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GNS Construction for C^*-Valued Positive Sesquilinear Maps on a quasi *-algebra

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Abstract. The GNS construction for positive invariant sesquilinear forms on quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is generalized to a class of positive sesquilinear maps from $\mathfrak{A} \times \mathfrak{A}$ into a C^*-algebra $\mathfrak{C}$. The result is a *-representation taking values in a space of operators acting on a certain quasi-normed $\mathfrak{C}$-module.

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1. Introduction and Basic Definitions

The Gelfand–Naimark–Segal construction is nowadays a common tool when studying the structure properties of locally convex *-algebras, since it provides *-representations of the given *-algebra into some space of operators acting in Hilbert space.

The basic idea consists in building up *-representations starting from a positive linear functional on a *-algebra, constructing a Hilbert space from it and then defining operators in a natural way using the multiplication of the given *-algebra.

This construction was given first in the case of C^*-algebras and produces bounded operators in Hilbert spaces, but the paper of Powers [8], in the early 1970s, puts in evidence its generality if one is willing to pay the price of dealing with *-algebras of unbounded operators. Since then this procedure has been generalized in many directions and in various ways: extensions to the case of partial *-algebras and quasi *-algebra have been considered, see Refs. [1] and [3]. In particular, it has appeared clear that when dealing with algebraic structures where the multiplication is only partially defined, it is convenient to replace positive linear functionals with positive sesquilinear forms enjoying certain *invariance* properties.

In this paper, we will analyze the possible generalization of the GNS construction for a quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$, see below for a formal definition,

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starting from a *positive sesquilinear (i.e., conjugate-bilinear) map* Φ taking its values in a C^*-algebra $\mathfrak{C}$. In this case, one expects that the image of a *-representation is a space of operators acting on some Hilbert C^*-module. As we will see in what follows, this is not always the case (this depends on the Cauchy–Schwartz-like inequality that Φ satisfies) and for this reason, we have introduced quasi Banach spaces whose norm is defined by a $\mathfrak{C}$-valued inner product, named, for short, quasi $B_{\mathfrak{C}}$-spaces.

Positive and completely positive maps on C^*-algebras or Operator algebras play an important role in many applications such as quantum theory, quantum information, quantum probability theory, and a lot of deep mathematical results have been obtained (see, e.g., [9]). On the other hand, it is now long time that the C^*-algebraic approach to quantum theories has been considered as too rigid framework where casting all objects of physical interest. For this reason, several possible generalizations have been proposed: quasi *-algebra, partial *-algebras, and so on. They reveal in fact to be more suited to cover situations where unbounded operator algebras occur. These facts provide, in our opinion, good motivations for the generalizations we are proposing here.

The paper is organized as follows. In Sect. 2, we analyze some properties of a $\mathfrak{C}$-valued positive sesquilinear map. It turns out that such a map defines a *quasi-inner product* on a given vector space $\mathfrak{X}$ and we study, in particular, the quasi $B_{\mathfrak{C}}$-space it generates. Section 3 is devoted to the construction of the *-representation associated to Φ. This is in fact a generalization of Paschke result [7, Theorem 5.2] which is the first involving Hilbert C^*-modules (as far as we know). The proofs we give are often adaptations to the case under consideration of the corresponding ones for positive sesquilinear *forms* but is not surprising at all, since all generalizations of the GNS representation are variants of the beautiful construction made by Gelfand, Naimark, and Segal. The main results of the paper are Theorem 3.2 and Corollary 3.10 which provides a representation of C^*-valued positive maps on unital *-algebras. Moreover, Corollary 2.4 and Corollary 3.13 illustrate also the applications to C^*-valued positive linear maps on (quasi) *-algebras. Examples coming mostly from the theory of noncommutative integration are discussed.

To keep the paper sufficiently self-contained, we begin with some preliminary definitions and facts.

A *quasi *-algebra* $(\mathfrak{A}, \mathfrak{A}_0)$ is a pair consisting of a vector space $\mathfrak{A}$ and a *-algebra $\mathfrak{A}_0$ contained in $\mathfrak{A}$ as a subspace and such that

- $\mathfrak{A}$ carries an involution $a \mapsto a^*$ extending the involution of $\mathfrak{A}_0$;
- $\mathfrak{A}$ is a bimodule over $\mathfrak{A}_0$ and the module multiplications extend the multiplication of $\mathfrak{A}_0$. In particular, the following associative laws hold:

$$(ca)d = c(ad); \quad a(cd) = (ac)d, \quad \forall\, a \in \mathfrak{A},\; c, d \in \mathfrak{A}_0;$$

- $(ac)^* = c^*a^*$, for every $a \in \mathfrak{A}$ and $c \in \mathfrak{A}_0$.

The *identity* or *unit element* of $(\mathfrak{A}, \mathfrak{A}_0)$, if any, is a necessarily unique element $\mathsf{e} \in \mathfrak{A}_0$, such that $a\mathsf{e} = a = \mathsf{e}a$, for all $a \in \mathfrak{A}$.

We will always suppose that

$$ac = 0,\ \forall c \in \mathfrak{A}_0 \Rightarrow a = 0$$
$$ac = 0,\ \forall a \in \mathfrak{A} \Rightarrow c = 0.$$

Clearly, both these conditions are automatically satisfied if $(\mathfrak{A}, \mathfrak{A}_0)$ has an identity e.

Definition 1.1. A quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is said to be *locally convex* if $\mathfrak{A}$ is a locally convex vector space, with a topology τ enjoying the following properties

- (lc1) $c \mapsto c^*,\ \ c \in \mathfrak{A}_0$, is continuous;
- (lc2) for every $c \in \mathfrak{A}_0$, the maps $a \mapsto ac$ and $a \mapsto ca$, from $\mathfrak{A}[\tau]$ into $\mathfrak{A}[\tau]$, $a \in \mathfrak{A}$, are continuous;
- (lc3) $\overline{\mathfrak{A}_0}^{\tau} = \mathfrak{A}$; i.e., $\mathfrak{A}_0$ is dense in $\mathfrak{A}[\tau]$.

The involution of $\mathfrak{A}_0$ extends by continuity to $\mathfrak{A}$. Moreover, if τ is a norm topology, with norm $\|\cdot\|$, and

(bq*) $\|a^*\| = \|a\|,\ \forall a \in \mathfrak{A}$

then, $(\mathfrak{A}, \mathfrak{A}_0)$ is called a *normed quasi *-algebra* and a *Banach quasi *-algebra* if the normed vector space $\mathfrak{A}[\|\cdot\|]$ is complete.

The simplest example of a locally convex quasi *-algebra is obtained by taking the completion $\mathfrak{A} := \widetilde{\mathfrak{A}_0}[\tau]$ of a locally convex *-algebra $\mathfrak{A}_0[\tau]$ with separately (but not jointly) continuous multiplication (this was, in fact, the case considered at an early stage of the theory, in view of applications to quantum physics).

In the whole paper, $\mathfrak{C}$ will denote a C^*-algebra with unit *1* and norm $\|\cdot\|_{\mathfrak{C}}$ and $\mathfrak{C}^+$ its positive cone. If θ is a continuous linear functional on $\mathfrak{C}$, we denote by $\|\theta\|^*_{\mathfrak{C}}$ the norm in the Banach dual of $\mathfrak{C}$. Let $\mathcal{S}(\mathfrak{C})$ denote the set of all positive linear functionals on $\mathfrak{C}$ such that $\|\theta\|^*_{\mathfrak{C}} = 1$. We recall that if $x \in \mathfrak{C}$,

$$\|x\|^2_{\mathfrak{C}} = \sup_{\theta \in \mathcal{S}(\mathfrak{C})} \theta(x^*x).$$

In particular, if x is a normal element of $\mathfrak{C}$ (i.e., $xx^* = x^*x$),

$$\|x\|_{\mathfrak{C}} = \sup_{\theta \in \mathcal{S}(\mathfrak{C})} |\theta(x)|. \tag{1.1}$$

Hence, if $\mathfrak{C}$ is a commutative C^*-algebra,

$$\|x\|_{\mathfrak{C}} = \sup_{\theta \in \mathcal{S}(\mathfrak{C})} |\theta(x)|, \quad \forall x \in \mathfrak{C}. \tag{1.2}$$

2. $\mathfrak{C}$-Valued Positive Sesquilinear Maps

In this section, we will study $\mathfrak{C}$-*valued positive sesquilinear maps on* $\mathfrak{X} \times \mathfrak{X}$ when $\mathfrak{X}$ is either simply a vector space or a right (left) module on $\mathfrak{C}$, or a (locally convex) quasi *-algebra which is a $\mathfrak{C}$-module. Throughout the section, we progressively add some hypotheses on Φ to get more results.

2.1. The Case of a Vector Space

Let $\mathfrak{X}$ be a complex vector space and Φ a $\mathfrak{C}$-valued positive sesquilinear map on $\mathfrak{X} \times \mathfrak{X}$

$$\Phi : (a, b) \in \mathfrak{X} \times \mathfrak{X} \to \Phi(a, b) \in \mathfrak{C};$$

i.e., a map with the properties

i) $\Phi(a, a) \in \mathfrak{C}^+$,
ii) $\Phi(\alpha a + \beta b, \gamma c) = \overline{\gamma}[\alpha\Phi(a, c) + \beta\Phi(b, c)]$,

with $a, b, c \in \mathfrak{X}$ and $\alpha, \beta, \gamma \in \mathbb{C}$.
The $\mathfrak{C}$-valued positive sesquilinear map Φ is called *faithful* if

$$\Phi(a, a) = 0_{\mathfrak{C}} \;\Rightarrow\; a = 0.$$

By property *i)*, it follows that

iii) $\Phi(b, a) = \Phi(a, b)^*$, for all $a, b \in \mathfrak{X}$.

In fact, let $\alpha \in \mathbb{C}$ and $a, b \in \mathfrak{X}$, then

$$0 \le \Phi(a + \alpha b, a + \alpha b) = \Phi(a, a) + |\alpha|^2\Phi(b, b) + \alpha\Phi(a, b) + \overline{\alpha}\Phi(b, a)$$

Since $\Phi(a + \alpha b, a + \alpha b)$, $\Phi(a, a)$ and $\Phi(b, b)$ are positive, hence hermitian, so it is $\alpha\Phi(a, b) + \overline{\alpha}\Phi(b, a)$; if we choose $\alpha = 1$ and $\alpha = i$ we get both

$$\Phi(a, b) + \Phi(b, a) = (\Phi(a, b) + \Phi(b, a))^* = \Phi(a, b)^* + \Phi(b, a)^*$$

and

$$i\Phi(a, b) - i\Phi(b, a) = (i\Phi(a, b) - i\Phi(b, a))^* = -i\Phi(a, b)^* + i\Phi(b, a)^*$$

hence

$$\Phi(a, b) - \Phi(b, a) = -\Phi(a, b)^* + \Phi(b, a)^*$$

if we add the first and the third equality, we get $\Phi(a, b) = \Phi(b, a)^*$.

