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Invariants for Second Type Almost Geodesic Mappings of Symmetric Affine Connection Space

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Abstract: This paper presents the results concerning a space of invariants for second type almost geodesic mappings. After discussing the general formulas of invariants for mappings of symmetric affine connection spaces, based on these formulas, invariants for second type almost geodesic mappings of symmetric affine connection spaces and Riemannian spaces are obtained, as well as their mutual connection. Also, one invariant of Thomas type and two invariants of Weyl type for almost geodesic mappings of the second type were attained.

Keywords: affine connection space; Riemannian space; almost geodesic mappings; invariants

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1. Introduction

In this research, invariants for almost geodesic mappings of the second type of a symmetric affine connection space are obtained. Invariants for second type almost geodesic mappings of a Riemannian space are obtained as a special case.

This research is based on symmetric affine connection spaces and Riemannian spaces in the sense of Eisenhart's definitions [1,2].

Transformations of affine connections of different symmetric affine connection and Riemannian spaces are studied in [3–9] and in many other scientific papers and monographs. The authors of [3–5,9–11] have considered a special transformation of a torsion-free affine connection space known as the second type almost geodesic mapping.

1.1. Symmetric Affine Connection Space in Eisenhart's Sense

An N-dimensional manifold $\mathcal{M}_N$ equipped with torsion-free affine connection $\overset{0}{\nabla}$, whose coefficients are $L^{\alpha}_{\underline{\beta\gamma}}$, $L^{\alpha}_{\underline{\beta\gamma}} = L^{\alpha}_{\underline{\gamma\beta}}$, is the symmetric affine connection space $\mathbb{A}_N$ (see [2–5]).

There are many authors that deal with symmetric affine connection spaces, as well as studying torsion-free affine connection spaces [2–5].

The covariant derivative of a tensor a^{α}_{β} with respect to the affine connection $\overset{0}{\nabla}$ in the direction of x^{γ} is defined as [3–5]

$$a^{\alpha}_{\beta|\gamma} = a^{\alpha}_{\beta,\gamma} + L^{\alpha}_{\underline{\delta\gamma}} a^{\delta}_{\beta} - L^{\epsilon}_{\underline{\beta\gamma}} a^{\alpha}_{\epsilon}, \tag{1}$$

for partial derivative $\partial/\partial x^{\gamma}$ denoted by comma.

From the alternation $a^{\alpha}_{\beta|\gamma|\delta} - a^{\alpha}_{\beta|\delta|\gamma}$, one Ricci identity is obtained (for details, see [3,5]). In this way, one curvature tensor of space $\mathbb{A}_N$ is defined

$$\overset{0}{R}{}^{\alpha}_{\beta\gamma\delta} = L^{\alpha}_{\underline{\beta\gamma},\delta} - L^{\alpha}_{\underline{\beta\delta},\gamma} + L^{\epsilon}_{\underline{\beta\gamma}} L^{\alpha}_{\underline{\epsilon\delta}} - L^{\epsilon}_{\underline{\beta\delta}} L^{\alpha}_{\underline{\epsilon\gamma}}. \tag{2}$$

The corresponding Ricci tensor $\overset{0}{R}_{\alpha\beta} = \overset{0}{R}{}^{\epsilon}_{\alpha\beta\epsilon}$ is

$$\overset{0}{R}_{\alpha\beta} = L^{\epsilon}_{\underline{\alpha\beta},\epsilon} - L^{\epsilon}_{\underline{\alpha\epsilon},\beta} + L^{\epsilon}_{\underline{\alpha\beta}} L^{\zeta}_{\underline{\epsilon\zeta}} - L^{\epsilon}_{\underline{\alpha\zeta}} L^{\zeta}_{\underline{\beta\epsilon}}. \tag{3}$$

The alternation of the Ricci tensor is

$$\overset{0}{R}_{[\alpha\beta]} = -L^{\epsilon}_{\underline{\alpha\epsilon},\beta} + L^{\epsilon}_{\underline{\beta\epsilon},\alpha}. \tag{4}$$

After involving the abbreviation $L^{\alpha}_{\underline{\beta\gamma}|\delta} = L^{\alpha}_{\underline{\beta\gamma},\delta} + L^{\alpha}_{\underline{\epsilon\delta}} L^{\epsilon}_{\underline{\beta\gamma}} - L^{\epsilon}_{\underline{\beta\delta}} L^{\alpha}_{\underline{\epsilon\gamma}} - L^{\epsilon}_{\underline{\gamma\delta}} L^{\alpha}_{\underline{\beta\epsilon}}$, we obtain

$$\overset{0}{R}_{[\alpha\beta]} = -L^{\epsilon}_{[\underline{\alpha\epsilon}|\beta]}. \tag{5}$$

