$(r,h)\leq(p\cap X,f\restriction X)\quad\text{and}\quad(r,h)\leq(q\cap X,g\restriction X).$$

Let $\overline{r}=r\cup\{X,N_0,\ldots,N_n\}$ and $\overline{h}=h\cup\{(X,\dot{a})\}$. It follows that $(\overline{r},\overline{h})$ extends (p,f) and (q,g). It is easy to see that i is onto, so in particular, the image is dense. This finishes the proof that $\mathbb{P}\cap Y$ and $(\mathbb{P}\cap X)*\mathbb{S}_{\in}(J(X),\mathcal{S}_{(X,Y)}[G_X])$ are forcing equivalent. ∎

The following lemma might seem artificial at first, but will come in handy when dealing with limit steps:

Lemma 85: *Let $Z,Y\in\mathcal{T}$ and $M\in\mathcal{S}$ such that $Z\in Y$ and $Z,Y\in M$. Let $(p,f),(q,g)\in\mathbb{P}\cap Y$ and $(r,h)\in\mathbb{P}\cap Z$ with the following properties:*

1. $Z,M\cap Y\in p$.
2. $\mathrm{dom}(f)\cap M\subseteq Z$.
3. $(q,g)\in M$.
4. $p\cap M\subseteq q$.
5. $(r,h)\leq(p\cap Z,f\restriction Z)$, $(q\cap Z,g\restriction Z)$.

Then (p,f) and (q,g) are compatible (in $\mathbb{P}\cap Y$). Furthermore, there is $(\overline{r},\overline{h})\in\mathbb{P}\cap Y$ such that $(\overline{r},\overline{h})\leq(p,f),(q,g)$, $\overline{r}\cap Z=r$ and $\overline{h}\restriction Z=h$.

Proof. Note that we have the following:

1. $q\in M\cap Y$.
2. $M\cap Y\in p$.
3. $q\leq p\cap(M\cap Y)=p\cap M$.

In this way, by Proposition 65, we can form $q\wedge p$. Since $q\wedge p$ is the largest common extension of p and q, it follows that $r\leq(q\wedge p)\cap Z$. Now, since $r\in Z$ and $Z\in q\wedge p$ (recall that $Z\in p$), by Proposition 62, we know that $\overline{r}=r\cup((q\wedge p)\setminus Z)$ is a condition of $\mathbb{P}_{\in}$.

Let $S=\mathrm{dom}(h)\cup\mathrm{dom}(g)\cup\mathrm{dom}(f)$. We know the following:

1. $\mathrm{dom}(g)\cap Z$, $\mathrm{dom}(f)\cap Z\subseteq\mathrm{dom}(h)$ and $\mathrm{dom}(h)\subseteq Z$.
2. $\mathrm{dom}(g)\subseteq M$.
3. $\mathrm{dom}(f)\cap\mathrm{dom}(g)\subseteq Z$ (recall that $\mathrm{dom}(f)\cap M\subseteq Z$ and $g\in M$).
4. $Z\notin\mathrm{dom}(f)$.

In this way,

$$S = \mathrm{dom}(h) \cup (\mathrm{dom}(g) \setminus Z) \cup (\mathrm{dom}(f) \setminus Z)$$

and this is a disjoint union. We now define $\overline{h} : S \longrightarrow \mathsf{H}(\theta)$ as follows:

(1) $h \subseteq \overline{h}$.
(2) If $W \in \mathrm{dom}(f) \setminus Z$, then $\overline{h}(W) = f(W)$.
(3) Let $W \in \mathrm{dom}(g) \setminus Z$. Define W^* the first transitive node of r above W if there is one, if not, let $W^* = Y$. Note that $g(W) \in M \cap W^*$. Let $[M \cap W^*, W^*)_r = \{N_0, \ldots, N_n\}$ (where $N_0 = M \cap W^*$). Since $J(W)$ is forced to be an $\mathcal{S}_{(W,W^+)}[G_W]$-proper forcing, we define $\overline{h}(W)$ as an extension of $g(W)$ that is forced to be $N_i[\dot{G}]$ generic for all $i \leq n$.