Definition 2.1. Let Φ be a positive sesquilinear $\mathfrak{C}$- valued map on $\mathfrak{X} \times \mathfrak{X}$ (with $\mathfrak{X}$ a complex vector space). We say that Φ satisfies a Cauchy–Schwarz inequality if

$$\|\Phi(a, b)\|^2_{\mathfrak{C}} \le \|\Phi(a, a)\|_{\mathfrak{C}}\|\Phi(b, b)\|_{\mathfrak{C}}, \quad \forall a, b \in \mathfrak{X}. \tag{2.1}$$

Example 2.2. Let $\mathfrak{X} = \mathfrak{C}$ and define

$$\Phi(a, b) = b^*a.$$

It is clear that Φ is a positive sesquilinear map of $\mathfrak{C} \times \mathfrak{C}$ into $\mathfrak{C}$. Φ satisfies (2.1):

$$\begin{aligned}\|\Phi(a, b)\|^2_{\mathfrak{C}} &= \|b^*a\|^2_{\mathfrak{C}} \le \|b\|^2_{\mathfrak{C}}\|a\|^2_{\mathfrak{C}} \\ &= \|a^*a\|_{\mathfrak{C}}\|b^*b\|_{\mathfrak{C}} = \|\Phi(a, a)\|_{\mathfrak{C}}\|\Phi(b, b)\|_{\mathfrak{C}}, \quad \forall a, b \in \mathfrak{C}.\end{aligned}$$

Lemma 2.3. *Let Φ be a $\mathfrak{C}$-valued positive sesquilinear map Φ on $\mathfrak{X} \times \mathfrak{X}$. Then*

(i) for all $a, b \in \mathfrak{X}$,

$$\|\Phi(a, b)\|_{\mathfrak{C}} \le 2\|\Phi(a, a)\|^{1/2}_{\mathfrak{C}}\|\Phi(b, b)\|^{1/2}_{\mathfrak{C}}.$$

(ii) If $\mathfrak{C}$ is commutative, then Φ satisfies the Cauchy–Schwarz inequality.

Proof. Let θ be a positive linear functional on $\mathfrak{C}$ and let $\varphi : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ be given by

$$\varphi(a,b) = \theta(\Phi(a,b)), \quad \forall a,b \in \mathfrak{X}.$$

Since Φ is sesquilinear and positive and by linearity and positivity of θ, it follows that φ is a positive sesquilinear form on $\mathfrak{X} \times \mathfrak{X}$. Hence, the classical Cauchy–Schwarz inequality holds true: for all $a, b \in \mathfrak{X}$, we have that

$$|\theta(\Phi(a,b))|^2 \leq \theta(\Phi(a,a))\theta(\Phi(b,b)), \quad \forall a,b \in \mathfrak{X}.$$

Then by (1.1), taking the supremum over $\theta \in \mathcal{S}(\mathfrak{C})$, we get the inequality

$$|\theta(\Phi(a,b))| \leq \|\Phi(a,a)\|_{\mathfrak{C}} \|\Phi(b,b)\|_{\mathfrak{C}}.$$

If $\mathfrak{C}$ is commutative, using (1.2), we get

$$\|\Phi(a,b)\|_{\mathfrak{C}} \leq \|\Phi(a,a)\|_{\mathfrak{C}}^{1/2} \|\Phi(b,b)\|_{\mathfrak{C}}^{1/2}, \quad \forall a,b \in \mathfrak{X}.$$

This proves (ii).

Let us come back to the general case. Without loss of generality, we can consider $\mathfrak{C}$ as a C^*-subalgebra of $\mathfrak{B}(\mathcal{H})$ (for some Hilbert space $\mathcal{H}$). Let $\|\cdot\|$ indicate the operator norm on $\mathfrak{B}(\mathcal{H})$. Then for all $x \in \mathcal{H}$,

$$\begin{aligned} |\langle \Phi(a,b)x|x\rangle|^2 &\leq \langle \Phi(a,a)x|x\rangle \langle \Phi(b,b)x|x\rangle \\ &\leq \|\Phi(a,a)\| \|\Phi(b,b)\| \|x\|_{\mathcal{H}}^4. \end{aligned}$$

Let now $x, y \in \mathcal{H}$ with $\|x\|_{\mathcal{H}} = \|y\|_{\mathcal{H}} = 1$. Then, by the polarization identity

$$\begin{aligned} |\langle \Phi(a,b)x|y\rangle| &= \frac{1}{4}\left|\sum_{i=0}^{3} i^k \langle \Phi(a,b)(x+i^k y)|x+i^k y\rangle\right| \\ &\leq \frac{1}{4}\sum_{k=0}^{3} \left|\langle \Phi(a,b)(x+i^k y)|x+i^k y\rangle\right| \\ &\leq \frac{1}{4}\sum_{k=0}^{3} \sqrt{\|\Phi(a,a)\|\|\Phi(b,b)\|}\,\|x+i^k y\|_{\mathcal{H}}^2 \\ &\leq \sqrt{\|\Phi(a,a)\|\|\Phi(b,b)\|}\,(\|x\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}}^2) \\ &= 2\sqrt{\|\Phi(a,a)\|\|\Phi(b,b)\|}, \end{aligned}$$

since $\sum_{k=0}^{3} \|x+i^k y\|_{\mathcal{H}}^2 = 4(\|x\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}}^2)$. Taking now the supremum over all unit vectors $x, y \in \mathcal{H}$, we get

$$\begin{aligned} \|\Phi(a,b)\|_{\mathfrak{C}} = \|\Phi(a,b)\| &\leq 2\|\Phi(a,a)\|^{1/2}\|\Phi(b,b)\|^{1/2} \\ &= 2\|\Phi(a,a)\|_{\mathfrak{C}}^{1/2}\|\Phi(b,b)\|_{\mathfrak{C}}^{1/2}. \end{aligned}$$

□

The Stinespring theorem [9, Theorem 1.2.7] yields an inequality for $\mathfrak{C}$-valued positive linear maps on C^*-algebras, see [9, Theorem 1.3.1]. Motivated by that result, we provide the following corollary.

Corollary 2.4. *Let* $\mathfrak{A}$ *be a* $*$*–algebra with unit* e *and let* ω *be a* $\mathfrak{C}$*-valued positive linear map on* $\mathfrak{A}$*. Then*

$$4\|\omega(\mathsf{e})\|_{\mathfrak{C}}\,\|\omega(a^*a)\|_{\mathfrak{C}} \geq \|\omega(a)\|_{\mathfrak{C}}^2 = \|\omega(a^*)\|_{\mathfrak{C}}\,\|\omega(a)\|_{\mathfrak{C}}, \quad \forall a \in \mathfrak{A}.$$

Proof. It suffices to apply Lemma 2.3 to $\Phi(a,b) = \omega(b^*a)$, $a,b \in \mathfrak{A}$, with $\mathfrak{X} = \mathfrak{A}$. □

Remark 2.5. If $\Phi(a,b) = \omega(b^*a)$, $a,b \in \mathfrak{A}$ satisfies the Cauchy–Schwarz inequality in the norm (e.g., if either $\mathfrak{C}$ is commutative or if $\mathfrak{A}$ is a $\mathfrak{C}$-module and ω is $\mathfrak{C}$-linear), then

$$\|\omega(\mathsf{e})\|_{\mathfrak{C}}\,\|\omega(a^*a)\|_{\mathfrak{C}} \geq \|\omega(a^*)\|_{\mathfrak{C}}\,\|\omega(a)\|_{\mathfrak{C}}, \quad \forall a \in \mathfrak{A}.$$

Definition 2.6. Let $\mathfrak{X}$ be a vector space. A $\mathfrak{C}$-valued faithful positive sesquilinear map Φ on $\mathfrak{X} \times \mathfrak{X}$ is said to be a *C^*-valued quasi-inner product*, and we often will write $\langle a|b\rangle_\Phi := \Phi(a,b)$, $a,b \in \mathfrak{X}$.

A C^*-valued quasi-inner product $\Phi : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{C}$ induces a quasi-norm $\|\cdot\|_\Phi$ on $\mathfrak{X}$:

$$\|a\|_\Phi := \sqrt{\|\langle a|a\rangle_\Phi\|_{\mathfrak{C}}} = \sqrt{\|\Phi(a,a)\|_{\mathfrak{C}}}, \quad a \in \mathfrak{X}.$$

This means that

$$\begin{aligned} &\|a\|_\Phi \geq 0, \qquad \forall a \in \mathfrak{X} \text{ and } \|a\|_\Phi = 0 \Leftrightarrow a = 0,\\ &\|\alpha a\|_\Phi = |\alpha|\|a\|_\Phi, \qquad \forall \alpha \in \mathbb{C}, a \in \mathfrak{X},\\ &\|a+b\|_\Phi \leq \sqrt{2}(\|a\|_\Phi + \|b\|_\Phi), \qquad \forall a,b \in \mathfrak{X}. \qquad (2.2)\end{aligned}$$

Indeed, by Lemma 2.3:

$$\begin{aligned}\|a+b\|_\Phi^2 &= \|\Phi(a+b,a+b)\|_{\mathfrak{C}}\\ &\leq \|\Phi(a,a)\|_{\mathfrak{C}} + 2\|\Phi(a,b)\|_{\mathfrak{C}} + \|\Phi(b,b)\|_{\mathfrak{C}}\\ &\leq \|a\|_\Phi^2 + 4\|a\|_\Phi\,\|b\|_\Phi + \|b\|_\Phi^2 \leq 2(\|a\|_\Phi + \|b\|_\Phi)^2, \forall a,b \in \mathfrak{X}.\end{aligned}$$

The space $\mathfrak{X}$ is then a quasi-normed space with respect to the quasi-norm $\|\cdot\|_\Phi$.

Definition 2.7. If the complex vector space $\mathfrak{X}$ is complete with respect to the quasi-norm $\|\cdot\|_\Phi$, $\mathfrak{X}$ will be said a *quasi-Banach space with* $\mathfrak{C}$*-valued quasi-inner product* or for short a *quasi* $B_{\mathfrak{C}}$*-space*.

Let $\mathfrak{X}$ be a quasi $B_{\mathfrak{C}}$-space and $\mathfrak{D}(X)$ a dense subspace of $\mathfrak{X}$. A linear map $X : \mathfrak{D}(X) \to \mathfrak{X}$ is Φ-*adjointable* if there exists a linear map X^* defined on a subspace $\mathfrak{D}(X^*) \subset \mathfrak{X}$ such that

$$\Phi(Xa,b) = \Phi(a,X^*b), \quad \forall a \in \mathfrak{D}(X), b \in \mathfrak{D}(X^*).$$

Let $\mathfrak{D}$ be a dense subspace of $\mathfrak{X}$ and let us consider the following families of linear operators acting on $\mathfrak{D}$:

$$\begin{aligned}\mathcal{L}^\dagger(\mathfrak{D},\mathfrak{X}) &= \{X\,\Phi\text{-adjointable}\,, \mathfrak{D}(X) = \mathfrak{D};\ \mathfrak{D}(X^*) \supset \mathfrak{D}\}\\ \mathcal{L}^\dagger(\mathfrak{D}) &= \{X \in \mathcal{L}^\dagger(\mathfrak{D},\mathfrak{X}) : X\mathfrak{D} \subset \mathfrak{D};\ X^*\mathfrak{D} \subset \mathfrak{D}\}\\ \mathcal{L}^\dagger(\mathfrak{D})_b &= \{Y \in \mathcal{L}^\dagger(\mathfrak{D});\ Y \text{ is bounded on } \mathfrak{D}\}.\end{aligned}$$

The involution in $\mathcal{L}^\dagger(\mathfrak{D},\mathfrak{X})$ is defined by $X^\dagger := X^* \upharpoonright \mathfrak{D}$, the restriction of X^*, the Φ-adjoint of X, to $\mathfrak{D}$.