1.2. Riemannian Space in Eisenhart's Sense

A special kind of symmetric affine connection spaces are referred to as Riemannian spaces. An N-dimensional manifold $\mathcal{M}_N$ equipped with symmetric metric tensor $\hat{g}$, whose components are $g_{\underline{\alpha\beta}}$, $g_{\underline{\alpha\beta}} = g_{\underline{\beta\alpha}}$, $\det\left[g_{\underline{\alpha\beta}}\right] \neq 0$, is the Riemannian space $\mathbb{R}_N$ (see [1,3,5]). Because of the regularity of matrix $\left[g_{\underline{\alpha\beta}}\right]$, the metric tensor with upper indices is defined as $\left[g^{\underline{\alpha\beta}}\right] = \left[g_{\underline{\alpha\beta}}\right]^{-1}$.

The affine connection coefficients of space $\mathbb{R}_N$ are the second kind of Christoffell symbols

$$\Gamma^{\alpha}_{\underline{\beta\gamma}} = \frac{1}{2} g^{\underline{\alpha\delta}} \left(g_{\underline{\beta\delta},\gamma} - g_{\underline{\beta\gamma},\delta} + g_{\underline{\delta\gamma},\beta}\right). \tag{6}$$

The second kind of Christoffell symbols, $\Gamma^{\alpha}_{\underline{\beta\gamma}}$, are symmetric by β and γ uniquely generate the torsion-free affine connection $\overset{0}{\nabla}{}^{g}$. With respect to this affine connection, one kind of covariant derivative of the tensor a^{α}_{β} in the direction of x^{γ} is [3,5]

$$a^{\alpha}_{\beta|^{g}\gamma} = a^{\alpha}_{\beta,\gamma} + \Gamma^{\alpha}_{\underline{\delta\gamma}} a^{\delta}_{\beta} - \Gamma^{\delta}_{\underline{\beta\gamma}} a^{\alpha}_{\delta}. \tag{7}$$

The curvature tensor and the Ricci tensor of space $\mathbb{R}_N$ are

$$\overset{0}{R}{}^{g}{}^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\underline{\beta\gamma},\delta} - \Gamma^{\alpha}_{\underline{\beta\delta},\gamma} + \Gamma^{\epsilon}_{\underline{\beta\gamma}} \Gamma^{\alpha}_{\underline{\epsilon\delta}} - \Gamma^{\epsilon}_{\underline{\beta\delta}} \Gamma^{\alpha}_{\underline{\epsilon\gamma}}, \tag{8}$$

$$\overset{0}{R}{}^{g}{}_{\alpha\beta} = \Gamma^{\epsilon}_{\underline{\alpha\beta},\epsilon} - \Gamma^{\epsilon}_{\underline{\alpha\epsilon},\beta} + \Gamma^{\epsilon}_{\underline{\alpha\beta}} \Gamma^{\zeta}_{\underline{\epsilon\zeta}} - \Gamma^{\epsilon}_{\underline{\alpha\zeta}} \Gamma^{\zeta}_{\underline{\beta\epsilon}}. \tag{9}$$

The Ricci tensor $\overset{0}{R}{}^{g}{}_{\alpha\beta}$ is symmetric by α and β, i.e., it holds the equation

$$\overset{0}{R}{}^{g}{}_{[\alpha\beta]} = 0. \tag{10}$$

1.3. Almost Geodesic Mappings

In an attempt to generalize the concept of geodesics, N. S. Sinyukov defined the almost geodesic curve of a space $\mathbb{A}_N$ as a curve $\ell = \ell(t)$ which satisfies the next system of partial differential equations [3–5,9–13]

$$\lambda^{\alpha}_{(2)} = a(t)\lambda^{\alpha} + b(t)\lambda^{\alpha}_{(1)}, \quad \lambda^{\alpha}_{(1)} = \lambda^{\alpha}_{|\beta}\lambda^{\beta}, \quad \lambda^{\alpha}_{(2)} = \lambda^{\alpha}_{(1)|\beta}\lambda^{\beta}, \tag{11}$$

where $\lambda = d\ell/dt$ is tangential vector to ℓ, and $a(t)$ and $b(t)$ are functions of t.

A curve $\ell = \ell(t)$ is an almost geodesic line of Riemannian space $\mathbb{R}_N$ if the following system of partial differential equations is satisfied

$$\lambda^{\alpha}_{(2)} = a(t)\lambda^{\alpha} + b(t)\lambda^{\alpha}_{(1)}, \quad \lambda^{\alpha}_{(1)} = \lambda^{\alpha}_{|^{g}\beta}\lambda^{\beta}, \quad \lambda^{\alpha}_{(2)} = \lambda^{\alpha}_{(1)|^{g}\beta}\lambda^{\beta}. \tag{12}$$

A mapping $f : \mathbb{A}_N \to \bar{\mathbb{A}}_N$ which any geodesic line of the space $\mathbb{A}_N$ transforms to an almost geodesic line of the space $\bar{\mathbb{A}}_N$ is the almost geodesic mapping of $\mathbb{A}_N$.