We claim that $(\overline{r}, \overline{h})$ is in $\mathbb{P} \cap Y$. Let $W \in \mathrm{dom}(\overline{h})$ and $N \in \mathcal{S}(\overline{r})$ such that $N \in W$ and $(W, N)_{\overline{r}} \cap \mathcal{T} = \emptyset$. We need to prove that $\overline{h}(W)$ is forced to be generic for $N[\dot{G}]$. If $W <_r Z$, then we are fine since $(r, h) \in \mathbb{P}$, so now we assume that $Z \leq_r W$.

Case 86: $W \in \mathrm{dom}(f) \setminus Z$.

Here we have that $\overline{h}(W) = f(W)$. If $N \in p$, then we are fine since $(p, f) \in \mathbb{P}$. Note that $N \notin M$ (in particular, $N \notin q$) because if this was not the case, then $W \in N \in M$, so $W \in M$. But this is impossible since $\mathrm{dom}(f) \cap M \subseteq Z$. We are now in the case that $N \in (q \wedge p) \setminus M$ and $N \notin p$. By Proposition 65, we know that there are $L \in \mathcal{S}(p) \cap M$ and $X \in \mathcal{T}(q)$ such that $N = L \cap X$. Since $L \in p$, we know that $f(W) = \overline{h}(W)$ is forced to be generic for L, so it is also generic for $N = L \cap X$ by Lemma 70.

Case 87: $W \in \mathrm{dom}(g) \setminus Z$.

Let W^* be as defined above. First, note that $[W, M \cap W^*)_{\overline{r}} \subseteq q$. This is because every element in this interval is above Z and is also in M. Since $\overline{h}(W)$ is (forced to be) an extension of $g(W)$, it means that we are fine with every node in this interval. Furthermore, it follows by the definition of $\overline{h}(W)$ that it is (forced to be) a generic condition for every interval in $[M \cap W^*, W^*)_{\overline{r}}$. In this way, $\overline{h}(W)$ is a generic condition for every node in $(W, W^*)_{\overline{r}}$ and this is enough by Lemma 70.

It follows that $(\overline{r}, \overline{h})$ is a condition and clearly $(\overline{r}, \overline{h}) \leq (p, f), (q, g)$. ∎

We now have all the tools to prove the following:

THEOREM 88: *Let $Y \in \mathcal{T}$.*

(1) *If $\overline{M}$ is a countable elementary submodel of a large enough structure such that $\mathbb{P}, Y \in \overline{M}$ and $M = \overline{M} \cap \mathsf{H}(\theta) \in \mathcal{S}$, then for every $(p, f) \in \mathbb{P} \cap Y$ if $M \cap Y \in p$, then (p, f) is $(\overline{M}, \mathbb{P} \cap Y)$-generic.*
(2) *$\mathbb{P} \cap Y$ is $\mathcal{S}$-proper.*

Proof. Before starting the proof, note that by Proposition 78 we get that point (1) implies point (2). We proceed by induction over Y. If Y is the smallest element of $\mathcal{T}$, then $\mathbb{P} \cap Y$ is the $\in$-collapse parametrized by $\mathcal{S}$, which is $\mathcal{S}$-proper (even in its stronger form stated in point (1)) by the argument of [52, Theorem 47].

For the successor step, let $Y = X^+$ and assume the theorem holds for X. Let $\overline{M}$ be a countable elementary submodel of a large enough structure such that $\mathbb{P}, Y \in \overline{M}$ and $M = \overline{M} \cap \mathsf{H}(\theta) \in \mathcal{S}$. We will assume that X is not trivial, since the other case is similar but easier. Let $(p_1, f_1) \in \mathbb{P} \cap Y$ be a condition such that $M \cap Y \in p_1$. We want to prove that (p_1, f_1) is $(\overline{M}, \mathbb{P} \cap Y)$-generic. Or equivalently, that every extension of (p_1, f_1) has a further extension that is $(\overline{M}, \mathbb{P} \cap Y)$-generic.