The sets $\mathcal{L}^\dagger(\mathfrak{D})$ and $\mathcal{L}^\dagger(\mathfrak{D})_b$ are *-algebras.

Remark 2.8. If $X \in \mathcal{L}^\dagger(\mathfrak{D},\mathfrak{X})$, then X is closable. By definition, X is adjointable. Let X^* be its adjoint with domain $\mathfrak{D}(X^*)$. We prove that X^* is closed. Indeed, suppose that $\{u_n\}_n$ is a sequence in $\mathfrak{D}(X^*)$ such that $\|u_n - u\|_\Phi \to 0$ for some $u \in \mathfrak{X}$ and $\|X^*u_n - v\|_\Phi \to 0$ for some $v \in \mathfrak{X}$. Clearly $\|u_n - u\|_\Phi \to 0$ is equivalent to $\Phi(u_n - u, u_n - u) \to 0$. Then by Lemma 2.3, we get, for every $y \in \mathfrak{X}$,

$$\|\Phi(u_n - u, y)\|^2_{\mathfrak{C}} \leq 4\|\Phi(u_n - u, u_n - u)\|_{\mathfrak{C}}\|\Phi(y,y)\|_{\mathfrak{C}} \to 0.$$

Hence, for every $z \in \mathfrak{D}$

$$\|\Phi(u_n, Xz)\|_{\mathfrak{C}} = \|\Phi(X^*u_n, z)\|_{\mathfrak{C}} \to \|\Phi(u, Xz)\|_{\mathfrak{C}}.$$

On the other hand,

$$\|\Phi(X^*u_n, z)\|_{\mathfrak{C}} \to \|\Phi(v, z)\|_{\mathfrak{C}}.$$

These relations imply that $u \in \mathfrak{D}(X^*)$ and $X^*u = v$. Thus, X^* is closed. Now apply this result to $X^{\dagger *}$ to obtain a closed extension of X.

Remark 2.9. $\mathcal{L}^\dagger(\mathfrak{D},\mathfrak{X})$ is also a *partial *-algebra* [1] with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger := X^* \upharpoonright \mathfrak{D}$ and the *(weak)* partial multiplication $\square$ defined whenever there exists $Y \in \mathcal{L}^\dagger(\mathfrak{D},\mathfrak{X})$ such that

$$\Phi(X_2 a, X_1 b) = \Phi(Ya, b), \quad \forall a, b \in \mathfrak{D}.$$

The element Y, if it exists, is unique. We put $Y = X_1 \square X_2$.

If Φ is not faithful, we can consider the set

$$N_\Phi = \{a \in \mathfrak{X} : \ \Phi(a,a) = 0_{\mathfrak{C}}\}.$$

Lemma 2.10. *N_Φ is a subspace of $\mathfrak{X}$.*

Proof. $N_\Phi = \big\{a \in \mathfrak{X} : \Phi(a,a) = 0_{\mathfrak{C}}\big\} = \big\{a \in \mathfrak{X} : \Phi(a,b) = 0_{\mathfrak{C}}, \ \forall \ b \in \mathfrak{X}\big\}$ is an easy consequence of Lemma 2.3. □

For the sake of simplicity, we denote by $\Lambda_\Phi(a)$ the coset containing $a \in \mathfrak{X}$; i.e., $\Lambda_\Phi(a) = a + N_\Phi$.

We define a $\mathfrak{C}$-valued positive sesquilinear map on $\mathfrak{X}/N_\Phi \times \mathfrak{X}/N_\Phi$ as follows:

$$\begin{aligned} &\langle\cdot|\cdot\rangle_\Phi : \mathfrak{X}/N_\Phi \times \mathfrak{X}/N_\Phi \to \mathfrak{C} \\ &\langle\Lambda_\Phi(a)|\Lambda_\Phi(b)\rangle_\Phi := \Phi(a,b) \end{aligned} \tag{2.3}$$

The associated quasi-norm is:

$$\|\Lambda_\Phi(a)\|_\Phi := \sqrt{\|\Phi(a,a)\|_{\mathfrak{C}}}, \quad a \in \mathfrak{X}. \tag{2.4}$$

It is easy to check that

Lemma 2.11. *The quotient space* $\mathfrak{X}/N_\Phi = \Lambda_\Phi(\mathfrak{X})$ *is a quasi-normed space.*

Denote by $\widetilde{\mathfrak{X}}$ the completion of $(\mathfrak{X}/N_\Phi, \|\cdot\|_\Phi)$.

Remark 2.12. We can extend $\langle\cdot|\cdot\rangle_\Phi$ defined in (2.3) to $\widetilde{\mathfrak{X}} \times \widetilde{\mathfrak{X}}$ by continuity, taking into account that, $\langle\cdot|\cdot\rangle_\Phi$ is jointly continuous by Lemma 2.3.

2.2. The Case of a Module over $\mathfrak{C}$

In this section, $\mathfrak{X}$ is a right module over $\mathfrak{C}$, and Φ will be a $\mathfrak{C}$-valued positive sesquilinear map on $\mathfrak{X} \times \mathfrak{X}$ such that

$$\|\Phi(ax, ax)\|_{\mathfrak{C}} \leq \|\Phi(a, a)\|_{\mathfrak{C}}\, \|x\|_{\mathfrak{C}}^2, \quad a \in \mathfrak{X}, x \in \mathfrak{C}. \tag{2.5}$$

If Φ is not faithful (thus N_Φ is not the zero subspace), we have

Lemma 2.13. *Let* Φ *be a* $\mathfrak{C}$*-valued positive sesquilinear map on* $\mathfrak{X} \times \mathfrak{X}$ *satisfying (2.5), then* $\mathfrak{X}/N_\Phi[\|\cdot\|_\Phi]$ *is a normed right* C^**-module over* $\mathfrak{C}$*.*

Proof. First we observe that, by (2.5), if $a \in N_\Phi$ and $x \in \mathfrak{C}$, then $ax \in N_\Phi$. This implies that $\Lambda_\Phi(a)x = \Lambda_\Phi(ax)$ for every $a \in \mathfrak{X}$ and $x \in \mathfrak{C}$. Moreover, we have

$$\|\Lambda_\Phi(ax)\|_\Phi \leq \|\Lambda_\Phi(a)\|_\Phi \|x\|_{\mathfrak{C}}, \quad a \in \mathfrak{X}, x \in \mathfrak{C}.$$

Indeed, from (2.5), we get

$$\begin{aligned}\|\Lambda_\Phi(ax)\|_\Phi^2 &= \|\Phi(ax, ax)\|_{\mathfrak{C}} \leq \|\Phi(a, a)\|_{\mathfrak{C}} \|x\|_{\mathfrak{C}}^2 \\ &= \|\Lambda_\Phi(a)\|_\Phi^2 \|x\|_{\mathfrak{C}}^2, \quad a \in \mathfrak{X}, x \in \mathfrak{C}.\end{aligned}$$

□

If property (2.5) holds, then the completion $\widetilde{\mathfrak{X}}$ of $(\mathfrak{X}/N_\Phi, \|\cdot\|_\Phi)$ is also a right Banach module over $\mathfrak{C}$; indeed, the right multiplication by elements in $\mathfrak{C}$ can be extended by continuity to $\widetilde{\mathfrak{X}}$:

$$\|ax\|_\Phi \leq \|a\|_\Phi \|x\|_{\mathfrak{C}}, \quad \forall a \in \widetilde{\mathfrak{X}}, x \in \mathfrak{C}.$$

Definition 2.14. Let Φ be a $\mathfrak{C}$-valued positive sesquilinear map on $\mathfrak{X} \times \mathfrak{X}$. The map Φ is $\mathfrak{C}$-*linear* if

$$\Phi(a, bx) = \Phi(a, b)x, \ \forall x \in \mathfrak{C}; a, b \in \mathfrak{X}.$$

Then Φ satisfies the Cauchy–Schwarz inequality as shown in [6, Sect. 1.2].

Remark 2.15. If Φ is a $\mathfrak{C}$-valued $\mathfrak{C}$-linear positive sesquilinear map on $\mathfrak{X} \times \mathfrak{X}$ then (2.5) holds. Indeed, recalling that if $c \in \mathfrak{C}^+$, then $t^*ct \leq \|c\|_{\mathfrak{C}} t^*t$, $t \in \mathfrak{C}$

$$\begin{aligned}\Phi(ax, ax) &= \Phi(ax, a)x = \Phi(a, ax)^*x = (\Phi(a, a)x)^*x \\ &= x^*\Phi(a, a)x \leq \|\Phi(a, a)\|_{\mathfrak{C}}\, x^*x,\end{aligned}$$

and recalling that the norm in a C^*-algebra preserves the order on positive elements, we get (2.5). In this case, in fact, $\mathfrak{X}/N_\Phi$ is a pre-Hilbert $\mathfrak{C}$-module (see [6, Definition 1.2.1]).

Remark 2.16. Let Φ be a $\mathfrak{C}$-valued $\mathfrak{C}$-linear positive sesquilinear map on $\mathfrak{X}\times\mathfrak{X}$, then

$$\Phi(b,a)\Phi(a,b) \leq \|\Phi(a,a)\|_{\mathfrak{C}}\, \Phi(b,b), \quad \forall a,b \in \mathfrak{X}.$$

This is another generalization of the Cauchy–Schwarz inequality, see [6, Proposition 1.2.4].

It is easy to see that the following Cauchy–Schwarz inequality holds.

Lemma 2.17. *Let Φ be a $\mathfrak{C}$-valued $\mathfrak{C}$-linear positive sesquilinear map on $\mathfrak{X}\times\mathfrak{X}$, then $\|\Phi(a,b)\|_{\mathfrak{C}} \leq \|\Lambda_\Phi(a)\|_\Phi\, \|\Lambda_\Phi(b)\|_\Phi$, for every $a,b \in \mathfrak{X}$.*

Remark 2.18. If Φ is faithful and satisfies the Cauchy–Schwarz inequality, then $\|\cdot\|_\Phi$ defined in (2.4) is not only a quasi-norm, but is a norm: $\|a\|_\Phi = 0$ implies that $a = 0$ and the triangular inequality holds true. In fact:

$$\begin{aligned} \|a+b\|_\Phi^2 &= \|\Phi(a+b,a+b)\|_{\mathfrak{C}} \\ &\leq \|\Phi(a,a)\|_{\mathfrak{C}} + 2\|\Phi(a,b)\|_{\mathfrak{C}} + \|\Phi(b,b)\|_{\mathfrak{C}} \\ &\leq \|a\|_\Phi^2 + 2\|a\|_\Phi\, \|b\|_\Phi + \|b\|_\Phi^2 = (\|a\|_\Phi + \|b\|_\Phi)^2. \end{aligned}$$

2.3. The Case of a Locally Convex quasi *-algebra

In this section, we will consider a locally convex quasi *-algebra $(\mathfrak{A},\mathfrak{A}_0)$ with unit e.