A mapping $f : \mathbb{R}_N \to \bar{\mathbb{R}}_N$ which any geodesic line of the space $\mathbb{R}_N$ transforms to an almost geodesic line of the space $\bar{\mathbb{R}}_N$ is the almost geodesic mapping of $\mathbb{R}_N$.

It is proved [5,9–11,13] that a mapping $f : \mathbb{A}_N \to \bar{\mathbb{A}}_N$ is almost geodesic if and only if in the common coordinate system $x^1, \dots, x^N$, the deformation tensor $P^{\alpha}_{\underline{\beta\gamma}} = \bar{L}^{\alpha}_{\underline{\beta\gamma}} - L^{\alpha}_{\underline{\beta\gamma}}$ satisfies identically with respect to $x^1, \dots, x^N$ and $\lambda^1, \dots, \lambda^N$ the conditions

$$\big(P^{\alpha}_{\underline{\beta\gamma}|\delta} + P^{\alpha}_{\underline{\epsilon\beta}}P^{\epsilon}_{\underline{\gamma\delta}}\big)\lambda^{\beta}\lambda^{\gamma}\lambda^{\delta} = bP^{\alpha}_{\underline{\beta\gamma}}\lambda^{\beta}\lambda^{\gamma} + a\lambda^{\alpha}. \tag{13}$$

In this equation, $\lambda^1, \dots, \lambda^N$ are components of some vector, and a and b are invariants depending on $x^1, \dots, x^N$ and $\lambda^1, \dots, \lambda^N$.

The expressions of invariant b as

$$b = b_{\alpha}\lambda^{\alpha}, \quad b = \frac{b_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}}{\sigma_{\gamma}\lambda^{\gamma}}, \quad b = \frac{b_{\alpha\beta\gamma}\lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma}}{\sigma_{\gamma\delta}\lambda^{\gamma}\lambda^{\delta}}, \tag{14}$$

correspond to three types of almost geodesic mappings of space $\mathbb{A}_N$. These types are π_1, π_2, and π_3.

After reducing the Equation (13) to the case of affine connections $\overset{0}{\nabla}{}^{g}$ and $\overset{0}{\bar{\nabla}}{}^{g}$ of Riemannian spaces $\mathbb{R}_N$ and $\bar{\mathbb{R}}_N$, the necessary and sufficient condition for a mapping $f : \mathbb{R}_N \to \bar{\mathbb{R}}_N$ to be almost geodesic is

$$\big(P^{g}{}^{\alpha}_{\underline{\beta\gamma}|^{g}\delta} + P^{g}{}^{\alpha}_{\underline{\epsilon\beta}}P^{g}{}^{\epsilon}_{\underline{\gamma\delta}}\big)\lambda^{\beta}\lambda^{\gamma}\lambda^{\delta} = bP^{g}{}^{\alpha}_{\underline{\beta\gamma}}\lambda^{\beta}\lambda^{\gamma} + a\lambda^{\alpha}. \tag{15}$$

for the deformation tensor $P^{g}{}^{\alpha}_{\underline{\beta\gamma}} = \bar{\Gamma}^{\alpha}_{\underline{\beta\gamma}} - \Gamma^{\alpha}_{\underline{\beta\gamma}}$. As in the case of almost geodesic mappings of symmetric affine connection spaces, there are three types of almost geodesic mappings of Riemannian spaces as well. These three types are determined with the expressions (14) of invariant b.

A mapping $f : \mathbb{A}_N \to \bar{\mathbb{A}}_N$ determined with the following system of partial differential equations

$$\begin{cases} \bar{L}^{\alpha}_{\underline{\beta\gamma}} = L^{\alpha}_{\underline{\beta\gamma}} + \psi_{\gamma}\delta^{\alpha}_{\beta} + \psi_{\beta}\delta^{\alpha}_{\gamma} + 2\sigma_{\gamma}F^{\alpha}_{\beta} + 2\sigma_{\beta}F^{\alpha}_{\gamma}, \\ F^{\alpha}_{\beta|\gamma} + F^{\alpha}_{\gamma|\beta} + 2\sigma_{\beta}F^{\delta}_{\gamma}F^{\alpha}_{\delta} + 2\sigma_{\gamma}F^{\delta}_{\beta}F^{\alpha}_{\delta} = \nu_{\gamma}\delta^{\alpha}_{\beta} + \nu_{\beta}\delta^{\alpha}_{\gamma} + \mu_{\gamma}F^{\alpha}_{\beta} + \mu_{\beta}F^{\alpha}_{\gamma}, \end{cases} \tag{16}$$

for 1-forms $\psi_{\alpha}, \sigma_{\alpha}, \mu_{\alpha}, \nu_{\alpha}$, and an affinor F^{α}_{β}, is the second type almost geodesic mapping [3,5,12]. The class of second-type almost geodesic mappings is marked as π_2.