Let $(p_2, f_2) \leq (p_1, f_1)$ and by Lemma 77, we can find an extension

$$(p, f) \leq (p_2, f_2)$$

such that $X \in \mathrm{dom}(f)$ (and in particular, $X \in p$). Note that $X, M \cap Y \in p$, so it follows that

$$M \cap X = (M \cap Y) \cap X$$

is in p. Recall that from Proposition 83, we have a dense embedding i from (a dense set of) $\mathbb{P} \cap Y$ to $\mathbb{P} \cap X * \mathbb{S}_\in(J(X), \mathcal{S}_{(X,Y)}[G])$. Here, we have that

$$i(p, f) = ((p \cap X, f \restriction X), (\{N[\dot{G}] \mid X \in N \in p\}), f(X)).$$

Note that $(M \cap Y)[\dot{G}]$ is in the second coordinate of $i(p, f)$. By the inductive hypothesis, we know that $(p \cap X, f \restriction X)$ is generic for $\mathbb{P} \cap X$. Also, by the previous remark and Proposition 50, it follows that the tail of $i(p, f)$ is forced to be generic for $\mathbb{S}_\in(J(X), \mathcal{S}_{(X,Y)}[G])$. This implies that $i(p, f)$ is generic for $\mathbb{P} \cap X * \mathbb{S}_\in(J(X), \mathcal{S}_{(X,Y)}[G])$, which entails that (p, f) is $(\overline{M}, \mathbb{P} \cap Y)$-generic.

We are now left in the case that Y is a limit node. Let $\overline{M}$ be a countable elementary submodel of a large enough structure such that $\mathbb{P}, Y \in \overline{M}$ and $M = \overline{M} \cap \mathsf{H}(\theta) \in \mathcal{S}$. Let $(p_1, f_1) \in \mathbb{P} \cap Y$ be a condition such that $M \cap Y \in p_1$.

We want to prove that (p_1, f_1) is $(\overline{M}, \mathbb{P} \cap Y)$-generic. Let $D \in \overline{M}$ be an open dense subset of $\mathbb{P} \cap Y$. We need to prove that $D \cap \overline{M}$ is predense below (p_1, f_1). Let $(\overline{p}, f) \leq (p_1, f_1)$ and we may as well assume that $(\overline{p}, f) \in D$. Since $Y \in \overline{M}$ and it is limit, by elementarity, we can find a transitive node Z such that $Z \in M \cap Y$ and $\overline{p} \cap M \subseteq Z$.

By Lemma 73, we can find $p \in \mathbb{P}_{\in}$ such that $\overline{p} \cup \{Z\} \subseteq p$ and (p, f) is a condition. Since we are not changing f, we have that $\mathrm{dom}(f) \cap M \subseteq Z$. We know that $Z, M \cap Y \in p$, so it follows that $M \cap Z = (M \cap Y) \cap Z$ is in p. By the inductive hypothesis, this implies that $(p \cap Z, f \restriction Z)$ is an $(\overline{M}, \mathbb{P} \cap Z)$-generic condition. We now define

$$D_Z = \{(q \cap Z, g \restriction Z) \mid p \cap M \subseteq q \wedge (q, g) \in D\}.$$

It is clear that $D_Z \in \overline{M}$ and $(p \cap Z, f \restriction Z) \in D_Z$. Since $(p \cap Z, f \restriction Z)$ is a generic condition, we conclude that there is $(q, g) \in \mathbb{P} \cap Y$ with the following properties:

(1) $(q, g) \in D \cap M$.
(2) $p \cap M \subseteq q$.
(3) $(q \cap Z, g \restriction Z)$ and $(p \cap Z, f \restriction Z)$ are compatible (in $\mathbb{P} \cap Z$).