Definition 2.19. We denote by $\mathcal{Q}_{\mathfrak{A}_0}^{\mathfrak{C}}(\mathfrak{A})$ the set of all $\mathfrak{C}$-valued positive sesquilinear maps on $\mathfrak{A}\times\mathfrak{A}$ that satisfy a property of invariance:

(I) $\Phi(ac,d) = \Phi(c,a^*d), \quad \forall\, a \in \mathfrak{A},\ c,d \in \mathfrak{A}_0$

and call $\Phi \in \mathcal{Q}_{\mathfrak{A}_0}^{\mathfrak{C}}(\mathfrak{A})$ a $\mathfrak{C}$-valued *invariant* positive sesquilinear map.

We maintain the same notations as before: then $\Lambda_\Phi(a)$ will denote the coset in $\mathfrak{A}/N_\Phi$, containing a.

Remark 2.20. We recall that

$$\lim_{n\to\infty} \Phi(a_n,a_n) = 0_{\mathfrak{C}} \Leftrightarrow \lim_{n\to\infty} \|\Lambda_\Phi(a_n)\|_\Phi = 0.$$

We will indicate by $\widetilde{\mathfrak{A}}$ the completion of $\mathfrak{A}/N_\Phi$ w.r.to $\|\cdot\|_\Phi$.

Proposition 2.21. *Let Φ be a $\mathfrak{C}$-valued positive sesquilinear map on $\mathfrak{A}\times\mathfrak{A}$. The following statements are equivalent:*

- *(i)* $\Lambda_\Phi(\mathfrak{A}_0) = \mathfrak{A}_0/N_\Phi$ *is dense in* $\widetilde{\mathfrak{A}}$.
- *(ii) If* $\{a_n\}_n$ *is a sequence of elements of* $\mathfrak{A}$ *such that:*
  - *(ii.a)* $\Phi(a_n,c)\to 0_{\mathfrak{C}}$, *as* $n\to\infty$, *for every* $c \in \mathfrak{A}_0$
  - *(ii.b)* $\Phi(a_n - a_m, a_n - a_m)\to 0_{\mathfrak{C}}$, *as* $n,m\to\infty$,

  then $\lim_{n\to\infty} \Phi(a_n,a_n) = 0_{\mathfrak{C}}$.

Proof. We proceed along the lines of [3, Proposition 2.3.2].
(i) $\Rightarrow$ *(ii)* Let $\{a_n\}_n \subset \mathfrak{A}$ be a sequence as required in *(ii)*. Then by *(ii.b)* and Remark 2.20, the sequence $\{\Lambda_\Phi(a_n)\}_n$ is Cauchy in the complete space

$\widetilde{\mathfrak{A}}$. Then there exists $\xi \in \widetilde{\mathfrak{A}}$ such that $\lim_{n\to\infty} \|\Lambda_\Phi(a_n) - \xi\|_\Phi = 0$. Now, by $(ii.a)$

$$\|\langle \xi | \Lambda_\Phi(c)\rangle_\Phi\|_{\mathfrak{C}} = \lim_{n\to\infty} \|\langle \Lambda_\Phi(a_n)|\Lambda_\Phi(c)\rangle_\Phi\|_{\mathfrak{C}} = \lim_{n\to\infty} \|\Phi(a_n, c)\|_{\mathfrak{C}} = 0,$$

for all $c \in \mathfrak{A}_0$, hence $\langle \xi|\Lambda_\Phi(c)\rangle_\Phi = 0_{\mathfrak{C}}$, $\forall c \in \mathfrak{A}_0$, i.e., ξ is orthogonal to $\Lambda_\Phi(\mathfrak{A}_0)$ dense subset of $\widetilde{\mathfrak{A}}$, thus $\xi = 0$. Finally,

$$\lim_{n\to\infty} \|\Phi(a_n, a_n)\|_{\mathfrak{C}} = \|\langle \xi|\xi\rangle_\Phi\|_{\mathfrak{C}} = \|\xi\|_\Phi^2 = 0$$

and this is equivalent to $\lim_{n\to\infty} \Phi(a_n, a_n) = 0_{\mathfrak{C}}$.
$(ii) \Rightarrow (i)$ Let $\xi \in \widetilde{\mathfrak{A}}$ be a vector which is orthogonal to $\Lambda_\Phi(\mathfrak{A}_0)$, i.e.,

$$\langle \xi|\Lambda_\Phi(c)\rangle_\Phi = 0_{\mathfrak{C}}, \quad \forall c \in \mathfrak{A}_0.$$

Suppose that $\{a_n\}_n \subset \mathfrak{A}$ is a sequence such that $\Lambda_\Phi(a_n) \overset{\|\cdot\|_\Phi}{\to} \xi$, i.e.,

$$\|\Lambda_\Phi(a_n) - \xi\|_\Phi \to 0, \text{ as } n \to \infty. \tag{2.6}$$

Then $\{a_n\}_n$ fulfills $(ii.a)$, indeed, for every $c \in \mathfrak{A}_0$

$$0 = \|\langle \xi|\Lambda_\Phi(c)\rangle_\Phi\|_{\mathfrak{C}} = \lim_{n\to\infty} \|\langle \Lambda_\Phi(a_n)|\Lambda_\Phi(c)\rangle_\Phi\|_{\mathfrak{C}} = \lim_{n\to\infty} \|\Phi(a_n, c)\|_{\mathfrak{C}},$$

hence $\Phi(a_n, c) \to 0_{\mathfrak{C}}$, as $n \to \infty$, for every $c \in \mathfrak{A}_0$; $(ii.b)$ follows because $\{\Lambda_\Phi(a_n)\}_n$ is a convergent sequence in a complete space; hence, it is Cauchy in $\widetilde{\mathfrak{A}}$ i.e., $\lim_{n,m\to\infty} \|\Lambda_\Phi(a_n - a_m)_\Phi\|_\Phi = 0$ which is equivalent of saying that

$$\lim_{n,m\to\infty} \Phi(a_n - a_m, a_n - a_m) = 0_{\mathfrak{C}}.$$

Thus, by hypothesis $\{a_n\}_n$ is such that $\lim_{n\to\infty} \Phi(a_n, a_n) = 0_{\mathfrak{C}}$; hence,

$$\lim_{n\to\infty} \|\Phi(a_n, a_n)\|_{\mathfrak{C}} = \lim_{n\to\infty} \|\Lambda_\Phi(a_n)\|_\Phi = 0.$$

Comparing with (2.6) we get $\|\xi\|_\Phi = 0$, hence $\xi = 0$. It follows that $\Lambda_\Phi(\mathfrak{A}_0)$ is dense in $\widetilde{\mathfrak{A}}$. □

Definition 2.22. We denote by $\mathcal{I}_{\mathfrak{A}_0}^{\mathfrak{C}}(\mathfrak{A})$ the subset of $\mathcal{Q}_{\mathfrak{A}_0}^{\mathfrak{C}}(\mathfrak{A})$ satisfying one of the conditions (i) or (ii) of Proposition 2.21.

2.4. Examples

Before going forth, we give some examples of C^*-valued positive sesquilinear maps. We denote by $\mathfrak{B}(\mathcal{H})$ the C^*-algebra of bounded operators on $\mathcal{H}$.

Example 2.23. Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$ and consider the *-algebra $\mathcal{L}^\dagger(\mathcal{D})_b$. Let $\mathcal{H}_1 \subset \mathcal{D}$ be a closed subspace of $\mathcal{H}$ and P be the orthogonal projection onto $\mathcal{H}_1$. Then $P\mathfrak{B}(\mathcal{H})P$ is a von Neumann algebra (see, e.g., [5, Corollary 5.5.7]) which can be identified with a subspace of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. If V is an operator in $P\mathfrak{B}(\mathcal{H})P$, let

$$\Phi : \mathcal{L}^\dagger(\mathcal{D})_b \times \mathcal{L}^\dagger(\mathcal{D})_b \to P\mathfrak{B}(\mathcal{H})P$$

be given by

$$\Phi(A, B) = V^* B^* A V.$$

Then Φ satisfies our assumptions, but in general Φ is not $P\mathfrak{B}(\mathcal{H})P$-linear. However, if $V = P$, then Φ is $P\mathfrak{B}(\mathcal{H})P$-linear.

Consider the quasi *-algebra $(\mathcal{L}^\dagger(\mathcal{D},\mathcal{H}),\mathcal{L}^\dagger(\mathcal{D})_b)$. Let V_i be positive bounded operators on $\mathcal{H}$. Define the sesquilinear form Φ on this quasi *-algebra by

$$\Phi(A,B)=\sum_{i=1}^{n}\langle Ax_i|Bx_i\rangle V_i, \quad x_1,...,x_n\in\mathcal{D}.$$

Then $\Phi\in\mathcal{I}^{\mathfrak{C}}_{\mathfrak{A}_0}(\mathfrak{A})$.

Example 2.24. Let $\mathfrak{M}$ be a von Neumann algebra and ρ a normal faithful finite trace on $\mathfrak{M}_+$. Consider the proper CQ*-algebra $(L^p(\rho),L^\infty(\rho))$ (see [3]). Let $W\in L^\infty(\rho)$ such that $W\geq 0$. For every $t\in[0,\|W\|]$, consider the function

$$f_t(s):=\begin{cases} s & \text{for } \ 0\leq s<t\\ t & \text{for } \ t\leq s\leq\|W\|.\end{cases}$$

Then $\|f_t\|_\infty\leq\|W\|$ and for each t_1,t_2 it is

$$\|f_{t_1}-f_{t_2}\|_\infty\leq|t_1-t_2|.$$

Moreover, if $\sigma(W)$ denotes the spectrum of the operator W, it is $f_t\restriction\sigma(W)\in C(\sigma(W))$. Then:

$$\|f_t(W)\|_{\frac{p}{p-2}}\leq\rho(\mathbb{I})\|f_t(W)\|_\infty=\rho(\mathbb{I})\|W\|$$

where $\rho(\mathbb{I})$ is the trace of the identity operator $\mathbb{I}$. Hence, $f_t(W)\in L^{\frac{p}{p-2}}(\rho)$, for each $t\in[0,\|W\|]$. Consider the right multiplication operator

$$R_W: X\in L^p(\rho)\to XW\in L^{\frac{p-1}{p}}(\rho).$$

Let $\Phi: L^p(\rho)\times L^p(\rho)\to C([0,\|W\|])$ be given by

$$\Phi(X,Y)(t)=\rho(X(R_{f_t(W)}Y)^*). \tag{2.7}$$

For each $t\in[0,\|W\|]$, $\Phi(\cdot,\cdot)(t)$ is a well-defined positive sesquilinear (scalar valued) form on $L^p(\rho)\times L^p(\rho)$ because $f_t(W)\in C^*(W)\subseteq L^\infty(\rho)$, with $C^*(W)$ the C^*-algebra generated by W. Moreover, $f_t(W)$ is a positive operator. To see that $\Phi(X,Y)$ is continuous, just observe that for $t_1,t_2\in[0,\|W\|]$, we have that

$$\begin{aligned}|\Phi(X,Y)(t_1)-\Phi(X,Y)(t_2)|&\leq\|X\|_p\|Y\|_p\|\|f_{t_1}(W)-f_{t_2}(W)\|_{\frac{p}{p-2}}\\ &\leq\|X\|_p\|Y\|_p\rho(\mathbb{I})(t_1-t_2).\end{aligned}$$