The mapping f has the property of reciprocity if the affinor F^{α}_{β} is an invariant for this mapping and the inverse mapping f^{-1} is an almost geodesic mapping of the second type. The basic equations for the second type almost geodesic mapping f, which has the property of reciprocity, are Equation (16) together with the condition

$$F^{\alpha}_{\gamma}F^{\gamma}_{\beta} = e\delta^{\alpha}_{\beta}, \quad e = \pm 1. \tag{17}$$

Almost geodesic mappings of symmetric affine connection space of the second type are elements of class π_2. The subclass of almost geodesic mappings of the second type, which have the property of reciprocity, is $\pi_2(e)$.

1.4. Invariants for Geometric Mappings

Important objects in mathematics are those ones which do not change after transformations. In differential geometry, several such objects have been determined [3,5,14–18].

After expressing the difference $\bar{L}^{\alpha}_{\underline{\beta\gamma}} - L^{\alpha}_{\underline{\beta\gamma}}$ as

$$\bar{L}^{\alpha}_{\underline{\beta\gamma}} - L^{\alpha}_{\underline{\beta\gamma}} = \bar{\omega}^{\alpha}_{\beta\gamma} - \omega^{\alpha}_{\beta\gamma}, \tag{18}$$

it was obtained $\big($see [17]$\big)$ that the geometrical objects

$$\overset{0}{\mathcal{T}}{}^{\alpha}_{\underline{\beta\gamma}} = L^{\alpha}_{\underline{\beta\gamma}} - \omega^{\alpha}_{\beta\gamma}, \tag{19}$$

$$\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta} = \overset{0}{R}{}^{\alpha}_{\beta\gamma\delta} - \omega^{\alpha}_{\beta\gamma|\delta} + \omega^{\alpha}_{\beta\delta|\gamma} + \omega^{\epsilon}_{\beta\gamma}\omega^{\alpha}_{\epsilon\delta} - \omega^{\epsilon}_{\beta\delta}\omega^{\alpha}_{\epsilon\gamma}, \tag{20}$$

are invariants for the mapping $f : \mathbb{A}_N \to \bar{\mathbb{A}}_N$ whose deformation tensor is given by (18). These invariants are the basic invariants of Thomas and Weyl type for mapping f, respectively.

In [19,20], two kinds of invariance of geometrical objects under mappings of non-symmetric affine connection spaces are defined. Non-symmetricity is not of great importance in the next definition.

Definition 1. *Let f be a mapping between two affine connection spaces, both symmetric or non-symmetric affine connected ones, and let $U^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$ be a geometrical object of type (p,q), $p,q \in \mathbb{N}_0$.*

1. *If the transformation f preserves value of the object $U^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$ but changes its form to $\bar{V}^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$, then the invariance for geometrical object $U^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$ under transformation f is* valued.
2. *If the transformation f preserves both the value and the form of geometrical object $U^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$, then the invariance for geometrical object under the transformation f is* total.

1.5. Motivation

Invariants for different geometric mappings of symmetric affine connection spaces and Riemannian spaces have been obtained [21,22].

Scalar curvature π of a $2n$-dimensional Riemannian manifold is defined in [23]. This scalar curvature and the corresponding scalar curvature obtained with respect to the corresponding complex metric are correlated in [23].

In cosmology [24,25], research on this topic starts with the Friedman–Lemaitre–Robertson–Walker (RLRW) metric

$$ds^2 = -a^2 d\eta^2 + a^2\big(dx^{1^2} + dx^{2^2} + dx^{3^2}\big), \tag{21}$$

where η is conformal time, x^1, x^2, x^3 are spatial coordinates, and $a = a(\eta)$ is the scale factor.

The perturbed FLRW metric is [24]

$$\begin{aligned} ds^2 = &-(1+2A)a^2 d\eta^2 + 2(\partial_i B)a^2 dx^i d\eta \\ &+ \left[\left(1 - 2\big(D + \frac{1}{3}\delta^{kl}(\partial_k\partial_l E)\big)\right)\delta_{ij} + 2(\partial_i\partial_j E)\right]a^2 dx^i dx^j, \end{aligned} \tag{22}$$

for $i,j,k,l = 1,2,3$, and scalar functions A, B, D, E.