By Lemma 85, we conclude that (q, g) and (p, f) are compatible. ∎

We can finally prove the following:

Theorem 89 (Neeman [37]): *$\mathbb{P}$ is $\mathcal{S}$-proper. In particular, if $\mathcal{S}$ is a club, then $\mathbb{P}$ is proper.*

Proof. We already know that all of the forcings $\mathbb{P} \cap Y$ are $\mathcal{S}$-proper (for $Y \in \mathcal{T}$). It remains to prove that $\mathbb{P}$ itself is $\mathcal{S}$-proper. Let $\overline{M}$ be a countable elementary submodel of a large enough structure such that $\mathbb{P} \in \overline{M}$ and $M = \overline{M} \cap \mathsf{H}(\theta) \in \mathcal{S}$. Let $(p_1, f_1) \in \mathbb{P} \cap \overline{M}$. By Proposition 78, we can find $(p, f) \leq (p_1, f_1)$ such that $M \in p$. We claim that (p, f) is an $(M, \mathbb{P})$-generic condition.

Let $A \in \overline{M}$ be a maximal antichain of $\mathbb{P}$ and $(p_2, f_2) \leq (p, f)$, we need to prove that $A \cap \overline{M}$ is predense below (p_2, f_2). By Proposition 74 and elementarity, we can find a transitive node $Y \in \mathcal{T} \cap \overline{M}$ such that $A \subseteq \mathbb{P} \cap Y$. Let $(q, g) \leq (p_2, f_2)$ such that $Y \in q$. Note that we also have that $M \cap Y$ is in q. In this way, by Theorem 88 we know that (q, g) is an $(\overline{M}, \mathbb{P} \cap Y)$-generic condition, so it is compatible with an element of $A \cap \overline{M}$. ∎

In the same way as $\mathbb{P}_{\in}$, the forcing $\mathbb{P}$ has the following properties:

Proposition 90:

(1) $\mathbb{P}$ *preserves* ω_1.
(2) *If* $\omega_1 < \kappa < \theta$, *then* $\mathbb{P}$ *collapses* κ *to* ω_1.
(3) $\mathbb{P}$ *has the* θ*-chain condition, so it preserves all cardinals that are larger than or equal to* θ.
(4) $\mathbb{P} \Vdash$ "$\omega_2 = \theta$".

We now introduce the following notion:

Definition 91: Let $G \subseteq \mathbb{P}$ be a generic filter. In $V[G]$, define

$$\mathcal{S}[G] = \{M[G] \mid M \in \mathcal{S} \wedge (M, \emptyset) \in G\}.$$

With a very similar proof to the one of Proposition 81 it is possible to show the following:

Proposition 92: $\mathbb{P} \Vdash$ "$\mathcal{S}[G]$ *is stationary in* $[\mathsf{H}(\omega_2)]^\omega$".

The reader wishing to know more about two type side conditions may consult [37], [38], [25], [13], [20], [14], [55], [19] and [56].

8. Entangled sets and two type side conditions

We developed all the tools needed in order to prove Theorem 7, it remains to combine them all together. We start with the following simple proposition:

Proposition 93: *Let* $E = \{e_\alpha \mid \in \omega_1\} \subseteq \mathbb{R}$ *be a* 2*-entangled set and* $\mathbb{P}$ *a strongly proper forcing.*

(1) *Let* $\dot{\mathcal{B}}$ *be a* $\mathbb{P}$*-name for an uncountable block-sequence of* $[\omega_1]^2$ *and* M *a countable elementary submodel of a large enough structure such that* $\mathbb{P}, E, \dot{\mathcal{B}} \in M$. *If* $p \in \mathbb{P}$ *is a strong* $(M, \mathbb{P})$*-generic condition,* $b \in [\omega_1]^2$ *is such that* $b \cap M = \emptyset$, $p \Vdash$ "$b \in \dot{\mathcal{B}}$" *and* $t : 2 \longrightarrow \{>, <\}$ *is a type, then* $p \Vdash$ "$\exists a \in \dot{\mathcal{B}}(T(a,b) = t)$".
(2) $\mathbb{P}$ *preserves* E.

