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A Unified Approach and Related Fixed-Point Theorems for Suzuki Contractions

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Abstract: This paper aims to give an extended class of contractive mappings combining types of Suzuki contractions α-admissible mapping and Wardowski F-contractions in b-metric-like spaces. Our results cover and generalize many of the recent advanced results on the existence and uniqueness of fixed points and fulfill the Suzuki-type nonlinear hybrid contractions on various generalized metrics.

Keywords: α-admissible mapping; Suzuki (α, F)-contraction; b-metric-like; fixed point

MSC: 47H10; 54H25; 54E50

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1. Introduction

The advancement of fixed point theory in the last few decades has mainly been related to introducing a new kind of generalized metric space and extensions of the Banach Contraction Principle. This principle has become known as a useful tool for establishing the existence and uniqueness of a fixed point for contractive mappings. In this direction, in 2012, Wardowski [1,2] introduced a new type of contraction named F-contraction as a generalization of this important principle in metric spaces. Subsequently, many researchers [3–10] further developed this new category by improving its properties and extending it in a more generalized setting. In the meantime, other recently defined concepts such as α-admissible mapping in [11] promoted in [12–16], Suzuki contraction widely used in [17–21], and formulations in partial metric spaces, metric-like spaces, b-metric spaces, and b-metric -like spaces underline their significance and offer a broader understanding in various contexts of the fixed point theory. For an extended introduction, we could mention many new theorems and corresponding classical results with applications in the above spaces, resulting in notions of interpolative and hybrid contractions; see [22–24].

In this paper, we introduce the notion of generalized Suzuki-type (α, F)-contraction via a set of implicit relations in the setting of b-metric-like spaces. It strictly extends the known generalizations of metric and b-metric spaces. Moreover, it presents a new approach and includes many types of contractions such as Suzuki F-contraction, interpolative, hybrid, and r-order hybrid contractions, exploring diverse fixed point theorems, implementations, and deduction for earlier and recent results.

2. Preliminaries

Definition 1. Ref. [16]. *Let S be a nonempty set and $v \geq 1$ be a given real number. A mapping $b : S \times S \to [0, +\infty)$ is called a b-metric-like if for all $s, \partial, \zeta \in S$, the following conditions are satisfied:*

$b(s, \partial) = 0$ *implies* $s = \partial$;
$b(s, \partial) = b(\partial, s)$;
$b(s, \partial) \leq v[b(s, \zeta) + b(\zeta, \partial)]$.
The pair (S, b) is called a b-metric-like space (for short b-m.l.s).

Note that by the first axiom of a definition, the self-distance of an arbitrary point $s \in S$ may be positive.

There are various examples of b-metric-like space in the reference literature. To illustrate them, we selected some from [16].

Example 1. *Let $S = [0, +\infty)$ and $b : S \times S = [0, +\infty)$ defined by $b(\partial, \zeta) = (\partial + \zeta)^2$ for all $\partial, \zeta \in S$. Then, (S, b) is a b-m.l.s with parameter $v = 2$ and b is not a b-metric on S.*

Example 2. *Let $S = [0, +\infty)$ and $b : S \times S = [0, +\infty)$ defined by $b(\partial, \zeta) = (\max\{\partial, \zeta\})^2$ for all $\partial, \zeta \in S$. Then, b is a b-metric-like on S with parameter $v = 2$ and it is not a b-metric or a metric-like on S.*

Definition 2. Ref. [16]. *Let (S, b) be a b-m.l.s with parameter v. Then, for any sequence $\{s_n\}$ in S the following applies:*

(a) *$\{s_n\}$ is said to be convergent to $s \in S$* if $\lim\limits_{n \to +\infty} b(s_n, s) = b(s, s)$;
(b) *$\{s_n\}$ is said to be a Cauchy sequence in (S, b) if $\lim\limits_{n,m \to +\infty} b(s_n, s_m)$ exists and is finite;*
(c) *(S, b) is called a complete b-m.l.s if, for every Cauchy sequence $\{s_n\}$ in S, there exists $s \in S$ such that $\lim\limits_{n,m \to +\infty} b(s_n, s_m) = \lim\limits_{n \to +\infty} b(s_n, s) = b(s, s)$.*

Remark 1. *In a b-metric-like-space:*

- *The limit of a convergent sequence is not necessarily unique.*
- *A convergent sequence need not be a Cauchy sequence.*

The following example characterizes and supports Definition 2.

Example 3. *Let $S = [0, +\infty)$ and $b : S \times S = [0, +\infty)$ defined by $b(s, \zeta) = \left(\max\{s^2, \zeta^2\}\right)^2$ for all $s, \zeta \in S$. Also let be the sequence $\{s_n\}$ as*

$$s_n = \begin{cases} 1 & if \;\; n \;\; \text{is even} \\ 0 & if \;\; n \;\; \text{is odd.} \end{cases}$$

For any $s \geq 1$ we have $\lim\limits_{n \to +\infty} b(s_n, s) = \lim\limits_{n \to +\infty} \left(\max\left\{(s_n)^2, s^2\right\}\right)^2 = s^4 = b(s, s)$. Therefore, the sequence $\{s_n\}$ is convergent where $s_n \to s$ for each $s \geq 1$, that is the limit of sequence, is not unique. Also, it is noted that $\lim\limits_{n,m \to +\infty} b(s_n, s_m)$ does not exist, so it is not Cauchy.

Definition 3. Ref. [16]. *Let (S, b) be a b-m.l.s with parameter v, and a function $P : S \to S$. We say that the function P is continuous if for each sequence $\{s_n\} \subset S$ the sequence $Ps_n \to Ps$ whenever $s_n \to s$ as $n \to +\infty$, that is if $\lim\limits_{n \to +\infty} b(s_n, s) = b(s, s)$ yields $\lim\limits_{n \to +\infty} b(Ps_n, Ps) = b(Ps, Ps)$.*

Remark 2. *In a b-m.l.s with parameter $v \geq 1$, if $\lim\limits_{n,m \to +\infty} b(s_n, s_m) = 0$ then the limit of the sequence $\{s_n\}$ is unique if it exists.*

Lemma 1. Ref. [14]. *Let (S, b) be a b-m.l.s with parameter $v \geq 1$. Then, the following applies:*

(a) *If* $b(s, \partial) = 0$, *then* $b(s, s) = b(\partial, \partial) = 0$;
(b) *If* $\{s_n\}$ *is a sequence such that* $\lim_{n\to+\infty} b(s_n, s_{n+1}) = 0$, *then we have* $\lim_{n\to+\infty} b(s_n, s_n) = \lim_{n\to+\infty} b(s_{n+1}, s_{n+1}) = 0$;
(c) *If* $s \neq \partial$, *then* $b(s, \partial) > 0$.