The perturbation of metric (21)→(22), $\hat{\hat{g}} \to \hat{g}$, induces the perturbation of Christoffell symbols and components of curvature tensor induced by the FLRW metric. If $g = \det\,[g_{\underline{\mu\nu}}]$,

$\mu,\nu = 0,1,2,3$, is the determinant of the perturbed metric then the corresponding Einstein–Hilbert action is

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\big(R^g + \mathcal{L}_M\big), \tag{23}$$

for the Einstein gravitational constant $\kappa = 2.08\times 10^{-43}N^{-1}$, scalar curvature R^g obtained from the perturbed metric, and the term $\mathcal{L}_M$ describing any matter fields from the theory. The action S should be invariant under the variation, i.e., it would be [24,25]

$$0 = \delta S = \int d^4\sqrt{-g}\left[\frac{1}{2\kappa}\left(\frac{\delta R^g}{\delta g^{\underline{\mu\nu}}} + \frac{R^g}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\underline{\mu\nu}}}\right) + \frac{1}{\sqrt{-g}}\frac{\delta\big(\sqrt{-g}\mathcal{L}_M\big)}{\delta g^{\underline{\mu\nu}}}\right]\delta g^{\underline{\mu\nu}},$$

for variational derivative $\delta/\delta g^{\underline{\mu\nu}}$. The last relation is equivalent to the equations of motion

$$R^g{}_{\mu\nu} - \frac{1}{2}R^g g_{\underline{\mu\nu}} = \kappa T_{\mu\nu}, \tag{24}$$

for the energy–momentum tensor $T_{\mu\nu}$.

The Weyl conformal curvature tensor with respect to the perturbed metric is [3,5,21]

$$C^{\pi}_{\mu\nu\sigma} = R^{g\,\pi}{}_{\mu\nu\sigma} + \frac{1}{2}\big(R^{g\,\pi}{}_{\sigma}g_{\underline{\mu\nu}} - R^{g\,\pi}{}_{\nu}g_{\underline{\mu\sigma}} + R^g{}_{\mu\nu}\delta^{\pi}_{\sigma} - R^g{}_{\mu\sigma}\delta^{\pi}_{\nu}\big) + \frac{1}{6}R^g\big(\delta^{\pi}_{\nu}g_{\underline{\mu\sigma}} - \delta^{\pi}_{\sigma}g_{\underline{\mu\nu}}\big). \tag{25}$$

The traces $C^{\alpha}_{\mu\nu\alpha}, C^{\alpha}_{\mu\alpha\nu}, C^{\alpha}_{\alpha\mu\nu}$ of the Weyl conformal tensor vanish. That means that it is not possible to contract the geometrical object $C^{\pi}_{\mu\nu\sigma}$ by π and some of the covariant indices, μ, ν, σ, to obtain a non-trivial invariant of the form $R^g{}_{\mu\nu} + \mathcal{D}_{\mu\nu}$, where $\mathcal{D}_{\mu\nu}$ is a tensor of the type $(0,2)$.

Motivated by the trace-free Weyl conformal tensor, R. Bach proposed a quadratic action [26]

$$S_2 = \int d^4x C_{\pi\mu\nu\sigma}C^{\pi\mu\nu\sigma}\sqrt{-g}, \tag{26}$$

which is invariant under the conformal group (the group of transformations from the space to itself that preserve angles). From the last action, the modified equations of motion are obtained.

With respect to the transformations in cosmology, and the methodology for obtaining the Einstein tensor $R^g{}_{\mu\nu} - \frac{1}{2}R^g g_{\underline{\mu\nu}}$, we are motivated to obtain invariants from the transformation of the curvature tensor under second type almost geodesic mappings. Unlike in the case of the Weyl conformal tensor, the trace of one of these invariants will not be identically equal to zero. For this reason, in future work, our results will be applicable for research in cosmology analogously, as in (26), but for linear cosmological models.

In the next part, the main aims of the paper are presented.

1. To review results about invariants for mappings of symmetric affine connection spaces obtained in [18].
2. To express the deformation tensor $P^{\alpha}_{\beta\gamma}$ of second type almost geodesic mapping $f : \mathbb{A}_N \to \bar{\mathbb{A}}_N$ in the form

$$P^{\alpha}_{\beta\gamma} = \psi_\gamma\delta^{\alpha}_{\beta} + \psi_\beta\delta^{\alpha}_{\gamma} + \bar{\rho}^{\alpha}_{\underline{\beta\gamma}} - \rho^{\alpha}_{\underline{\beta\gamma}},$$

   for tensors $\bar{\rho}^{\alpha}_{\underline{\beta\gamma}}$ and $\rho^{\alpha}_{\underline{\beta\gamma}}$ symmetric by β and γ and obtain the corresponding basic invariants (19 and 20) for almost geodesic mappings of second type of space $\mathbb{A}_N$.
3. To obtain the corresponding invariants for second type almost geodesic mappings of Riemannian space $\mathbb{R}_N$.