To see that $\Lambda_\Phi(L^\infty(\rho))$ is dense in $L^p(\rho)/N_\Phi$, just observe that for each $t\in[0,\|W\|]$, we have that, for every sequence $\{X_n\}_n\subseteq L^\infty(\rho)$ and $X\in L^p(\rho)$, with $X_n\to X$:

$$\begin{aligned}|\Phi(X_n-X,X_n-X)(t)|&\leq\|X_n-X\|_p^2\,\|f_t(W)\|_{\frac{p}{p-2}}\\ &\leq\|X_n-X\|_p^2\,\|W\|\rho(\mathbb{I}).\end{aligned}$$

Remark 2.25. The previous example holds also in the case $p=2$ ($\frac{p}{p-2}=\infty$). In this case, $\|f_t(W)\|_{\frac{p}{p-2}}=\|f_t(W)\|_\infty\leq\|W\|$ and

$$|\Phi(X,Y)(t_1)-\Phi(X,Y)(t_2)|\leq\|X\|_2\|Y\|_2(t_1-t_2), \quad t_1,t_2\in[0,\|W\|]$$

$$|\Phi(X_n - X, X_n - X)(t)| \leq \|X_n - X\|_2^2 \, \|W\|,$$

for every sequence $\{X_n\}_n \subseteq L^\infty(\rho)$ and $X \in L^2(\rho)$, with $X_n \to X$.

Example 2.26. Let $\mathfrak{M}$ be a von Neumann algebra and ρ a normal faithful finite trace on $\mathfrak{M}_+$. Let W be as in the previous example. Consider the space $L^2([0, \|W\|], \mathcal{B}(\mathcal{H}))$ with respect to the Gel'fand-Pettis integral (see [4]) and the weakly *-measurable operator-valued functions $A_t \in L^2([0, \|W\|], \mathcal{B}(\mathcal{H}))$ such that $A_t \geq 0$ for a.e. $t \in [0, \|W\|]$. Let the right multiplication operator and the function f_t be defined as in Example 2.24 and

$$\Phi : L^p(\rho) \times L^p(\rho) \to \mathcal{B}(\mathcal{H})$$

be defined as

$$\Phi(X, Y) = \int_0^{\|W\|} \rho(X(R_{f_t(W)}Y)^*) A_t dt.$$

Then $\Phi(X, Y)$ is well-defined since

$$\langle \rho(X(R_{f_t(W)}Y)^*) A_t h_1 | h_2 \rangle = \rho(X(R_{f_t(W)}Y)^*) \langle A_t h_1 | h_2 \rangle$$

is a measurable function of t for every fixed $h_1, h_2 \in \mathcal{H}$; hence, the function $\rho(X(R_{f_t(W)}Y)^*) A_t$ is weakly *-measurable. Recall from Example 2.24 that for every $X, Y \in L^p(\rho)$, we have that the function $t \to \rho(X(R_{f_t(W)}Y)^*)$ is continuous on $[0, \|W\|]$, hence $\sup_{t \in [0, \|W\|]} |\rho(X(R_{f_t(W)}Y)^*)| < \infty$. Moreover, put

$$|||A_t|||_2 = \left\| \int_0^{\|W\|} A_t^* A_t dt \right\|^{1/2} = \left(\sup_{h \in \mathcal{H}, \|h\| \leq 1} \int_0^{\|W\|} \|A_t(h)\|^2 dt \right)^{1/2};$$

since for every $h \in \mathcal{H}$, with $\|h\| \leq 1$

$$\begin{aligned} &\int_0^{\|W\|} |\rho(X(R_{f_t(W)}Y)^*)|^2 \|A_t h\|^2 dt \\ &\quad \leq \sup_{t \in [0, \|W\|]} |\rho(X(R_{f_t(W)}Y)^*)|^2 \int_0^{\|W\|} \|A_t h\|^2 dt \\ &\quad \leq \sup_{t \in [0, \|W\|]} |\rho(X(R_{f_t(W)}Y)^*)|^2 |||A_t|||_2^2 < +\infty \end{aligned}$$

it follows that $\rho(X(R_{f_t(W)}Y)^*) A_t \in L^2([0, \|W\|], \mathcal{B}(\mathcal{H}))$. It is straightforward to check that $\Phi(X, X) \geq 0$ for all $X \in L^p(\rho)$ using our choice of $A_t \geq 0$; moreover, it is

$$\Phi(CX, Y) = \Phi(X, C^* Y), \quad \forall X, Y \in L^\infty(\rho),\ C \in L^p(\rho).$$

If now we take a sequence $\{X_n\}_n \subseteq L^\infty(\rho)$ and $X \in L^p(\rho)$, by the above argument, we deduce that, for every $h \in \mathcal{H}$, with $\|h\| \leq 1$,

$$\begin{aligned}\int_0^{\|W\|} & (\rho((X_n - X)(R_{f_t(W)}(X_n - X))^*))^2 \|A_t h\|^2 dt \\ & \leq \sup_{t\in[0,\|W\|]} \rho((X_n - X)(R_{f_t(W)}(X_n - X))^*)^2 |\!|\!|A_t|\!|\!|_2^2 \\ & \leq \sup_{t\in[0,\|W\|]} \|X_n - X\|_p^4 \|f_t(W)\|^2_{\frac{p}{p-2}} |\!|\!|A_t|\!|\!|_2^2 \\ & \leq \|X_n - X\|_p^4 \|W\|^2 \rho(\mathbb{I}) |\!|\!|A_t|\!|\!|_2^2, \quad \forall n \in \mathbb{N}.\end{aligned}$$

This shows that $\Lambda_\Phi(L^\infty(\rho))$ is dense in $L^p(\rho)/N_\Phi$.

Example 2.27. Consider $(L^p(\rho), L^\infty(\rho))$ and let$\{\Phi_n\}_n$ be a family of $\mathfrak{C}$-valued invariant positive sesquilinear maps with $\Lambda_{\Phi_n}(\mathfrak{A}_0)$ dense in $\mathfrak{A}/N_{\Phi_n}$ and such that there exists $M > 0$ for which

$$\|\Phi_n(X,Y)\|_{\mathfrak{C}} \leq M \|X\|_p \|Y\|_p$$

for all $X, Y \in L^p(\rho)$. Define now

$$\Phi : L^p(\rho) \times L^p(\rho) \to \mathfrak{C}$$

by

$$\Phi(X,Y) = \sum_{n=1}^{\infty} x_n \Phi_n(X,Y) x_n^*,$$

for all $X, Y \in L^p(\rho)$ and $\{x_n\}_n \subseteq \mathfrak{C}$ such that $\sum_{n=1}^\infty \|x_n\|^2 < \infty$. Then

$$\|\Phi(X,Y)\|_{\mathfrak{C}} \leq M \|X\|_p \|Y\|_p \sum_{n=1}^{\infty} \|x_n\|^2, \quad X, Y \in L^p(\rho).$$

It is easy to verify that Φ is a $\mathfrak{C}$-valued invariant positive sesquilinear map with $\Lambda_\Phi(\mathfrak{A}_0)$ dense in $\mathfrak{A}/N_\Phi$.

3. Construction of *-Representations

An important tool for the study of the structure of a locally convex quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is the Gelfand–Naimark–Segal (GNS) construction for an invariant positive sesquilinear (ips) form on $\mathfrak{A} \times \mathfrak{A}$. The aim of this section is to extend this construction starting from a $\mathfrak{C}$-valued positive sesquilinear maps on $\mathfrak{A} \times \mathfrak{A}$ when $(\mathfrak{A}, \mathfrak{A}_0)$ is a locally convex quasi *-algebra with unit e.

Definition 3.1. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit e. Let $\mathfrak{D}_\Pi$ be a dense subspace of a certain quasi $B_{\mathfrak{C}}$-space $\mathfrak{X}$ with $\mathfrak{C}$-valued quasi-inner product $\langle\cdot|\cdot\rangle_{\mathfrak{X}}$. A linear map Π from $\mathfrak{A}$ into $\mathcal{L}^{\dagger}(\mathfrak{D}_\Pi, \mathfrak{X})$ is called a **–representation* of $(\mathfrak{A}, \mathfrak{A}_0)$, if the following properties are fulfilled:

- (i) $\Pi(a^*) = \Pi(a)^\dagger := \Pi(a)^* \upharpoonright \mathfrak{D}_\Pi, \quad \forall\, a \in \mathfrak{A}$;
- (ii) for $a \in \mathfrak{A}$ and $c \in \mathfrak{A}_0$, $\Pi(a) \,\Box\, \Pi(c)$ is well–defined and $\Pi(a) \,\Box\, \Pi(c) = \Pi(ac)$.

We assume that for every *-representation Π of $(\mathfrak{A}, \mathfrak{A}_0)$, $\Pi(\mathsf{e}) = \mathbb{I}_{\mathfrak{D}_\Pi}$, the identity operator on the space $\mathfrak{D}_\Pi$.

The *-representation Π is said to be

- *closable* if there exists $\widetilde{\Pi}$ closure of Π defined as $\widetilde{\Pi}(a) = \overline{\Pi(a)} \upharpoonright \widetilde{\mathfrak{D}}_\Pi$ where $\widetilde{\mathfrak{D}}_\Pi$ is the completion under the graph topology t_Π defined by the seminorms $\xi \in \mathfrak{D}_\Pi \to \|\xi\|_{\mathfrak{X}} + \|\Pi(a)\xi\|_{\mathfrak{X}}$, $a \in \mathfrak{A}$, where $\|\cdot\|_{\mathfrak{X}}$ is the norm induced by the inner product of $\mathfrak{X}$;
- *closed* if $\mathfrak{D}_\Pi[t_\Pi]$ is complete;
- *cyclic* if there exists $\xi \in \mathfrak{D}_\Pi$ such that $\Pi(\mathfrak{A}_0)\xi$ is dense in $\mathfrak{X}$ in its quasi-norm topology.

Theorem 3.2. *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit e and $\Phi \in \mathcal{Q}^{\mathfrak{C}}_{\mathfrak{A}_0}(\mathfrak{A})$. The following statements are equivalent:*

(i) $\Phi \in \mathcal{I}^{\mathfrak{C}}_{\mathfrak{A}_0}(\mathfrak{A})$.