Lemma 2. Ref. [16]. *Let* (S, b) *be a* b*-metric-like space with parameter* $v \geq 1$, *and suppose that* $\{s_n\}$ *is* b*-convergent to* s *with* $b(s, s) = 0$. *Then, for each* $\partial \in S$, *we have*

$$v^{-1} b(s, \partial) \leq \liminf_{n\to+\infty} b(s_n, \partial) \leq \limsup_{n\to+\infty} b(s_n, \partial) \leq v b(s, \partial).$$

Definition 4. Ref. [1]. *Let* (S, b) *be a metric space and* $P : S \to S$ *be a mapping. Then,* P *is called an*

F*-contraction if there exists a function* $F : (0, +\infty) \to R$ *such that*
(F_1). F *is strictly increasing on* $(0, +\infty)$;
(F_2) *For each sequence* $\{\alpha_n\}$ *of positive numbers,*

$$\lim_{n\to+\infty} \alpha_n = 0 \text{ ifandonlyif } \lim_{n\to+\infty} F(\alpha_n) = -\infty$$

(F_3) *There exists* $c \in (0, 1)$ *such that* $\lim_{\alpha\to 0^+} \alpha^c F(\alpha) = 0$;
(F_4) *There exists* $\tau > 0$ *such that*

$$\tau + F(b(Ps, P\partial)) \leq F(b(s, \partial))$$

for all $s, \partial \in S$ *with* $b(Ps, P\partial) > 0$.

For examples that show the class of F-contraction, the reader can confront the extended literature in [1–3,6,7,9,12].

Definition 5. Ref. [11]. *Let* S *be a non-empty set. Let* $P : S \to S$ *and* $\alpha : S \times S \to R^+$ *be given functions. We say that* P *is an* α*-admissible mapping if* $\alpha(s, \partial) \geq 1$ *implies that* $\alpha(Ps, P\partial) \geq 1$ *for all* $s, \partial \in S$.

Definition 6. Ref. [18]. *Let* $S \neq \phi$ *and* $\alpha : S \times S \to [0, +\infty)$ *be a function,* $\{s_n\}$ *be a sequence in* S *and* $s \in S$. *Then,* S *is called* α*-regular if for any* $n \in N$: $\alpha(s_n, s_{n+1}) \geq 1$ *and* $\{s_n\}$ *converges to* s, *then* $\alpha(s_n, s) \geq 1$.

Lemma 3. Ref. [14]. *Let* (S, b) *be complete b-m.l.s with parameter* $v \geq 1$, *let* $\{s_n\}$ *be a sequence such that* $\lim_{n\to+\infty} b(s_n, s_{n+1}) = 0$. *If for the sequence* $\{s_n\}$, $\lim_{n,m\to+\infty} b(s_n, s_m) \neq 0$, *then there exists* $\varepsilon > 0$ *and sequences* $\{m_k\}_{k=1}^{+\infty}$ *and* $\{n_k\}_{k=1}^{+\infty}$ *of natural numbers with* $n_k > m_k > k$, *(positive integers) such that* $b(s_{m_k}, s_{n_k}) \geq \varepsilon$, $b(s_{m_k}, s_{n_k-1}) < \varepsilon$, $\varepsilon/v^2 \leq \limsup_{k\to+\infty} b(s_{m_k-1}, s_{n_k-1}) \leq \varepsilon v$, $\varepsilon/v \leq \limsup_{k\to+\infty} b(s_{n_k-1}, s_{m_k}) \leq \varepsilon v^2$ *and* $\varepsilon/v \leq \limsup_{k\to+\infty} b(s_{m_k-1}, s_{n_k}) \leq \varepsilon v^2$.

In the sequel, let we represent some notations and properties.
Suppose that $F : R^+ \to R$ satisfied the conditions:

(a) F is continuous and non-decreasing;
(b) For any sequence of positive real numbers, $\{\alpha_n\}$ if $\lim_{n\to+\infty} F(\alpha_n) = -\infty$ then $\lim_{n\to+\infty} \alpha_n = 0$.

This family of all functions $F : R^+ \to R$ will be denoted by $\mathcal{F}$.

And W_4 is the set of all continuous functions $\kappa : [0, +\infty)^4 \to [0, +\infty)$, satisfying the conditions:

κ is non-decreasing in respect to each variable;

$$\kappa(u, u, u, u) \leq u \text{ for } u \in [0, +\infty).$$

Let be the mappings $P : S \to S$ and $\alpha : S \times S \to [0, +\infty)$. We say that P satisfies admissible convergence property (for short AC-property), if for every sequence $\{s_n\}$ in S such that $\alpha(s_n, s_{n+1}) \geq 1$ for all $n \in N$ and $\{s_n\}$ converges to s, then $\alpha(s, Ps) \geq 1$.

3. Results

In this main section, among an enormous work presented for types of Suzuki contractions and interesting generalizations established in various spaces, we propose our new general definitions and related theorems concerning such contractive mappings.

Definition 7. *Let (S, b) be a complete b-m.l.s with parameter $v \geq 1$, $P : S \to S$ be a self-mapping, and there exist $F \in \mathcal{F}$, $\tau > 0$ and $\alpha : S \times S \to [0, +\infty)$. Then, P is called a generalized α-admissible Suzuki-type (α, F)-contraction if the following condition is satisfied:*

$$\frac{1}{2v} b(s, Ps) < b(s, u)$$

implies

$$\alpha(s, Ps)\alpha(u, Pu)F\Big(v^3 b(Ps, Pu)\Big) + \tau \leq F(\Omega(s, u)), \tag{1}$$

for all $s, u \in S$, $\alpha(s, u) \geq 1$ and $b(Ps, Pu) > 0$, where
$\Omega(s, u) = \kappa\Big[b(s, u), b(s, Ps), b(u, Pu), \frac{b(s,Pu)+b(u,Ps)}{4v}\Big]$ *for some $\kappa \in W_4$.*

Definition 8. *Let (S, b) be a b-m.l.s with parameter $v \geq 1$, $P : S \to S$ be a self-mapping and $\alpha : S \times S \to [0, +\infty)$. Then, P is an α-admissible Hardy–Rogers Suzuki-type interpolative (α, F)-contraction, if there exist $F \in \mathcal{F}$, $\tau > 0$ and $a_1, a_2, a_3, a_4 \in (0, 1)$ with $0 < a_1 + a_2 + a_3 + a_4 < 1$, such that*

$$\frac{1}{2v} b(s, Ps) < b(s, u)$$

implies

$$\alpha(s, Ps)\alpha(u, Pu)F\Big(v^3 b(Ps, Pu)\Big) + \tau \leq F(\Omega_a(s, u)) \tag{2}$$

for all $s, u \in S \backslash Fix(P)$, $\alpha(s, u) \geq 1$ and $b(Ps, Pu) > 0$, where

$$\Omega_a(s, u) = (b(s, u))^{a_1} \cdot (b(s, Ps))^{a_2} \cdot (b(u, Pu))^{a_3} \cdot \left(\frac{b(s, Pu) + b(u, Ps)}{4v}\right)^{a_4}.$$