2. Review of Basic and Derived Invariants

Let us consider a mapping $f : \mathbb{A}_N \to \bar{\mathbb{A}}_N$ whose deformation tensor $P^{\alpha}_{\underline{\beta\gamma}}$ is

$$P^{\alpha}_{\underline{\beta\gamma}} = \psi_{\gamma}\delta^{\alpha}_{\beta} + \psi_{\beta}\delta^{\alpha}_{\gamma} + \bar{\rho}^{\alpha}_{\underline{\beta\gamma}} - \rho^{\alpha}_{\underline{\beta\gamma}}, \tag{27}$$

for tensors $\rho^{\alpha}_{\underline{\beta\gamma}}$ and $\bar{\rho}^{\alpha}_{\underline{\beta\gamma}}$ symmetric by covariant indices. The forthcoming theorem is going to be proved.

Theorem 1. *Let $f : \mathbb{A}_N \to \bar{\mathbb{A}}_N$ be a mapping of symmetric affine connection space $\mathbb{A}_N$, whose deformation tensor is given by (18).*

The geometrical objects

$$\overset{0}{\mathcal{T}}{}^{\alpha}_{\underline{\beta\gamma}} = L^{\alpha}_{\underline{\beta\gamma}} - \rho^{\alpha}_{\underline{\beta\gamma}} - \frac{1}{N+1}\Big(\big(L^{\delta}_{\underline{\gamma\delta}} - \rho^{\delta}_{\underline{\gamma\delta}}\big)\delta^{\alpha}_{\beta} + \big(L^{\delta}_{\underline{\beta\delta}} - \rho^{\delta}_{\underline{\beta\delta}}\big)\delta^{\alpha}_{\gamma}\Big), \tag{28}$$

$$\begin{aligned} \overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta} = {} & \overset{0}{R}{}^{\alpha}_{\beta\gamma\delta} - \rho^{\alpha}_{\underline{\beta\gamma}|\delta} + \rho^{\alpha}_{\underline{\beta\delta}|\gamma} + \rho^{\epsilon}_{\underline{\beta\gamma}}\rho^{\alpha}_{\underline{\epsilon\delta}} - \rho^{\epsilon}_{\underline{\beta\delta}}\rho^{\alpha}_{\underline{\epsilon\gamma}} + \frac{1}{N+1}\delta^{\alpha}_{\beta}\big(\overset{0}{R}_{[\gamma\delta]} + \rho^{\epsilon}_{[\underline{\gamma\epsilon}|\delta]}\big) \\ & - \frac{1}{(N+1)^2}\delta^{\alpha}_{\gamma}\bigg((N+1)\Big(L^{\epsilon}_{\underline{\beta\epsilon}|\delta} - \rho^{\epsilon}_{\underline{\beta\epsilon}|\delta} + \rho^{\epsilon}_{\underline{\beta\delta}}\big(L^{\zeta}_{\underline{\epsilon\zeta}} - \rho^{\zeta}_{\underline{\epsilon\zeta}}\big)\Big) + \big(L^{\epsilon}_{\underline{\beta\epsilon}} - \rho^{\epsilon}_{\underline{\beta\epsilon}}\big)\big(L^{\zeta}_{\underline{\delta\zeta}} - \rho^{\zeta}_{\underline{\delta\zeta}}\big)\bigg) \\ & + \frac{1}{(N+1)^2}\delta^{\alpha}_{\delta}\bigg((N+1)\Big(L^{\epsilon}_{\underline{\beta\epsilon}|\gamma} - \rho^{\epsilon}_{\underline{\beta\epsilon}|\gamma} + \rho^{\epsilon}_{\underline{\beta\gamma}}\big(L^{\zeta}_{\underline{\epsilon\zeta}} - \rho^{\zeta}_{\underline{\epsilon\zeta}}\big)\Big) + \big(L^{\epsilon}_{\underline{\beta\epsilon}} - \rho^{\epsilon}_{\underline{\beta\epsilon}}\big)\big(L^{\zeta}_{\underline{\gamma\zeta}} - \rho^{\zeta}_{\underline{\gamma\zeta}}\big)\bigg), \end{aligned} \tag{29}$$

are the basic invariants of Thomas and Weyl type for the mapping f.