*(ii) There exist a quasi $B_{\mathfrak{C}}$-space $\mathfrak{X}_\Phi$ with $\mathfrak{C}$-valued inner product $\langle\cdot|\cdot\rangle_{\mathfrak{X}_\Phi}$, a dense subspace $\mathfrak{D}_\Phi \subseteq \mathfrak{X}_\Phi$ and a closed cyclic *-representation $\Pi_\Phi : \mathfrak{A} \to \mathcal{L}^\dagger(\mathfrak{D}_\Phi, \mathfrak{X}_\Phi)$ with cyclic vector ξ_Φ such that*

$$\langle \Pi_\Phi(a)x|y\rangle_{\mathfrak{X}_\Phi} = \langle x|\Pi_\Phi(a^*)y\rangle_{\mathfrak{X}_\Phi}, \quad \forall x, y \in \mathfrak{D}_\Phi, a \in \mathfrak{A}$$

and such that

$$\Phi(a, b) = \langle \Pi_\Phi(a)\xi_\Phi|\Pi_\Phi(b)\xi_\Phi\rangle_{\mathfrak{X}_\Phi}, \quad \forall a, b \in \mathfrak{A}.$$

Proof. The proof proceeds along the lines of that one of [3, Proposition 2.4.1].

i) $\Rightarrow$ *ii)* Let $\Phi \in \mathcal{I}^{\mathfrak{C}}_{\mathfrak{A}_0}(\mathfrak{A})$. The completion $\widetilde{\mathfrak{A}}$ of $\Lambda_\Phi(\mathfrak{A})$ is, as we have seen, a quasi $B_{\mathfrak{C}}$-space with quasi-norm $\|\cdot\|_\Phi$ induced by the quasi-inner product $\langle\cdot|\cdot\rangle_\Phi$: $\|a\|_\Phi = \|\langle a|a\rangle_\Phi\|^{1/2}_{\mathfrak{C}}$, $a \in \widetilde{\mathfrak{A}}$. For any $a \in \mathfrak{A}$ and $c \in \mathfrak{A}_0$ put

$$\Pi^\circ_\Phi(a)(\Lambda_\Phi(c)) := \Lambda_\Phi(ac).$$

Let $c \in N_\Phi$ and $\{c_n\}_n \subset \mathfrak{A}_0$ such that

$$\Phi(c_n - ac, c_n - ac) \to 0, \quad \text{as } n \to \infty.$$

By the invariance of Φ

$$\|\Phi(ac, d)\|^2_{\mathfrak{C}} = \|\Phi(c, a^*d)\|^2_{\mathfrak{C}} \leq 2\|\Phi(c, c)\|_{\mathfrak{C}}\|\Phi(a^*d, a^*d)\|_{\mathfrak{C}} = 0, \forall d \in \mathfrak{A}_0,$$

and by (2.2) we get

$$\begin{aligned}\|\Phi(ac, ac)\|_{\mathfrak{C}} &\leq 2(\|\Phi(ac, c_n)\|_{\mathfrak{C}} + \|\Phi(ac, ac - c_n)\|_{\mathfrak{C}}) \\ &\leq 4\|\Phi(ac, ac)\|^{1/2}_{\mathfrak{C}}\|\Phi(ac - c_n, ac - c_n)\|^{1/2}_{\mathfrak{C}} \to 0, \text{ as } n \to \infty.\end{aligned}$$

Hence, we get $\Phi(ac, ac) = 0_{\mathfrak{C}}$. Thus, $ac \in N_\Phi$ and for every $a \in \mathfrak{A}$, the operator $\Pi^\circ_\Phi(a)$ is well-defined from $\Lambda_\Phi(\mathfrak{A}_0)$ into $\widetilde{\mathfrak{A}}$. Further, for every $a \in \mathfrak{A}, c, d \in \mathfrak{A}_0$

$$\begin{aligned}\langle \Pi^\circ_\Phi(a)(\Lambda_\Phi(c))|\Lambda_\Phi(d)\rangle_\Phi &= \Phi(ac, d) = \Phi(c, a^*d) \\ &= \langle \Lambda_\Phi(c)|\Lambda_\Phi(a^*d)\rangle_\Phi \\ &= \langle \Lambda_\Phi(c)|\Pi^\circ_\Phi(a^*)(\Lambda_\Phi(d))\rangle_\Phi,\end{aligned}$$

hence $\Pi^\circ_\Phi(a^*) = \Pi^\circ_\Phi(a)^\dagger$ and for every $a \in \mathfrak{A}, c, d, f \in \mathfrak{A}_0$.

$$\begin{aligned}\langle \Pi^\circ_\Phi(ac)(\Lambda_\Phi(f)) | \Lambda_\Phi(d)\rangle_\Phi &= \Phi(acf, d) = \Phi(cf, a^*d)\\ &= \langle \Pi^\circ_\Phi(c)(\Lambda_\Phi(f)) | \Pi^\circ_\Phi(a^*)(\Lambda_\Phi(d))\rangle_\Phi\end{aligned}$$

By the definition given in Remark 2.9, we conclude that $\Pi^\circ_\Phi(a) \,\square\, \Pi^\circ_\Phi(c)$ is well-defined, and therefore

$$\Pi^\circ_\Phi(ac) = \Pi^\circ_\Phi(a) \,\square\, \Pi^\circ_\Phi(c), \quad \forall a \in \mathfrak{A}, c \in \mathfrak{A}_0.$$

Hence, Π°_Φ is a *-representation and $\Pi^\circ_\Phi \upharpoonright \Lambda_\Phi(\mathfrak{A}_0)$ maps $\Lambda_\Phi(\mathfrak{A}_0)$ into itself.

The operator $\Pi^\circ_\Phi(a)$ is closable: $\Pi^\circ_\Phi(a^*)$ is adjointable, $\Pi^\circ_\Phi(a^*)^*$ is a closed extension of $\Pi^\circ_\Phi(a)$ (see Remark 2.8). Denote by $\overline{\Pi^\circ_\Phi(a)}$ its closure and let $\mathfrak{D}_\Phi$ denote the completion of $\Lambda_\Phi(\mathfrak{A}_0)$ in the graph topology t_Π and for each $a \in \mathfrak{A}$ let $\Pi_\Phi(a) = \overline{\Pi^\circ_\Phi(a)} \upharpoonright \mathfrak{D}_\Phi$. Then Π_Φ is a closed *-representation of $(\mathfrak{A}, \mathfrak{A}_0)$. Finally, since $(\mathfrak{A}, \mathfrak{A}_0)$ has a unit e, it follows that $\Lambda_\Phi(\mathsf{e}) = \mathsf{e} + N_\Phi$ is a cyclic vector and $\Pi_\Phi(\mathsf{e}) = \mathbb{I}_{\mathfrak{D}_\Phi}$.

ii) $\Rightarrow$ *i)* To prove that the $\mathfrak{C}$-valued sesquilinear form

$$\Phi(a, b) = \langle \Pi_\Phi(a)\xi_\Phi | \Pi_\Phi(b)\xi_\Phi\rangle_{\mathfrak{X}_\Phi}, \quad \forall a, b \in \mathfrak{A}$$

(where ξ_Φ is a cyclic vector for Π_Φ) is in $\mathcal{I}^{\mathfrak{C}}_{\mathfrak{A}_0}(\mathfrak{A})$ it suffices to prove that it is positive, invariant, and $\Lambda_\Phi(\mathfrak{A}_0)$ is dense in $\widetilde{\mathfrak{A}}$. By definition, it is positive. Since Π_Φ is a *-representation, we get that

$$\begin{aligned}\Phi(ac, d) &= \langle \Pi_\Phi(a)\Pi_\Phi(c)\xi_\Phi | \Pi_\Phi(d)\xi_\Phi\rangle_\Phi\\ &= \langle \Pi_\Phi(c)\xi_\Phi | \Pi_\Phi(a^*)\Pi_\Phi(d)\xi_\Phi\rangle_\Phi = \Phi(c, a^*d).\end{aligned}$$

By hypothesis, $\Pi_\Phi(\mathfrak{A}_0)\xi_\Phi$ is dense in $\mathfrak{X}_\Phi = \widetilde{\mathfrak{A}}$, hence for every $a \in \mathfrak{A}$ there exists $\{c_n\}_n \subset \mathfrak{A}_0$ such that

$$\|\Pi_\Phi(a)\xi_\Phi - \Pi_\Phi(c_n)\xi_\Phi\|_\Phi \to 0, \quad n \to \infty.$$

Hence,

$$\|\Lambda_\Phi(a) - \Lambda_\Phi(c_n)\|^2_\Phi = \Phi(a - c_n, a - c_n) = \|\Pi_\Phi(a)\xi_\Phi - \Pi_\Phi(c_n)\xi_\Phi\|^2_\Phi \to 0$$

as $n \to \infty$. This implies that $\Lambda_\Phi(\mathfrak{A}_0)$ is dense in $\widetilde{\mathfrak{A}}$ and concludes the proof. □

Definition 3.3. The triple $(\Pi_\Phi, \Lambda_\Phi, \mathfrak{X}_\Phi)$ constructed in Theorem 3.2 is called the GNS construction for Φ, and Π_Φ is called the GNS representation of $\mathfrak{A}$ corresponding to Φ.

Proposition 3.4. *Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be a quasi *-algebra with unit* e *and* $\Phi \in \mathcal{I}^{\mathfrak{C}}_{\mathfrak{A}_0}(\mathfrak{A})$. *Then the GNS construction* $(\Pi_\Phi, \Lambda_\Phi, \mathfrak{X}_\Phi)$ *is unique up to unitary equivalence.*

If we consider a normed quasi *-algebra $(\mathfrak{A}[\|\cdot\|], \mathfrak{A}_0)$, then the underlying *-algebra $\mathfrak{A}_0$ is dense (in this norm) in $\mathfrak{A}$; hence, we get automatically the density of $\Lambda_\Phi(\mathfrak{A}_0)$ in $\widetilde{\mathfrak{A}}$ when Φ is bounded.

Corollary 3.5. *Let* $(\mathfrak{A}[\|\cdot\|], \mathfrak{A}_0)$ *be a normed quasi *-algebra and* $\Phi \in \mathcal{Q}^{\mathfrak{C}}_{\mathfrak{A}_0}(\mathfrak{A})$ *be such that* Φ *is bounded with respect to* $\|\cdot\|$. *Then* $\Phi \in \mathcal{I}^{\mathfrak{C}}_{\mathfrak{A}_0}(\mathfrak{A})$.