Definition 9. *Let (S, b) be a b-m.l.s with parameter $v \geq 1$, $P : S \to S$ be a self-mapping and $\alpha : S \times S \to [0, +\infty)$. Then, P is an α-admissible Hardy–Rogers Suzuki-type r-order hybrid (α, F)-contraction, if there exist $F \in \mathcal{F}$, $\tau > 0$ such that:*

$$\frac{1}{2v} b(s, Ps) < b(s, u)$$

implies

$$\alpha(s, Ps)\alpha(u, Pu)F\Big(v^3 b(Ps, Pu)\Big) + \tau \leq F(\Omega_a^r(s, u)) \tag{3}$$

for all $s, u \in S \backslash Fix(P)$, $\alpha(s, u) \geq 1$, $r \geq 0$, $a_i \geq 0$, $i = 1, 2, 3, 4$ such that $0 < a_1 + a_2 + a_3 + a_4 = 1$ and $b(Ps, Pu) > 0$, where

$$\Omega_a^r(s, u) = \begin{cases} \left[a_1(b(s, u))^r + a_2(b(s, Ps))^r + a_3(b(u, Pu))^r + a_4\left(\frac{b(s,Pu)+b(u,Ps)}{4v}\right)^r\right]^{\frac{1}{r}} & for\ r > 0,\ s \neq u \\ (b(s, u))^{a_1}(b(s, Ps))^{a_2}(b(u, Pu))^{a_3}\left(\frac{b(s,Pu)+b(u,Ps)}{4v}\right)^{a_4} & for\ r = 0\,; s, u \in S \backslash Fix(P). \end{cases}$$

Remark 3. *Some notable cases of introduced definitions include the following:*

- $v = 1$ *corresponds to all in metric setting.*
- $F : (0, +\infty) \to R$ *can be fixed regarding its condition, and it turns to a generalized Suzuki-type* α*-contractions.*
- $\alpha(s, u) = 1$, *leads to types of Suzuki* (v, F)*-contraction defined in a* b*-m.l.s.*
- *All cases above taken simultaneously.*

Theorem 1. *Let* (S, b) *be a complete b-m.l.s with parameter* $v \geq 1$ *and* P *a generalized* α*-admissible Suzuki-type F-contraction. Assume that*

J1. *there exists* $s_0 \in S$ *with* $\alpha(s_0, Ps_0) \geq 1$;

J2. P *is* α*-admissible and satisfies AC-property.*

Then, P has a fixed point in S, and it is unique if $\alpha(s, Ps) \geq 1$, *for all* $s \in Fix(P)$.

Proof. Let be $s_{n+1} = Ps_n$ for $n \in N \cup \{0\}$, the Picard sequence induced by function P with initial point $s_0 \in S$ with $\alpha(s_0, Ps_0) \geq 1$. Let we perform the general case where $s_n \neq s_{n+1}$ for each $n \in N \cup \{0\}$, (that is the same with $0 < b(s_n, Ps_n)$). Since P is α-admissible, then $\alpha(s_0, Ps_0) = \alpha(s_0, s_1) \geq 1$ implies $\alpha(Ps_0, Ps_1) = \alpha(s_1, s_2) \geq 1$. Repeating this process we obtain $\alpha(s_n, s_{n+1}) \geq 1$. Hence, we have $\alpha(s_n, s_{n+1}) \geq 1$, and $s_n \neq s_{n+1}$ implies $\frac{1}{2v} b(s_n, Ps_n) < b(s_n, Ps_n) = b(s_n, s_{n+1})$.□

Therefore, we apply Condition (1) of theorem

$$\begin{aligned}
&F(b(s_{n+1}, s_{n+2})) + \tau \leq \\
&\leq \alpha(s_n, Ps_n)\alpha(s_{n+1}, Ps_{n+1})F\big(v^3 b(Ps_n, Ps_{n+1})\big) + \tau \\
&\leq F\bigg(\kappa\bigg(b(s_n, s_{n+1}), b(s_n, Ps_n), b(s_{n+1}, Ps_{n+1}), \tfrac{b(s_n, Ps_{n+1}) + b(s_{n+1}, Ps_n)}{4v}\bigg)\bigg) \\
&= F\bigg(\kappa\bigg(b(s_n, s_{n+1}), b(s_n, s_{n+1}), b(s_{n+1}, s_{n+2}), \tfrac{b(s_n, s_{n+2}) + b(s_{n+1}, s_{n+1})}{4v}\bigg)\bigg) \\
&\leq F\bigg(\kappa\bigg(b(s_n, s_{n+1}), b(s_n, s_{n+1}), b(s_{n+1}, s_{n+2}), \tfrac{v(b(s_n, s_{n+1}) + b(s_{n+1}, s_{n+2})) + 2vb(s_{n+1}, s_n)}{4v}\bigg)\bigg) \\
&\leq F\bigg(\kappa\bigg(b(s_n, s_{n+1}), b(s_n, s_{n+1}), b(s_{n+1}, s_{n+2}), \tfrac{b(s_{n+1}, s_{n+2}) + 3b(s_{n+1}, s_n)}{4}\bigg)\bigg)
\end{aligned} \tag{4}$$

If suppose that

$$b(s_n, s_{n+1}) < b(s_{n+1}, s_{n+2}),$$

then from Inequality (4), we obtain

$$F(b(s_{n+1}, s_{n+2})) + \tau < F(\kappa(b(s_{n+1}, s_{n+2}), b(s_{n+1}, s_{n+2}), b(s_{n+1}, s_{n+2}), b(s_{n+1}, s_{n+2}))),$$

that implies

$$\begin{aligned}
F(b(s_{n+1}, s_{n+2})) &\leq F(\kappa(b(s_{n+1}, s_{n+2}), b(s_{n+1}, s_{n+2}), b(s_{n+1}, s_{n+2}), b(s_{n+1}, s_{n+2}))) - \tau \\
&< F(\kappa(b(s_{n+1}, s_{n+2}), b(s_{n+1}, s_{n+2}), b(s_{n+1}, s_{n+2}), b(s_{n+1}, s_{n+2}))).
\end{aligned}$$

And from property of F have

$$b(s_{n+1}, s_{n+2}) < \kappa(b(s_{n+1}, s_{n+2}), b(s_{n+1}, s_{n+2}), b(s_{n+1}, s_{n+2}), b(s_{n+1}, s_{n+2})).$$