If $\bar{\rho}^{\alpha}_{\underline{\beta\gamma}}\bar{\rho}^{\delta}_{\underline{\epsilon\zeta}} = \rho^{\alpha}_{\underline{\beta\gamma}}\rho^{\delta}_{\underline{\epsilon\zeta}}$, for the geometrical objects $\rho^{\alpha}_{\underline{\beta\gamma}}$ and $\bar{\rho}^{\alpha}_{\underline{\beta\gamma}}$ used in the basic Equation (27), the invariant $\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta}$ reduces to

$$\begin{aligned} \overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta} = {} & \overset{0}{R}{}^{\alpha}_{\beta\gamma\delta} - \rho^{\alpha}_{\underline{\beta\gamma}|\delta} + \rho^{\alpha}_{\underline{\beta\delta}|\gamma} + \frac{1}{N+1}\delta^{\alpha}_{\beta}\big(\overset{0}{R}_{[\gamma\delta]} + \rho^{\epsilon}_{[\underline{\gamma\epsilon}|\delta]}\big) \\ & - \frac{1}{(N+1)^2}\delta^{\alpha}_{\gamma}\Big((N+1)\big[L^{\epsilon}_{\underline{\beta\epsilon}|\delta} - \rho^{\epsilon}_{\underline{\beta\epsilon}|\delta} + \rho^{\epsilon}_{\underline{\beta\delta}}L^{\zeta}_{\underline{\epsilon\zeta}}\big] + L^{\epsilon}_{\underline{\beta\epsilon}}L^{\zeta}_{\underline{\delta\zeta}} - L^{\epsilon}_{\underline{\beta\epsilon}}\rho^{\zeta}_{\underline{\delta\zeta}} - L^{\epsilon}_{\underline{\delta\epsilon}}\rho^{\zeta}_{\underline{\beta\zeta}}\Big) \\ & + \frac{1}{(N+1)^2}\delta^{\alpha}_{\delta}\Big((N+1)\big[L^{\epsilon}_{\underline{\beta\epsilon}|\gamma} - \rho^{\epsilon}_{\underline{\beta\epsilon}|\gamma} + \rho^{\epsilon}_{\underline{\beta\gamma}}L^{\zeta}_{\underline{\epsilon\zeta}}\big] + L^{\epsilon}_{\underline{\beta\epsilon}}L^{\zeta}_{\underline{\gamma\zeta}} - L^{\epsilon}_{\underline{\beta\epsilon}}\rho^{\zeta}_{\underline{\gamma\zeta}} - L^{\epsilon}_{\underline{\gamma\epsilon}}\rho^{\zeta}_{\underline{\beta\zeta}}\Big). \end{aligned} \tag{30}$$

The derived invariant of Weyl type for the mapping f is the geometrical object

$$\begin{aligned} \overset{0}{W}{}^{\alpha}_{\beta\gamma\delta} = {} & \overset{0}{R}{}^{\alpha}_{\beta\gamma\delta} + \frac{1}{N+1}\delta^{\alpha}_{\beta}\big(\overset{0}{R}_{[\gamma\delta]} + \rho^{\epsilon}_{[\underline{\gamma\epsilon}|\delta]}\big) + \frac{N}{N^2-1}\delta^{\alpha}_{[\gamma}\overset{0}{R}_{\beta\delta]} + \frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\overset{0}{R}_{\delta]\beta} \\ & - \rho^{\alpha}_{\underline{\beta\gamma}|\delta} + \rho^{\alpha}_{\underline{\beta\delta}|\gamma} + \rho^{\epsilon}_{\underline{\beta\gamma}}\rho^{\alpha}_{\underline{\epsilon\delta}} - \rho^{\epsilon}_{\underline{\beta\delta}}\rho^{\alpha}_{\underline{\epsilon\gamma}} \\ & - \frac{1}{N-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\delta]}||\epsilon} + \frac{1}{N-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\delta]}}\rho^{\zeta}_{\underline{\epsilon\zeta}} - \frac{1}{N-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\zeta}}\rho^{\zeta}_{\underline{\delta]\epsilon}} \\ & + \frac{N}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\epsilon}|\delta]} + \frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\delta]\epsilon}|\beta}. \end{aligned} \tag{31}$$