Proof. If $\Phi \in \mathcal{Q}^{\mathfrak{C}}_{\mathfrak{A}_0}(\mathfrak{A})$ is bounded, then the subspace $\Lambda_\Phi(\mathfrak{A}_0)$ is dense in $\widetilde{\mathfrak{A}}$. Indeed, if $a \in \mathfrak{A}$, there exists a sequence $\{c_n\}_n \subset \mathfrak{A}_0$, such that $c_n \to a$ in $\mathfrak{A}$ as $n \to \infty$. Then, we have

$$\begin{aligned}\|\Lambda_\Phi(a) - \Lambda_\Phi(c_n)\|^2_\Phi &= \|\Phi(a - c_n, a - c_n)\|_{\mathfrak{C}} \\ &\leq \|\Phi\|^2 \|a - c_n\|^2 \to 0, \text{ as } n \to \infty.\end{aligned}$$

□

Corollary 3.6. *Let $\Phi \in \mathcal{Q}^{\mathfrak{C}}_{\mathfrak{A}_0}(\mathfrak{A})$ and $\mathfrak{A}$ be also a right module over $\mathfrak{C}$ and let Φ satisfy (2.5), then the quasi $B_{\mathfrak{C}}$-space $\mathfrak{X}_\Phi$ in Theorem 3.2 is also a Banach right module over $\mathfrak{C}$ and $\Pi_\Phi(a)$ is a $\mathfrak{C}$-linear operator for all $a \in \mathfrak{A}$.*

Proof. By Lemma 2.13, $\mathfrak{A}/N_\Phi[\|\cdot\|_\Phi]$ is a normed right C^*-module over $\mathfrak{C}$. The right multiplication by an element of $\mathfrak{C}$ can be extended by continuity to the completion $\widetilde{\mathfrak{A}}$ of $\mathfrak{A}/N_\Phi$; hence, $\widetilde{\mathfrak{A}}$ is a Banach right module over $\mathfrak{C}$. Further, $\Pi^\circ_\Phi(a)$ is a $\mathfrak{C}$-linear operator for every $a \in \mathfrak{A}$:

$$\begin{aligned}\Pi^\circ_\Phi(a)(\Lambda_\phi(cx)) &= \Lambda_\Phi(acx) = (\Lambda_\Phi(ac))x \\ &= [\Pi^\circ_\Phi(a)(\Lambda_\Phi(c)]x, \quad \forall a \in \mathfrak{A}, c \in \mathfrak{A}_0, x \in \mathfrak{C}.\end{aligned}$$

□

Definition 3.7. The $\mathfrak{C}$-valued positive sesquilinear map Φ on $\mathfrak{A} \times \mathfrak{A}$ is called *admissible* if, for every $a \in \mathfrak{A}$, there exists some $\gamma_a > 0$ such that

$$\|\Phi(ac, ac)\|_{\mathfrak{C}} \leq \gamma_a \|\Phi(c, c)\|_{\mathfrak{C}}, \quad \forall c \in \mathfrak{A}_0.$$

Remark 3.8. If $\Phi \in \mathcal{Q}^{\mathfrak{C}}_{\mathfrak{A}_0}(\mathfrak{A})$ is admissible, then the *-representation Π_Φ constructed from Φ is bounded. Indeed:

$$\|\Pi_\Phi(a)\Lambda_\Phi(c)\|^2_\Phi = \|\Phi(ac, ac)\|_{\mathfrak{C}} \leq \gamma_a \|\Phi(c, c)\|_{\mathfrak{C}} = \gamma_a \|\Lambda_\Phi(c)\|^2_\Phi,$$

for every $a \in \mathfrak{A}, c \in \mathfrak{A}_0$.

Corollary 3.9. *Let Φ be a $\mathfrak{C}$-linear form in $\mathcal{Q}^{\mathfrak{C}}_{\mathfrak{A}_0}(\mathfrak{A})$ and $\mathfrak{A}$ be also a right module over $\mathfrak{C}$. Then the quasi $B_{\mathfrak{C}}$-space $\mathfrak{X}_\Phi$ in Theorem 3.2 is also a right Hilbert module over $\mathfrak{C}$ and $\Pi_\Phi(a)$ is a $\mathfrak{C}$-linear operator for all $a \in \mathfrak{A}$.*

Proof. If Φ is $\mathfrak{C}$-linear, then the assertion follows from Lemma 2.17. □

As an application of what we have seen until now, if $\mathfrak{A}$ is a *–algebra with unit e, every $\mathfrak{C}$-valued positive linear map ω on $\mathfrak{A}$ (i.e., such that $\omega(a^*a) \in \mathfrak{C}^+$, for all $a \in \mathfrak{A}$) is representable.

Corollary 3.10. *Let $\mathfrak{A}$ be a *–algebra with unit e and let ω be a $\mathfrak{C}$-valued positive linear map on $\mathfrak{A}$. Then there exist a quasi $B_{\mathfrak{C}}$-space $\mathfrak{X}_\omega$ whose quasi-norm is induced by a $\mathfrak{C}$-valued quasi-inner product $\langle\cdot|\cdot\rangle_{\mathfrak{X}_\omega}$, a dense subspace $\mathfrak{D}_\omega \subseteq \mathfrak{X}_\omega$, and a closed cyclic *–representation Π_ω of $\mathfrak{A}$ with domain $\mathfrak{D}_\omega$, such that*

$$\omega(b^*ac) = \langle \Pi_\omega(a)\Lambda_\omega(c)|\Lambda_\omega(b)\rangle_{\mathfrak{X}_\omega}, \quad \forall a, b, c \in \mathfrak{A}$$

where $\Lambda_\omega(a)$ is the coset of the quotient $\mathfrak{A}/N_\omega$ and $N_\omega = \{a \in \mathfrak{A}|\,\omega(a^*a) = 0_{\mathfrak{C}}\}$. Moreover, there exists a cyclic vector η_ω, such that

$$\omega(a) = \langle \Pi_\omega(a)\eta_\omega|\eta_\omega\rangle_{\mathfrak{X}_\omega}, \quad \forall\ a \in \mathfrak{A}.$$

The representation is unique up to unitary equivalence.

Proof. The assertion can be proved by applying Theorem 3.2 and Proposition 3.4 to $\Phi : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{C}$ defined as $\Phi(a, b) = \omega(b^*a)$, for all $a, b \in \mathfrak{A}$, considering $\mathfrak{A}_0 = \mathfrak{A}$. Indeed, Φ is positive and invariant: $\Phi(a, a) = \omega(a^*a) \in \mathfrak{C}^+$ and $\Phi(ac, d) = \omega(d^*(ac)) = \omega((a^*d)^*c) = \Phi(c, a^*d)$ for all $a, c, d \in \mathfrak{A}$ and naturally $\Lambda_\Phi(\mathfrak{A}) = \mathfrak{A}/N_\Phi$ is dense in its completion. Moreover, for each $a \in \mathfrak{A}$, we have that

$$\omega(a) = \Phi(a, e) = \langle \Pi_\Phi(a)\xi_\Phi | \xi_\Phi \rangle_{\mathfrak{X}_\Phi},$$

where Π_Φ is the closed *-representation and ξ_Φ is the cyclic vector from Theorem 3.2. Now, we put $\Pi_\omega := \Pi_\Phi$, $\eta_\omega := \xi_\Phi$ and $\Lambda_\omega = \Lambda_\Phi$. □

Remark 3.11. If in addition $\mathfrak{A}$ is a $\mathfrak{C}$-bimodule and

$$\|\omega(x^*a^*ax)\|_{\mathfrak{C}} \le \|\omega(a^*a)\|_{\mathfrak{C}} \|x\|^2_{\mathfrak{C}}, \quad \forall a \in \mathfrak{A}, x \in \mathfrak{C}$$

then $\mathfrak{X}_\Phi$ is a right quasi-Banach module over $\mathfrak{C}$ and $\Pi_\omega(a)$ is $\mathfrak{C}$-linear for all $a \in \mathfrak{A}$.

Remark 3.12. If in addition $\mathfrak{A}$ is a $\mathfrak{C}$-bimodule and

$$\omega(ax) = \omega(a)x, \quad \forall a \in \mathfrak{A}, x \in \mathfrak{C}$$

then $\mathfrak{X}_\Phi$ is a right Hilbert $\mathfrak{C}$-module and $\Pi_\omega(a)$ is $\mathfrak{C}$-linear for all $a \in \mathfrak{A}$.

The following corollary gives a result of *-representability of $\mathfrak{C}$-valued bounded linear positive map on $(\mathfrak{A}, \mathfrak{A}_0)$.

Corollary 3.13. *Let $(\mathfrak{A}[\|\cdot\|], \mathfrak{A}_0)$ be a unital normed quasi *-algebra and ω be a $\mathfrak{C}$-valued bounded linear positive map on $(\mathfrak{A}, \mathfrak{A}_0)$ ($\omega(c^*c) \ge 0$, for every $c \in \mathfrak{A}_0$). If there exists $M > 0$ such that $\|\omega(d^*c)\|_{\mathfrak{C}} \le M\|c\|\|d\|$, for all $c, d \in \mathfrak{A}_0$, then there exists a quasi $B_{\mathfrak{C}}$-space $\mathfrak{X}_\Phi$ whose quasi-norm is induced by a $\mathfrak{C}$-valued quasi-inner product $\langle\cdot|\cdot\rangle_{\mathfrak{X}_\Phi}$, a dense subspace $\mathfrak{D}_\omega \subseteq \mathfrak{X}_\Phi$, and a closed cyclic *–representation Π_ω of $(\mathfrak{A}, \mathfrak{A}_0)$ with domain $\mathfrak{D}_\omega$ and cyclic vector η_ω, such that*

$$\omega(a) = \langle \Pi_\omega(a)\eta_\omega | \eta_\omega \rangle_{\mathfrak{X}_\Phi}, \quad \forall\ a \in \mathfrak{A},$$

and

$$\omega(b^*ac) = \langle \Pi_\omega(a)\Lambda_\omega(c) | \Lambda_\omega(b) \rangle_{\mathfrak{X}_\Phi}, \quad \forall a \in \mathfrak{A},\ \forall b, c \in \mathfrak{A}_0.$$

The representation is unique up to unitary equivalence.

Proof. Define $\Phi_0 : (b, c) \in \mathfrak{A}_0 \times \mathfrak{A}_0 \to \Phi_0(b, c) = \omega(c^*b) \in \mathfrak{C}$. Then Φ_0 is a $\mathfrak{C}$-valued bounded positive sesquilinear map on $\mathfrak{A}_0 \times \mathfrak{A}_0$ and

$$\Phi_0(bc, d) = \Phi_0(c, b^*d), \quad \forall b, c, d \in \mathfrak{A}_0.$$

Since $\mathfrak{A}_0$ is dense in $\mathfrak{A}$, it is easy to prove that Φ_0 can be extended by continuity, to a $\mathfrak{C}$-valued bounded positive sesquilinear map on $\mathfrak{A} \times \mathfrak{A}$. Hence, $\Lambda_\Phi(\mathfrak{A}_0)$ is dense in $\mathfrak{A}/N_\Phi$, since Φ is bounded and $\mathfrak{A}_0 \subset \mathfrak{A}$ densely. If $a \in \mathfrak{A}$ and $\{c_n\}_n \subset \mathfrak{A}_0$ with $c_n \to a$ as $n \to \infty$, then also $c_n^* \to a^*$ as $n \to \infty$; since $(\mathfrak{A}, \mathfrak{A}_0)$ is a normed quasi *-algebra, we have also:

$$c_n c \to ac, \text{ and } c_n^* b \to a^* b, \quad n \to \infty,\ b, c \in \mathfrak{A}_0.$$

Hence, since Φ is bounded, we get that Φ is invariant because

$$\Phi(ac, b) = \lim_{n\to\infty} \Phi_0(c_n c, b) = \lim_{n\to\infty} \Phi_0(c, c_n^* b) = \Phi(c, a^* b).$$

Therefore, by Theorem 3.2, Φ is *-representable and

$$\Phi(ac, b) = \Phi(c, a^* b) = \langle \Pi_\Phi(a)\Lambda_\Phi(c) | \Lambda_\Phi(b)\rangle_{\mathfrak{X}_\Phi}, \quad \forall a \in \mathfrak{A}, b, c \in \mathfrak{A}_0.$$