That is a contradiction based with property of κ. So we have that

$$b(s_{n+1}, s_{n+2}) \leq b(s_n, s_{n+1}) \text{ for all } n \in N. \tag{5}$$

Then, using the result (5), Inequality (4) yields

$$F(b(s_{n+1}, s_{n+2})) \leq F(b(s_n, s_{n+1})) - \tau \tag{6}$$

Repeating this in general we obtain

$$\begin{aligned} F(b(s_{n+1}, s_{n+2})) &\le F(b(s_n, s_{n+1})) - \tau \\ &\le F(b(s_{n-1}, s_n)) - 2\tau \\ &\le F(b(s_{n-2}, s_{n-1})) - 3\tau \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\le F(b(s_0, s_1)) - n\tau. \end{aligned}$$

As per above we have $\lim_{n\to+\infty} F(b(s_n, s_{n+1})) = -\infty$ since $\tau > 0$. Hence, in view of property of F we obtain

$$\lim_{n\to+\infty} b(s_n, s_{n+1}) = 0 \tag{7}$$

Next, we show that $\lim_{n,m\to+\infty} b(s_n, s_m) = 0$. Let we suppose that $\lim_{n,m\to+\infty} b(s_n, s_m) > 0$. Then, by Lemma 3, there exist $\varepsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$, such that $b(s_{m_k}, s_{n_k}) \ge \varepsilon, b(s_{m_k}, s_{n_k-1}) < \varepsilon$ and $\frac{\varepsilon}{v^2} \le \limsup_{k\to+\infty} b(s_{m_k-1}, s_{n_k-1}) \le \varepsilon\, v$, $\frac{\varepsilon}{v} \le \limsup_{k\to+\infty} b(s_{n_k-1}, s_{m_k}) \le \varepsilon$,

$$\frac{\varepsilon}{v} \le \limsup_{k\to+\infty} b(s_{m_k-1}, s_{n_k}) \le \varepsilon v^2 \tag{8}$$

From (7) and (8) we observe $\frac{1}{2v} b(s_{n_k}, Ps_{n_k}) < \frac{\varepsilon}{2v} < b(s_{m_k}, s_{n_k})$ and $0 < \varepsilon \le b(s_{m_k}, s_{n_k}) = b(Ps_{m_k-1}, Ps_{n_k-1})$

Hence, applying Condition (1), we obtain

$$\begin{aligned} F\big(v^3 b(Ps_{m_k}, Ps_{n_k})\big) + \tau &\le \alpha(s_{m_k}, Ps_{m_k})\alpha(s_{n_k}, Ps_{n_k}) F\big(b(Ps_{m_k}, Ps_{n_k})\big) + \tau \\ &\le F\left(\kappa(b(s_{m_k}, s_{n_k}), b(s_{m_k}, Ps_{m_k}), b(s_{n_k}, Ps_{n_k}), \tfrac{b(s_{m_k}, Ps_{n_k}) + b(s_{n_k}, Ps_{m_k})}{4v})\right) \\ &= F\left(\kappa\left[b(s_{m_k}, s_{n_k}), b(s_{m_k}, s_{m_k+1}), b(s_{n_k}, s_{n_k+1}), \tfrac{b(s_{m_k}, s_{n_k+1}) + b(s_{n_k}, s_{m_k+1})}{4v}\right]\right), \end{aligned} \tag{9}$$

that implies

$$\begin{aligned} &F\big(v^3 b(Ps_{m_k}, Ps_{n_k})\big) + \tau \le \\ &\le F\left(\kappa\left[b(s_{m_k}, s_{n_k}), b(s_{m_k}, s_{m_k+1}), b(s_{n_k}, s_{n_k+1}), \tfrac{b(s_{m_k}, s_{n_k+1}) + b(s_{n_k}, s_{m_k+1})}{4v}\right]\right). \end{aligned} \tag{10}$$

By taking the limit superior as $k \to +\infty$ along (10), we write

$$\begin{aligned} &\limsup_{k\to+\infty} F\big(v^3 b(Ps_{m_k}, Ps_{n_k})\big) + \tau \le \\ &\le \limsup_{k\to+\infty} F\left(\kappa\left[b(s_{m_k}, s_{n_k}), b(s_{m_k}, s_{m_k+1}), b(s_{n_k}, s_{n_k+1}), \tfrac{b(s_{m_k}, s_{n_k+1}) + b(s_{n_k}, s_{m_k+1})}{4v}\right]\right) \\ &\le F\left(\kappa\left[\limsup_{k\to\infty} b(s_{m_k}, s_{n_k}), \limsup_{k\to\infty} b(s_{m_k}, s_{m_k+1}), \limsup_{k\to\infty} b(s_{n_k}, s_{n_k+1}), \tfrac{\limsup_{x\to\infty} b(s_{m_k}, s_{n_k+1}) + \limsup_{x\to\infty} b(s_{n_k}, s_{m_k+1})}{4v}\right]\right). \end{aligned} \tag{11}$$

From (11), by using Lemmas 3 and result (7), we conclude

$$\begin{aligned} F\left(v^3 \tfrac{\varepsilon}{v^2}\right) &\le F\left(\limsup_{n\to+\infty} v^3 b(s_{m_k+1}, s_{n_k+1})\right) + \tau \\ &\le F\left(\kappa\left[\varepsilon v, 0, 0, \tfrac{\varepsilon + \varepsilon v^2}{4v}\right]\right) \le F\left(\kappa\left[\varepsilon v, 0, 0, \tfrac{\varepsilon v}{2}\right]\right) \le F(\varepsilon v). \end{aligned} \tag{12}$$

Hence, the acquired inequality

$$F(\varepsilon v) + \tau < F(\varepsilon v)$$

is a contradiction since $\varepsilon > 0$ and $\tau > 0$.

Hence,

$$\lim_{n,m\to+\infty} b(s_n, s_m) = 0. \tag{13}$$

The sequence $\{s_n\}$ is a b-Cauchy sequence such that $\lim\limits_{n,m\to\infty} b(s_n, s_m) = 0$. Since (S, b) is a b-complete *b-m.l.s*, there is $s^* \in S$, such that

$$b(s^*, s^*) = \lim_{n\to\infty} b(s_n, s^*) = \lim_{n,m\to\infty} b(s_n, s_m) = 0. \tag{14}$$

Further let we show that

$$\frac{1}{2v} b(s_n, Ps_n) < b(s_n, s^*) \ or \ \frac{1}{2v} b\Big(Ps_n, P^2 s_n\Big) < b(Ps_n, s^*) \tag{15}$$

On the contrary, there exists $m \in N$ with

$$\frac{1}{2v} b(s_m, Ps_m) \geq b(s_m, s^*) \ and \ \frac{1}{2v} b\Big(Ps_m, P^2 s_m\Big) \geq b(Ps_m, s^*). \tag{16}$$

Therefore,

$$2vb(s_m, s^*) \leq b(s_m, Ps_m) \leq vb(s_m, s^*) + vb(s^*, Ps_m),$$

and it gives

$$b(s_m, s^*) \leq b(s^*, Ps_m). \tag{17}$$

Using (14) and (17), we conclude

$$\begin{aligned} b\big(Ps_m, P^2 s_m\big) &= b(s_{m+1}, s_{m+2}) \\ &< b(s_m, s_{m+1}) \\ &= b(s_m, Ps_m) \\ &\leq vb(s_m, s^*) + vb(s^*, Ps_m) \\ &\leq 2vb(s^*, Ps_m). \end{aligned}$$

and inequality above implies that

$$\frac{1}{2v} b\Big(Ps_m, P^2 s_m\Big) < b(s^*, Ps_m),$$

that is a contradiction due to (16). Hence, Inequality (15) is true.