Proof. It is clear that the first point implies the second. Let $\dot{\mathcal{B}}, M, p, b$ and t as above. Define D as the set of all $r \in \mathbb{P} \cap M$ such that one of the following conditions hold:

(1) $r \perp p$.
(2) There is $a \in [\omega_1]^2$ such that $T(a,b) = t$ and $r \Vdash_{\mathbb{P}}$ "$\exists a \in \dot{\mathcal{B}}(T(a,b) = t)$".

We claim that D is dense in $\mathbb{P}\cap M$. Let $r \in \mathbb{P}\cap M$; if r is incompatible with p we are done, so assume this is not the case. Define

$$A = \{d \in [\omega_1]^2 \mid \exists q \leq r(q \Vdash_{\mathbb{P}} \text{“}d \in \dot{\mathcal{B}}\text{”})\},$$

which is clearly an element of M. Since r and q are compatible, it follows that $b \in A$. By Proposition 18, there is $d \in A \cap M$ such that $T(d,b) = t$. By elementarity, we can find $q \in M$ extending r such that $q \Vdash$“$d \in \dot{\mathcal{B}}$”. It is clear that q is an extension of r that is in D, so this set is dense.

Since p is a strong $(M,\mathbb{P})$-generic condition, it follows that there is $q \in D$ such that q and p are compatible. This finishes the proof of the first point. ∎

We can now prove the preservation theorem for 2-entangled sets under Neeman iteration:

Theorem 94: *Let θ be an inaccessible cardinal, J, $\mathcal{S}$, $\mathcal{T}$, $\mathbb{P}_\in$ and $\mathbb{P}$ as in the previous section, with $\mathcal{S}$ a club. Let $E = \{e_\alpha \mid \alpha \in \omega_1\}$ be a 2-entangled set. If for every $X \in \mathcal{T}$, either X is trivial or $\mathbb{P}\cap X \Vdash$“$J(X)$ preserves E”, then $\mathbb{P}$ preserves E.*

Proof. By the last section, we know that $\mathbb{P}$ has the θ-chain condition and it does not collapse ω_1, so it will be enough to prove that if $Y \in \mathcal{T}$, then $\mathbb{P}\cap Y$ preserves E. We proceed by induction on Y.

If Y is the smallest element of $\mathcal{T}$, then $\mathbb{P}\cap Y$ is strongly proper (since it is an $\in$-collapse). This case is taken care of by Proposition 93. The successor case follows by Proposition 83 and by Proposition 53. There remains the case where Y is a limit model. The argument follows closely the one from Theorem 88.

Let $(p_1, f_1) \in \mathbb{P}\cap Y$, $\dot{\mathcal{B}}$ a $\mathbb{P}\cap Y$-name for an uncountable block-sequence of pairs of countable ordinals and $t : 2 \longrightarrow \{>,<\}$ a type. We need to prove that we can extend (p_1, f_1) to a condition that forces that t is realized in $\dot{\mathcal{B}}$. Let $\overline{M}$ be a countable elementary submodel of a large enough structure such that $\mathbb{P}, Y, E, \dot{\mathcal{B}}, (p_1, f_1) \in \overline{M}$ and $M = \overline{M} \cap \mathsf{H}(\theta) \in \mathcal{S}$. By Proposition 78, we can find $(p_2, f) \leq (p_1, f_1)$ such that $M \cap Y \in p_2$. We may further assume that there is $b \in [\omega_1]^2$ such that $b \cap M = \emptyset$ and