Finally, if $a \in \mathfrak{A}$ and $\{c_n\}_n \subset \mathfrak{A}_0$ with $c_n \to a$ as $n \to \infty$ then

$$\omega(a) = \lim_{n\to\infty} \omega(c_n) = \lim_{n\to\infty} \Phi_0(c_n, \mathsf{e}) = \Phi(a, \mathsf{e}) = \langle \Pi_\Phi(a)\xi_\Phi | \xi_\Phi\rangle_{\mathfrak{X}_\Phi}.$$

Further, if $a \in \mathfrak{A}$ and $\{c_n\}_n \subseteq \mathfrak{A}_0$ is such that $c_n \to a$ in $\mathfrak{A}$ as $n \to \infty$, then if $b, c \in \mathfrak{A}_0$, $c_n c \to ac$ as $n \to \infty$ and, hence, $b^* c_n c \to b^* ac$ as $n \to \infty$; therefore,

$$\begin{aligned}\omega(b^* ac) &= \lim_{n\to\infty} \omega(b^* c_n c) = \lim_{n\to\infty} \Phi_0(c_n c, b)\\ &= \Phi(ac, b) = \langle \Pi_\Phi(a)\Lambda_\Phi(c) | \Lambda_\Phi(b)\rangle_{\mathfrak{X}_\Phi}.\end{aligned}$$

Now, denote $\Pi_\omega := \Pi_\Phi$, $\Lambda_\omega := \Lambda_\Phi$ and $\eta_\omega := \xi_\Phi$. As for the uniqueness of the *-representation, it follows from Proposition 3.4. □

Remark 3.14. Assume that ω satisfies the assumptions of Corollary 3.13. Then if $a \in \mathfrak{A}$, for every sequence $\{c_n\}_n \subseteq \mathfrak{A}_0$ with $c_n \to a$ as $n \to \infty$, we have that

$$4\|\omega(\mathsf{e})\|_{\mathfrak{C}} \lim_{n\to\infty} \|\omega(c_n^* c_n)\|_{\mathfrak{C}} \geq \|\omega(a)\|_{\mathfrak{C}}^2.$$

Indeed, since $\|\omega(d^* c)\|_{\mathfrak{C}} \leq M\|c\|\|d\|$ for all $c, d \in \mathfrak{A}_0$, then it is not hard to see that $\{\omega(c_n^* c_n)\}$ is a Cauchy sequence in $\mathfrak{C}$, hence the limit

$$\lim_{n\to\infty} \|\omega(c_n^* c_n)\|_{\mathfrak{C}}$$

does exist. Moreover, by the boundedness of ω, we have that

$$\lim_{n\to\infty} \omega(c_n) = \omega(a).$$

By Corollary 2.4 we obtain that

$$4\|\omega(\mathsf{e})\|_{\mathfrak{C}} \|\omega(c_n^* c_n)\|_{\mathfrak{C}} \geq \|\omega(c_n)\|_{\mathfrak{C}}^2, \quad \forall n \in \mathbb{N}.$$

By taking the limits on both sides of the previous inequality, we get the desired one.

In the following example, we show that there really exist nontrivial C^*-valued positive maps on unital normed quasi *-algebra that satisfy the conditions of Corollary 3.13.

Example 3.15. Let W be a positive operator in $L^\infty(\rho)$ and $f_t(W)$ be as in Example 2.24. Then

$$|\rho(X f_t(W))| \leq \|X\|_p \|f_t(W)\|_{\frac{p}{p-1}} \leq \|X\|_p (\rho(\mathbb{I}))^{\frac{p-1}{p}} \|f_t(W)\|_\infty,$$

for all $X \in L^p(\rho)$; the first inequality is due to the Hölder inequality, the second one follows from the functional calculus applied to the positive operator $f_t(W)$ which gives that $\|f_t(W)\|_\infty^{\frac{p}{p-1}} \mathbb{I} \geq (f_t(W))^{\frac{p}{p-1}}$ since $\|f_t(W)\|_\infty^{\frac{p}{p-1}} 1 \geq (id)^{\frac{p}{p-1}}$ where 1 is the constant function 1 on $[0, \|f_t(W)\|_\infty]$ and $id(t) = t$

for all $t \in [0, \|f_t(W)\|_\infty]$. Therefore, $\rho((f_t(W))^{\frac{p}{p-1}}) \leq \|f_t(W)\|_\infty^{\frac{p}{p-1}} \rho(\mathbb{I})$ so $\|f_t(W)\|_{\frac{p}{p-1}} \leq (\rho(\mathbb{I}))^{\frac{p-1}{p}} \|f_t(W)\|_\infty$. It is not hard to see that the map

$$\omega : A \in L^p(\rho) \to \omega(A) := \rho(Af_t(W)) \in C([0, \|W\|])$$

is a well-defined ($\|(f_{t_1} - f_{t_2})(W)\|_\infty = |t_1 - t_2|$) bounded linear positive map on $(L^p(\rho), L^\infty(\rho))$ with values on the C^*-algebra $C([0, \|W\|])$ and

$$\begin{aligned} \|\omega(X^*Y)\|_{\mathfrak{C}} &\leq \|X\|_p \|Y\|_p \sup_{t \in [0, \|W\|]} \|f_t(W)\|_{\frac{p}{p-2}} \\ &= \|X\|_p \|Y\|_p \sup_{t \in [0, \|W\|]} \|f_t(W)\|_\infty (\rho(\mathbb{I}))^{\frac{p-1}{p}} \\ &= \|X\|_p \|Y\|_p \|W\| (\rho(\mathbb{I}))^{\frac{p-1}{p}}, \quad \forall X, Y \in L^\infty(\rho). \end{aligned}$$

Similarly, given $A_t \in L^2([0, \|W\|], \mathcal{B}(\mathcal{H}))$ such that $A_t \geq 0$ for a.e. $t \in [0, \|W\|]$, we can consider the map $\Omega : L^p(\rho) \to \mathfrak{B}(\mathcal{H})$ given by

$$\Omega(X) = \int_0^{\|W\|} \rho(Xf_t(W)) A_t dt.$$

Here we consider Gel'fand-Pettis integral and $A_t \in L^2([0, \|W\|], \mathfrak{B}(\mathcal{H}))$. Since, for each $h \in \mathcal{H}$ with $\|h\| \leq 1$

$$\begin{aligned} &\int_0^{\|W\|} |\rho(Xf_t(W))|^2 \, \|A_t h\|^2 dt \\ &\quad \leq \|X\|_p \rho(\mathbb{I}) \|W\|_\infty \int_0^{\|W\|} \|A_t h\|^2 dt \\ &\quad \leq \|X\|_p \rho(\mathbb{I}) \|W\|_\infty \|\|A_t\|\|_2^2, \end{aligned}$$

it follows that $\rho(Xf_t(W))A_t \in L^2([0, \|W\|], \mathfrak{B}(\mathcal{H}))$, for all $X \in L^p(\rho)$. Moreover, for all $h_1, h_2 \in \mathcal{H}$ with $\|h_i\| \leq 1$, $i \in \{1, 2\}$, we obtain that

$$\begin{aligned} &\left| \left\langle \int_0^{\|W\|} \rho(X^*Yf_t(W)) A_t dt \, h_1 \mid h_2 \right\rangle \right| \\ &= \left| \int_0^{\|W\|} \langle \rho(X^*Yf_t(W)) A_t \, h_1 | h_2 \rangle dt \right| \\ &\leq \int_0^{\|W\|} |\langle \rho(X^*Yf_t(W)) A_t \, h_1 | h_2 \rangle| \, dt \\ &\leq \int_0^{\|W\|} \|\rho(X^*Yf_t(W)) A_t \, h_1\| \, \|h_2\| dt \\ &\leq \left(\int_0^{\|W\|} \|\rho(X^*Yf_t(W)) A_t \, h_1\|^2 dt \right)^{1/2} \|W\|^{1/2} \\ &\leq \|X\|_p \|Y\|_p \|W\| \rho(\mathbb{I}) \left(\int_0^{\|W\|} \|A_t h_1\|^2 dt \right)^{1/2} \|W\|^{1/2} \\ &\leq \|X\|_p \|Y\|_p \|W\|^{3/2} \rho(\mathbb{I}) \|\|A_t\|\|_2, \end{aligned}$$

hence,

$$\left\| \int_0^{\|W\|} \rho(X f_t(W)) A_t \, dt \right\| \leq \|X\|_p \|Y\|_p \|W\|^{3/2} \rho(\mathbb{I}) \| A_t \|_2,$$

for all $X, Y \in L^p(\rho)$. It follows that Ω satisfies the conditions of Corollary 3.13.

Let $\Phi \in \mathcal{I}_{\mathfrak{A}_0}^{\mathfrak{C}}(\mathfrak{A})$ and ϑ a state on $\mathfrak{C}$. Then $\phi := \vartheta \circ \Phi$, (i.e., $\phi(a, b) = \vartheta(\Phi(a, b))$, for every $a, b \in \mathfrak{A}$) is an invariant positive sesquilinear *form* on $\mathfrak{A} \times \mathfrak{A}$. We have

$$N_\phi = \{a \in \mathfrak{A} : \phi(a, a) = 0\} = \{a \in \mathfrak{A}; \phi(a, a) \in N_\vartheta\}.$$

Moreover we have

$$|\phi(a, a)| = |\vartheta(\Phi(a, a))| \leq \|\Phi(a, a)\|_{\mathfrak{C}}.$$

This implies that the map

$$T_{\phi,\Phi} : \Lambda_\Phi(a) \in \mathfrak{X}_\Phi \to \lambda_\phi(a) \in \mathcal{H}_\phi$$

is well-defined and bounded, where $\mathcal{H}_\phi$ is the Hilbert space completion of $\mathfrak{A}/N_\phi$ with respect to $\|\cdot\|_\phi$, with $\lambda_\phi(a)$ the coset containing a, $\|\lambda_\phi(a)\|_\phi = \phi(a, a)^{1/2}$. From this, we deduce that $\lambda_\phi(\mathfrak{A}_0)$ is dense in $\mathcal{H}_\phi$. Thus, a GNS *-representation constructed from the invariant positive sesquilinear (ips-)form ϕ is possible (see [3, Proposition 2.4.1]). Let us denote it by π_ϕ. Then we have for $a \in \mathfrak{A}$ and $b \in \mathfrak{A}_0$,

$$\pi_\phi(a)\lambda_\phi(b) = \lambda_\phi(ab) = T_{\phi,\Phi}\Lambda_\Phi(ab) = T_{\phi,\Phi}\Pi_\Phi(a)\Lambda_\Phi(b).$$

On the other hand,

$$\pi_\phi(a)\lambda_\phi(b) = \pi_\phi(a) T_{\phi,\Phi}\Lambda_\Phi(b).$$

Therefore,

$$T_{\phi,\Phi}\Pi_\Phi(a) = \pi_\phi(a) T_{\phi,\Phi}, \quad \forall a \in \mathfrak{A}.$$

Hence π_ϕ and Π_Φ are intertwined with bounded intertwining operator $T_{\phi,\Phi}$, (see [2, Definition 1.3.1]).

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