Therefore,

$$\frac{1}{2v} b(s_n, Ps_n) < b(s_n, s^*) \ , b(Ps_n, Ps^*) > 0 \text{ and } \alpha(s^*, Ps^*) \geq 1.$$

So, we are in conditions to apply (1), and we have

$$\begin{aligned} F\big(v^3 b(s_{n+1}, Ps^*)\big) + \tau &< \alpha(s_n, Ps_n)\alpha(s^*, Ps^*) F\big(v^3 b(Ps_n, Ps^*)\big) + \tau \\ &\leq F\Big(\kappa\Big(b(s_n, s^*), b(s_n, Ps_n), b(s^*, Ps^*), \tfrac{b(s_n, Ps^*) + b(s^*, Ps_n)}{4v}\Big)\Big) \\ &= F\Big(\kappa\Big(b(s_n, s^*), b(s_n, s_{n+1}), b(s^*, Ps^*), \tfrac{b(s_n, Ps^*) + b(s^*, s_{n+1})}{4v}\Big)\Big) \end{aligned} \tag{18}$$

Taking limit superior as $n \to +\infty$ in Inequality (18) and keeping in mind (7), (14), and Lemma 3, we obtain

$$\begin{aligned} F\big(v^2 b(s^*, Ps^*)\big) + \tau &\leq F\Big(\kappa\Big(0, 0, b(s^*, Ps^*), \tfrac{vb(s^*, Ps^*) + 0}{4v}\Big)\Big) \\ &\leq F(\kappa(b(s^*, Ps^*), b(s^*, Ps^*), b(s^*, Ps^*), b(s^*, Ps^*))) \\ &\leq F(b(s^*, Ps^*)), \end{aligned}$$

that implies

$$F\left(v^2 b(s^*, Ps^*)\right) + \tau \leq F(b(s^*, Ps^*)). \tag{19}$$

Inequality (19) leads to a contradiction, so $b(s^*, Ps^*) = 0$, and $s^* \in S$ is a fixed point of function P. Also, the uniqueness of the fixed point can be proved easily from Condition (1) of the theorem.

Let there be two different fixed points named u, s with $\alpha(s, Ps) \geq 1, \alpha(u, Pu) \geq 1$. Then

$$u \neq s \; implies \; 0 < b(u,s) = b(Pu, Ps),$$

and

$$u \neq s \; implies \; 0 = \frac{1}{2v} b(u, Pu) < b(u,s).$$

Hence, being in the conditions of theorem and in view of (14), Inequality (1) implies

$$\begin{aligned} F\left(v^3 b(u,s)\right) + \tau &< \alpha(u, Pu)\alpha(s, Ps) F\left(v^3 b(Pu, Ps)\right) + \tau \\ &\leq F\left(\kappa\left(b(u,s), b(u, Pu), b(s, Ps), \tfrac{b(u,Ps)+b(s,Pu)}{4v}\right)\right) \\ &= F\left(\kappa\left(b(u,s), b(u,u), b(s,s), \tfrac{b(u,s)+b(s,u)}{4v}\right)\right) \\ &= F\left(\kappa\left(b(u,s), 0, 0, \tfrac{b(u,s)}{2v}\right)\right) \\ &\leq F(\kappa(b(u,s), b(u,s), b(u,s), b(u,s))). \end{aligned} \tag{20}$$

From (20), we obtain

$$F\left(v^3 b(u,s)\right) + \tau \leq F(\kappa(b(u,s), b(u,s), b(u,s), b(u,s))),$$

which implies

$$\begin{aligned} F\left(v^3 b(u,s)\right) &\leq F(\kappa(b(u,s), b(u,s), b(u,s), b(u,s))) - \tau \\ &< F(\kappa(b(u,s), b(u,s), b(u,s), b(u,s))). \end{aligned} \tag{21}$$

The above inequality implies $b(u,s) < \kappa(b(u,s), b(u,s), b(u,s), b(u,s))$ that is a contradiction due to property of κ. Therefore, the fixed point is unique.

Example 4. *Let (S, b) be a complete b-m.l.s with parameter $v = 2 > 1$, where the b-m.l. distance is given as $b(s,u) = (s+u)^2$. Define the mappings P and α by*

$$P(s) = \begin{cases} s/8 & for \;\; s \in [0,1] \\ 3s & for \;\; s > 1 \end{cases}, \alpha(s,u) = \begin{cases} 1 & for \, s, u \in [0,1] \\ 0 & otherwise \end{cases}.$$

Clearly, the mapping P is α-admissible. And for $s, u \in [0,1]$, we have

$$\frac{1}{2v} b(s, Ps) = \frac{1}{4} b\left(s, \frac{s}{8}\right) = \frac{1}{4}\left(s + \frac{s}{8}\right)^2 \leq s^2 \leq (s+u)^2 = b(s,u).$$

Take $F \in \mathcal{F}$, such as $F(t) = \ln t$, $\kappa(u_1, u_2, u_3, u_4) = \max\{u_1, u_2, u_3, u_4\}$ and $\tau = \ln 8$. We see that for $s, u \in S$ that $\alpha(s,u) = 1$, we have $s, u \in [0,1]$ and one can compute:

$$\begin{aligned} \alpha(s, Ps)\alpha(u, Pu) v^3 F(b(Ps, Pu)) + \tau &= \ln\left(v^3 b(Ps, Pu)\right) + \ln 8 = \ln\left(8\left(\tfrac{s}{8} + \tfrac{u}{8}\right)^2\right) + \ln 8 \\ &= \ln\left(\tfrac{8}{64}(s+u)^2\right) + \ln 8 \leq \ln\left(\tfrac{1}{8} b(s,u)\right) + \ln 8 \\ &= \ln(b(s,u)) - \ln 8 + \ln 8 \\ &\leq \ln\left(\max\left\{b(s,u), b(s, Ps), b(u, Pu), \tfrac{b(s,Pu)+b(u,Ps)}{4v}\right\}\right) \\ &= \ln\left(\kappa\left[b(s,u), b(s, Ps), b(u, Pu), \tfrac{b(s,Pu)+b(u,Ps)}{4v}\right]\right) \\ &= \ln(\Omega(s,u)) \\ &= F(\Omega(s,u)). \end{aligned}$$

Then, P is a generalized Suzuki-type (α, F)-contraction, and assumptions of theorem 1 are satisfied. Hence, $s = 0$ is the unique fixed point for P.