$$(p_2, f) \Vdash \text{“}b \in \dot{\mathcal{B}}\text{”}.$$

Since Y is a limit model and $Y \in M$, we can find $Z \in M\cap Y$ such that $p_2 \cap M \subseteq Z$. Now, by Lemma 73, we can find $p \in \mathbb{P}_\in$ such that $p_2 \cup \{Z\} \subseteq p$ and (p, f) is a condition. Since we are not changing f, we have that $\mathrm{dom}(f) \cap M \subseteq Z$. We know that $Z, M \cap Y \in p$, so it follows that $M \cap Z = (M \cap Y) \cap Z$ is in p. It follows by Theorem 88 that $(p \cap Z, f \restriction Z)$ is an $(\overline{M}, \mathbb{P} \cap Z)$-generic condition. We now define

$$\dot{\mathcal{A}} = \{(a, (q \cap Z, g \restriction Z)) \mid a \in [\omega_1]^2 \wedge p \cap M \subseteq q \wedge (q, g) \Vdash \text{“}a \in \dot{\mathcal{B}}\text{”}\}.$$

It is clear that $\dot{\mathcal{A}} \in \overline{M}$ and is a $\mathbb{P} \cap Z$-name for a subset of $[\omega_1]^2$. We know that $(p \cap Z, f \restriction Z)$ is an $(M, \mathbb{P} \cap Z)$-generic condition and

$$(p \cap Z, f \restriction Z) \Vdash \text{“}b \in \dot{\mathcal{A}}\text{”}.$$

By the inductive hypothesis and Proposition 20, we know that there are $(q, g) \in \mathbb{P} \cap Y$ and $a \in [\omega_1]^2$ with the following properties:

(1) $(q, g), a \in M$.
(2) $p \cap M \subseteq q$.
(3) $(q \cap Z, g \restriction Z)$ and $(p \cap Z, f \restriction Z)$ are compatible (in $\mathbb{P} \cap Z$).
(4) $T(a, b) = t$.
(5) $(q, g) \Vdash$“$a \in \dot{\mathcal{B}}$”.

By Lemma 85, we conclude that (q, g) and (p, f) are compatible; this finishes the proof. ∎

Recall the following notion introduced by Solovay:

Definition 95: Let θ be a cardinal. We say that θ is **supercompact** if for every cardinal λ, there are M and j with the following properties:

(1) M is a transitive inner model.
(2) $j : V \longrightarrow M$ is an elementary embedding.
(3) $\mathrm{crit}(j) = \theta$.
(4) $j(\theta) > \lambda$.
(5) $[M]^\lambda \subseteq M$.

The following is a remarkable theorem of Laver:

Theorem 96 (Laver, [29]): *Let θ be a supercompact cardinal. There is a function $J : \theta \longrightarrow \mathsf{H}(\theta)$ such that for every set X, there is an elementary embedding $j : V \longrightarrow M$ such that $j(J)(\theta) = X$.*

A function as above is called a **Laver sequence** or **Laver diamond**. We can finally prove the promised result:

THEOREM 97 (LC): *There is a model of* $\mathsf{ZFC} + \mathsf{MA} + \mathsf{PID} + \mathfrak{c} = \omega_2$ *in which there is a* 2*-entangled set.*

Proof. We start with a model of GCH in which there is a supercompact cardinal θ. Let $K : \theta \longrightarrow \mathsf{H}(\theta)$ be a Laver sequence. Fix $<_w$ a well-order of $\mathsf{H}(\theta)$. Let $\mathcal{S}$ be the set of all countable elementary submodels of $(\mathsf{H}(\theta), \in, <_w, K)$ and $\mathcal{T}$ the set of all $\mathsf{H}(\lambda)$ that are elementary submodels of $(\mathsf{H}(\theta), \in, <_w, K)$ and that λ has uncountable cofinality. Fix $E = \{e_\alpha \mid \alpha \in \omega_1\} \subseteq \mathbb{R}$ a 2-entangled set (which exists by the Continuum Hypothesis). Recursively, we define a function $J : \theta \longrightarrow \mathsf{H}(\theta)$ and $\mathbb{P}$ as follows:

(1) $\mathbb{P} = \mathbb{P}(J)$ is the Neeman iteration using J and $\mathcal{S}, \mathcal{T}$ as parameters.
(2) If $\alpha < \theta$ and α is not a cardinal of uncountable cofinality or $\mathsf{H}(\alpha) \notin \mathcal{T}$, then $J(\alpha) = \emptyset$.
(3) If $X = \mathsf{H}(\alpha)$ is in $\mathcal{T}$, then we do as follows:
    (a) If $K(X) = \dot{\mathcal{J}}_X$ is a $\mathbb{P} \cap X$-name for a P-ideal where the second possibility of the P-ideal dichotomy does not hold, then $J(X)$ is the $\mathbb{P} \cap X$-name for $\mathbb{P}(\dot{\mathcal{J}}_X)$.
    (b) If $K(X) = \dot{\mathbb{Q}}_X$ is a $\mathbb{P} \cap X$-name of a ccc partial order that preserves E, then $J(X) = \dot{\mathbb{Q}}_X$.
    (c) If $K(X) = \dot{\mathbb{Q}}_X$ is a $\mathbb{P} \cap X$-name of a ccc partial order that does not preserve E, then $J(X)$ is a $\mathbb{P} \cap X$-name of a proper forcing that preserves E and adds an uncountable antichain to $\dot{\mathbb{Q}}_X$.
    (d) In any other case, let $J(X)$ be the $\mathbb{P} \cap X$-name of the trivial forcing.

The previous construction is well defined since $\mathbb{P} \cap X$ only depends on $J \restriction X$. Recall that $\mathbb{P}$ forces $\mathfrak{c} = \omega_2$. Since every iterand of $\mathbb{P}$ preserves E, by Theorem 94, E will still be a 2-entangled set after forcing with $\mathbb{P}$. Finally, $\mathbb{P}$ forces the P-ideal dichotomy and Martin's axiom by the same argument as the one of [37, Lemma 6.14] (see also [25, Lemma 3.20]). ∎

9. Open questions

We finish the paper with some open questions. The results of this article suggest the following:

Problem 98: Is there a "natural" cardinal invariant $\mathfrak{j}$ such that under PID, the statement "There are no 2-entangled sets" (OGA, $\mathsf{BA}(\omega_1)$) is equivalent to "$\omega_1 < \mathfrak{j}$"?

It is a well-known theorem of the second author that PFA implies $\mathfrak{c} = \omega_2$ (see [10]). We can ask the following:

Problem 99: Does PID + MA_{ω_1} imply that $\mathfrak{c} = \omega_2$?

In fact, the following is not known:

Problem 100: Does PID imply that $\mathfrak{c} \leq \omega_2$?

It is very possible that under the *P*-ideal dichotomy the statements "$\mathfrak{c} = \omega_2$" and "$\mathfrak{c} > \omega_1$" are equivalent. Regarding PID and cardinal invariants, the following is a crucial problem:

Problem 101: Find a cardinal invariant $\mathfrak{j}$ such that under PID the statements "There is an S-space" and "$\mathfrak{j} > \omega_1$" are equivalent.

Two good candidates for the problem above are $\mathfrak{p}$ and $\mathfrak{b}$. The second author proved that $\mathfrak{b} = \omega_1$ implies that there is an S-space (see [47, Chapters 0 and 2]), but there might be a more optimal hypothesis. It is currently unknown if $\mathfrak{p} = \omega_1$ implies that there is an S-space. It is also unknown if forcing with a Suslin tree always adds an S-space (see [58] for a partial result).

We do not know about the veracity of PFA^+ in the models constructed using Neeman's iteration.

Problem 102: If $\mathbb{P}(J)$ is a Neeman iteration forcing PFA, does it necessarily force PFA^+?

Problem 103: Is it possible to force PFA^+ using Neeman's iteration?

The reader wishing to learn more about PFA^+ and some applications may consult [9], [21] and [27] among others.

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