Theorem 2. *Let (S, b) be a complete b-m.l.s with parameter $v \geq 1$ and P a generalized α-admissible Suzuki-type F-contraction. Assume that*

J1. P is α-admissible and there exists $s_0 \in S$ with $\alpha(s_0, Ps_0) \geq 1$;
J1. S is α regular and for every sequence $\{s_n\}$ in S such that $\alpha(s_n, s_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$, we have $\alpha(s_m, s_n) \geq 1$ for all $m, n \in N$ with $m < n$.

Then, P has a fixed point, and it is unique if $\alpha(s, u) \geq 1$, for all $s, u \in Fix(P)$.

Proof. Using J1, we define the Picard sequence $s_{n+1} = Ps_n$ for $n \in N \cup \{0\}$, induced by function P with initial point $s_0 \in S$ with $\alpha(s_0, Ps_0) \geq 1$. From the theorem above, it is concluded that the sequence $\{s_n\}$ is a b-Cauchy sequence such that $\lim_{n,m\to\infty} b(s_n, s_m) = 0$. And there is $s^* \in S$, such that

$$b(s^*, s^*) = \lim_{n\to\infty} b(s_n, s^*) = \lim_{n,m\to\infty} b(s_n, s_m) = 0.$$

In the same, we can show that

$$\frac{1}{2v} b(s_n, Ps_n) < b(s_n, s^*) \ or \ \frac{1}{2v} b\Big(Ps_n, P^2 s_n\Big) < b(Ps_n, s^*)$$

Since S is regular, there exists a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $\alpha(s_{n_k}, s) \geq 1$ for all $k \in$ N. Similarly holds

$$\frac{1}{2v} b(s_{n_k}, Ps_{n_k}) < b(s_{n_k}, s^*) \ or \ \frac{1}{2v} b\Big(Ps_{n_k}, P^2 s_{n_k}\Big) < b(Ps_{n_k}, s^*).$$

Therefore, applying (1), we have

$$\begin{aligned} F(v^3 b(s_{n_k+1}, Ps^*)) + \tau &< \alpha(s_{n_k}, Ps_{n_k}) \alpha(s^*, Ps^*) F(v^3 b(Ps_{n_k}, Ps^*)) + \tau \\ &\leq F\left(\kappa\left(b(s_{n_k}, s^*), b(s_{n_k}, Ps_{n_k}), b(s^*, Ps^*), \frac{b(s_{n_k}, Ps^*) + b(s^*, Ps_{n_k})}{4s}\right)\right) \\ &= F\left(\kappa\left(b(s_{n_k}, s^*), b(s_{n_k}, s_{n_k+1}), b(s^*, Ps^*), \frac{b(s_{n_k}, Ps^*) + b(s^*, s_{n_k+1})}{4v}\right)\right). \end{aligned} \tag{22}$$

Taking limit superior in Inequality (22) and keeping in mind lemma 3, we obtain

$$\begin{aligned} F(v^2 b(s^*, Ps^*)) + \tau &\leq F\left(\kappa\left(0, 0, b(s^*, Ps^*), \frac{vb(s^*, Ps^*) + 0}{4v}\right)\right) \\ &\leq F(\kappa(b(s^*, Ps^*), b(s^*, Ps^*), b(s^*, Ps^*), b(s^*, Ps^*))) \\ &\leq F(b(s^*, Ps^*)), \end{aligned}$$

which implies

$$F\Big(v^2 b(s^*, Ps^*)\Big) + \tau \leq F(b(s^*, Ps^*)). \tag{23}$$

Inequality (23) leads to a contradiction, so $b(s^*, Ps^*) = 0$, and $s^* \in S$ is a fixed point of function P. □

Let be two different fixed points named u, s with $\alpha(u, s) \geq 1$.
and

$$u \neq s \, implies \, 0 < b(u, s) = b(Pu, Ps),$$

also

$$u \neq s \, implies \, 0 = \frac{1}{2v} b(u, Pu) < b(u, s)$$

Hence, Inequality (1) can be written as

$$\begin{aligned} F\big(v^3 b(u,s)\big)+\tau &< \alpha(u,Pu)\alpha(s,Ps)F\big(v^3 b(Pu,Ps)\big)+\tau \\ &\leq F\Big(\kappa\Big(b(u,s),b(u,Pu),b(s,Ps),\tfrac{b(u,Ps)+b(s,Pu)}{4v}\Big)\Big) \\ &= F\Big(\kappa\Big(b(u,s),b(u,u),b(s,s),\tfrac{b(u,s)+b(s,u)}{4v}\Big)\Big) \\ &= F\Big(\kappa\Big(b(u,s),0,0,\tfrac{b(u,s)}{2v}\Big)\Big) \\ &\leq F(\kappa(b(u,s),b(u,s),b(u,s),b(u,s))). \end{aligned} \tag{24}$$

From (24), we obtain

$$F\Big(v^3 b(u,s)\Big)+\tau \leq F(\kappa(b(u,s),b(u,s),b(u,s),b(u,s))),$$

that is

$$\begin{aligned} F\big(v^3 b(u,s)\big) &\leq F(\kappa(b(u,s),b(u,s),b(u,s),b(u,s)))-\tau \\ &< F(\kappa(b(u,s),b(u,s),b(u,s),b(u,s))), \end{aligned} \tag{25}$$

which implies $b(u,s)<\kappa(b(u,s),b(u,s),b(u,s),b(u,s))$ that is a contradiction due to property of κ. Therefore, the fixed point is unique.

Corollary 1. *Let (S,b) be a complete b-m.l.s with parameter $v\geq 1$ and $P:S\to S$. If there exist $F\in\mathcal{F},\tau>0$ such that*

$$\frac{1}{2v}b(s,Ps)<b(s,u)$$

implies

$$F\Big(v^3 b(Ps,Pu)\Big)+\tau\leq F(\Omega(s,u)),$$

for all $s,u\in S$, $b(Ps,Pu)>0$ where
$\Omega(s,u)=\kappa\Big[b(s,u),b(s,Ps),b(u,Pu),\frac{b(s,Pu)+b(u,Ps)}{4v}\Big]$ *for some $\kappa\in W_4$,*
then P has a unique fixed point in S.

Proof. Corollary can be obtained from Theorem 1 by taking $\alpha(s,u)=1$.□

Now we will propose some new results belonging to the class of Suzuki (α,F)-contractions.

Theorem 3. *Let we have (S,b) a b-m.l.s with parameter $v\geq 1$, $P:S\to S$, and $\alpha:S\times S\to[0,+\infty)$. If P is an Hardy–Rogers Suzuki- type interpolative (α,F)-contraction, satisfying conditions J_1, J_2, then P has a fixed point in S. And it is unique if $\alpha(s,Ps)\geq 1$, for all $s\in Fix(P)$.*

Proof. It is a consequence of Theorem 1 by taking $\kappa\in W_4$ as $\kappa(\gamma_1,\gamma_2,\gamma_3,\gamma_4)={\gamma_1}^{a_1}\cdot{\gamma_2}^{a_2}\cdot{\gamma_3}^{a_3}\cdot{\gamma_4}^{1-a_1-a_2-a_3}$; where $a_1,a_2,a_3\in(0,1)$ and $a_1+a_2+a_3<1$. □

Theorem 4. *Let we have (S,b) a b-m.l.s with parameter $v\geq 1$, $P:S\to S$, and $\alpha:S\times S\to[0,+\infty)$. If there exist $F\in\mathcal{F}$, $\tau>0$ such that hold condition J_1, J_2 and:*

$$\frac{1}{2v}b(s,Ps)<b(s,u)$$

implies

$$\begin{aligned} &\alpha(s,Ps)\alpha(u,Pu)F\big(v^3 b(Ps,Pu)\big)+\tau\leq \\ &F\left(\left[a_1(b(s,u))^r+a_2(b(s,Ps))^r+a_3(b(u,Pu))^r+a_4\left(\tfrac{b(s,Pu)+b(u,Ps)}{4v}\right)^r\right]^{\frac{1}{r}}\right) \end{aligned}$$

for all $s, u \in S$, $\alpha(s, u) \geq 1$, $b(Ps, Pu) > 0$ *and* $r > 0$.

Then, P has a fixed point in S. And it is unique if $\alpha(s, Ps) \geq 1$, *for all* $s \in Fix(P)$.

Proof. It comes from Theorem 1 by taking $\kappa \in W_4$ as $\kappa(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = [a_1\gamma_1{}^r + a_2\gamma_2{}^r + a_3\gamma_3{}^r + a_4\gamma_4{}^r]^{\frac{1}{r}}$, $r > 0$; where $0 < a_1 + a_2 + a_3 + a_4 < 1$. □

Theorem 5. *Let* (S, b) *be a b-m.l.s with parameter* $v \geq 1$, *and mappings* $P : S \to S$, $\alpha : S \times S \to [0, +\infty)$. *If the following conditions are satisfied*

- *There exists* $s_0 \in S$ *with* $\alpha(s_0, Ps_0) \geq 1$;
- *P is α-admissible mapping and satisfies AC-property;*
- *Pis a Hardy–Rogers Suzuki –type r-order hybrid* (α, F)*-contraction.*

Then, P has a fixed point in S. And it is unique if $\alpha(s, Ps) \geq 1$, *for all* $s \in Fix(P)$.

Proof. The $L_a^r(s, u)$ can be represented as

$$\Omega_a^r(s, u) = \begin{cases} \kappa_1(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (a_1\gamma_1{}^r + a_2\gamma_2{}^r + a_3\gamma_3{}^r + a_4\gamma_4{}^r)^{\frac{1}{r}} & for\, r > 0, \\ \kappa_2(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \gamma_1{}^{a_1} \cdot \gamma_2{}^{a_2} \cdot \gamma_3{}^{a_3} \cdot \gamma_4{}^{a_4} & for\, r = 0\,. \end{cases}$$

where κ_1, $\kappa_2 \in W_4$, $a_1, a_2, a_3, a_4 \geq 0$ with $0 < a_1 + a_2 + a_3 + a_4 = 1$, $r \geq 0$. Hence, the proof can be classified in Theorem 1. □

Corollary 2. *Let* (S, b) *be a b-m.l.s with parameter* $v \geq 1$, *and* $P : S \to S$. *If there exist* $F \in \mathcal{F}$, $\tau > 0$ *such that*

$$\frac{1}{2v} b(s, Ps) < b(s, u)$$

implies

$$F\Big(v^3 b(Ps, Pu)\Big) + \tau \leq F(\Omega_a{}^r(s, u))$$

for all $s, u \in S \backslash Fix(P)$, $r \geq 0$, $b(Ps, Pu) > 0$ *and* $a_i \geq 0$, $i = 1, 2, 3, 4$ *such that* $0 < a_1 + a_2 + a_3 + a_4 = 1$,*where*

$$\Omega_a^r(s, u) = \begin{cases} \Big[a_1(b(s, u))^r + a_2(b(s, Ps))^r + a_3(b(u, Pu))^r + a_4\Big(\frac{b(s,Pu)+b(u,Ps)}{4v}\Big)^r\Big]^{\frac{1}{r}} & for\ r > 0,\ s \neq u \\ (b(s, u))^{a_1}(b(s, Ps))^{a_2}(b(u, Pu))^{a_3}\Big(\frac{b(s,Pu)+b(u,Ps)}{4v}\Big)^{a_4} & for\ r = 0\,; s, u \in S\backslash Fix(P). \end{cases}$$

Then, P has a fixed point in S.

Proof. It can be derived from Theorem 5 by taking $\alpha(s, u) = 1$. □

Corollary 3. *Let* (S, b) *be a b-m.l.s with parameter* $v \geq 1$, *and* $P : S \to S$, $\alpha : S \times S \to [0, +\infty)$. *If there exist* $F \in \mathcal{F}$, $\tau > 0$ *such that hold condition* J_1, J_2 *and*

$$\frac{1}{2v} b(s, Ps) < b(s, u)$$

implies

$$\tau + \alpha(s, Ps)\alpha(u, Pu)v^3 b(Ps, Pu) \leq (b(s, u))^{a_1} \cdot (b(s, Ps))^{a_2} \cdot (b(u, Pu))^{a_3} \cdot \left(\frac{b(s, Pu) + b(u, Ps)}{4v}\right)^{a_4}$$

for all $s, u \in S \backslash Fix(P)$, $b(Ps, Pu) > 0$ *and* $a_i \geq 0$, $i = 1, 2, 3, 4$ *such that* $0 < a_1 + a_2 + a_3 + a_4 = 1$,

Then, P has a fixed point in S. And it is unique if $\alpha(s, Ps) \geq 1$, *for all* $s \in Fix(P)$.

Proof. The proof is called done by Theorem 1, if we take $F \in \mathcal{F}$, as $F(x) = x$ and

$$\kappa(u_1, u_2, u_3, u_4) = (u_1)^{a_1} \cdot (u_2)^{a_2} \cdot (u_3)^{a_3} \cdot (u_4)^{1-(a_1+a_2+a_3)}. \square$$

Corollary 4. *Let (S, b) be a b-m.l.s with parameter $v \geq 1$, and $P : S \to S$. If there exist $F \in \mathcal{F}$, $\tau > 0$ such that:*

$$\frac{1}{2v} b(s, Ps) < b(s, u)$$

implies

$$\tau + v^3 b(Ps, Pu) \leq (b(s, u))^{a_1} \cdot (b(s, Ps))^{a_2} \cdot (b(u, Pu))^{a_3} \cdot \left(\frac{b(s, Pu) + b(u, Ps)}{4v} \right)^{a_4}$$

for all $s, u \in S \backslash Fix(P)$, $b(Ps, Pu) > 0$ and $a_i \geq 0$, $i = 1, 2, 3, 4$ such that $0 < a_1 + a_2 + a_3 + a_4 = 1$, Then, P has a fixed point in S.

Proof. It is generated by corollary 3 by taking $\alpha(s, u) = 1. \square$

4. Application

The study of the existence and finding of the solution of differential and integral equations is a longstanding problem, so one of the main tools of the solution is developed and consists of the application of the fixed point method. Many researchers have employed various contractions in different metric spaces to define the necessary conditions for a variety of types of linear and nonlinear integral equations

In this supported section, by employing *b-m.l.s,* the purpose is to prove the existence and uniqueness of the solution for the following integral equation of the form:

$$x(t) = h(t, x(t)) + \lambda \int_0^t G(t, \rho) h(\rho, x(\rho)) d\rho; \quad t \in [0, 1], \tag{26}$$

where $0 \leq t \leq 1$ and given continuous functions $G : [0.1] \times R \to [0, +\infty)$, $h : [0.1] \times R \to R$.

Consider $Y = C([0, 1], R)$ the set of real continuous functions defined on $[0, 1]$ with the b-metric-like $b(s, u) = \sup_{t \in [0,1]} |s(t) + u(t)|^m$ for all $s, u \in Y$. The pair (Y, b) is a complete *b-m.l.s* with parameter $v = 2^{m-1}$.

Define the mapping $P : Y \to Y$ for all $x \in C[0, 1]$, by

$$Px(t) = h(t, x(t)) + \lambda \int_0^t G(t, \rho) h(\rho, x(\rho)) d\rho.$$

Associated with the following hypotheses:

(i) The mapping $P : Y \to Y$ is continuous;

There exists constant $A > 0$ such that $h : [0, 1] \times R \to R$ satisfies $h(\rho, v(\rho)) + h(\rho, u(\rho)) \leq A|v(\rho) + u(\rho)|$ for $t, \rho \in [0, 1]$;

(ii) The constants λ, A and function G satisfy condition $\lambda \sup \int_0^t G(t, \rho) d\rho < \frac{1}{A \sqrt[m]{v^4}} - 1$ for $t, \rho \in (0, 1)$.

Theorem 6. *If for the integral Equation (26) assume the assertions: (i), (ii), (iii) then, the integral Equation (26) has a unique solution $x(t)$ in Y.*

Proof. Solving Equation (26) is equivalent to find $x(t) \in Y$ which is a fixed point of function P. And for all $t \in [0, 1]$, and $x, u \in Y$ we have

$$
\begin{aligned}
v^3\sigma_b(Px(t),Pu(t)) &= v^3|Px(t)+Pu(t)|^m \\
&= v^3\left|h(t,x(t))+\lambda\int_0^t G(t,\rho)h(\rho,x(\rho))d\rho+h(t,u(t))+\lambda\int_0^t G(t,\rho)h(\rho,u(\rho))d\rho\right|^m \\
&= v^3\left|[h(t,x(t))+h(t,u(t))]+\lambda\left[\int_0^t G(t,\rho)h(\rho,x(\rho))d\rho+\int_0^t G(t,\rho)h(\rho,u(\rho))d\rho\right]\right|^m \\
&= v^3\left|[h(t,x(t))+h(t,u(t))]+\lambda\left[\int_0^t G(t,\rho)(h(\rho,x(\rho))+h(\rho,u(\rho)))d\rho\right]\right|^m \\
&\le v^3\left|A|x(t)+u(t)|+\lambda\left[\int_0^t G(t,\rho)A|x(\rho)+u(\rho)|d\rho\right]\right|^m \\
&= v^3\left|A\left(|x(t)+u(t)|^m\right)^{\frac{1}{m}}+\lambda\left[\int_0^t G(t,\rho)A\left(|x(\rho)+u(\rho)|^m\right)^{\frac{1}{m}}d\rho\right]\right|^m \\
&= v^3\left|A(b(x,u))^{\frac{1}{m}}+\lambda\left[\int_0^t G(t,\rho)A(b(x,u))^{\frac{1}{m}}d\rho\right]\right|^m \\
&= v^3\left|A(b(x,u))^{\frac{1}{m}}+\lambda A(b(x,u))^{\frac{1}{m}}\int_0^t G(t,\rho)d\rho\right|^m \\
&= v^3\left|A(b(x,u))^{\frac{1}{m}}\left(1+\lambda\int_0^t G(t,\rho)d\rho\right)\right|^m \\
&= v^3A^m b(x,u)\left|1+\lambda\int_0^t G(t,\rho)d\rho\right|^m \\
&\le v^3A^m b(x,u)\left|\frac{1}{A\sqrt[m]{v^4}}\right|^m \\
&\le \frac{b(x,u)}{v}
\end{aligned} \tag{27}
$$

Further, taking $\kappa \in W_4$ as $\kappa(\gamma_1,\gamma_2,\gamma_3,\gamma_4)=\gamma_1$ we can obtain:

$$v^3b(Px(t),Pu(t)) \le \frac{1}{v}L(x,u).$$

Consequently, by choosing: $F(\zeta)=\ln(\zeta)$, $\tau=\ln v$ we deduce

$$\frac{1}{2v}b(x,Px)\le b(x,u)\ \ implies\ \ F\left(v^3b(Px,Pu)\right)+\tau\le F(L(x,u))$$

which implies that P is a generalized Suzuki-type (v,F)-contraction. Thus, Corollary 1 is applicable and $x(t)$ is the fixed point of P which is the solution of the integral Equation (26). □

5. Conclusions

In this work, via the help of an implicit class of functions, we introduced new contractive mappings, including linear and nonlinear contractions revised under the name of Suzuki contractions. Our approach generates and resumes more well-known fixed point theorems; it shows the validity and generality of the proof of the main result for broad setting theorems. Also, it provides a further extension on the recent published work in the fixed point theory.

A future focus holds for the following:

- A consistent improvement in the inequality condition can be considered in terms of "v";
- The extension of the established Suzuki results in extended *b-m.l.s;* rectangular metric space; and b-rectangular metric spaces elaborating with the notion of α-admissible mappings.

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