If $\bar{\rho}^{\alpha}_{\underline{\beta\gamma}}\bar{\rho}^{\delta}_{\underline{\epsilon\zeta}} = \rho^{\alpha}_{\underline{\beta\gamma}}\rho^{\delta}_{\underline{\epsilon\zeta}}$, the derived invariant of Weyl type for the mapping f is

$$\begin{aligned} \overset{0}{W}{}^{\alpha}_{\beta\gamma\delta} = {} & \overset{0}{R}{}^{\alpha}_{\beta\gamma\delta} + \frac{1}{N+1}\delta^{\alpha}_{\beta}\big(\overset{0}{R}_{[\gamma\delta]} + \rho^{\epsilon}_{[\underline{\gamma\epsilon}|\delta]}\big) + \frac{N}{N^2-1}\delta^{\alpha}_{[\gamma}\overset{0}{R}_{\beta\delta]} + \frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\overset{0}{R}_{\delta]\beta} \\ & - \rho^{\alpha}_{\underline{\beta\gamma}|\delta} + \rho^{\alpha}_{\underline{\beta\delta}|\gamma} - \frac{1}{N-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\delta]}||\epsilon} + \frac{N}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\epsilon}|\delta]} + \frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\delta]\epsilon}|\beta}. \end{aligned} \tag{32}$$

The invariants for mapping f listed in this theorem are totalled.

Proof. After contracting the Equation (27) by α and γ, one obtains

$$\psi_{\beta} = \frac{1}{N+1}\big(\bar{L}^{\delta}_{\underline{\beta\delta}} - \bar{\rho}^{\delta}_{\underline{\beta\delta}}\big) - \frac{1}{N+1}\big(L^{\delta}_{\underline{\beta\delta}} - \rho^{\delta}_{\underline{\beta\delta}}\big).$$

Hence, the Equation (27) transforms to

$$\bar{L}^{\alpha}_{\underline{\beta\gamma}} = L^{\alpha}_{\underline{\beta\gamma}} + \bar{\rho}^{\alpha}_{\underline{\beta\gamma}} - \rho^{\alpha}_{\underline{\beta\gamma}} + \frac{1}{N+1}\delta^{\alpha}_{\gamma}\Big(\big(\bar{L}^{\delta}_{\underline{\gamma\delta}} - \bar{\rho}^{\delta}_{\underline{\gamma\delta}}\big)\delta^{\alpha}_{\beta} + \big(\bar{L}^{\delta}_{\underline{\beta\delta}} - \bar{\rho}^{\delta}_{\underline{\beta\delta}}\big)\delta^{\alpha}_{\gamma}\Big) - \frac{1}{N+1}\Big(\big(L^{\delta}_{\underline{\gamma\delta}} - \rho^{\delta}_{\underline{\gamma\delta}}\big)\delta^{\alpha}_{\beta} + \big(L^{\delta}_{\underline{\beta\delta}} - \rho^{\delta}_{\underline{\beta\delta}}\big)\delta^{\alpha}_{\gamma}\Big). \tag{33}$$

After comparing the Equations (33) with (18), we obtain

$$\omega^{\alpha}_{\beta\gamma} = \rho^{\alpha}_{\underline{\beta\gamma}} + \frac{1}{N+1}\Big(\big(L^{\delta}_{\underline{\gamma\delta}} - \rho^{\delta}_{\underline{\gamma\delta}}\big)\delta^{\alpha}_{\beta} + \big(L^{\delta}_{\underline{\beta\delta}} - \rho^{\delta}_{\underline{\beta\delta}}\big)\delta^{\alpha}_{\gamma}\Big). \tag{34}$$

Hence, the basic invariants of Thomas and Weyl type for mapping [17,18] f are

$$\overset{0}{\mathcal{T}}{}^{\alpha}_{\underline{\beta\gamma}} = L^{\alpha}_{\underline{\beta\gamma}} - \rho^{\alpha}_{\underline{\beta\gamma}} - \frac{1}{N+1}\Big(\big(L^{\delta}_{\underline{\gamma\delta}} - \rho^{\delta}_{\underline{\gamma\delta}}\big)\delta^{\alpha}_{\beta} + \big(L^{\delta}_{\underline{\beta\delta}} - \rho^{\delta}_{\underline{\beta\delta}}\big)\delta^{\alpha}_{\gamma}\Big),$$

$$\begin{aligned}\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta} ={}& \overset{0}{R}{}^{\alpha}_{\beta\gamma\delta} - \rho^{\alpha}_{\underline{\beta\gamma}|\delta} + \rho^{\alpha}_{\underline{\beta\delta}|\gamma} + \rho^{\epsilon}_{\underline{\beta\gamma}}\rho^{\alpha}_{\underline{\epsilon\delta}} - \rho^{\epsilon}_{\underline{\beta\delta}}\rho^{\alpha}_{\underline{\epsilon\gamma}} + \frac{1}{N+1}\delta^{\alpha}_{\beta}\big(\overset{0}{R}_{[\gamma\delta]} + \rho^{\epsilon}_{[\underline{\gamma\epsilon}|\delta]}\big)\\
&- \frac{1}{(N+1)^2}\delta^{\alpha}_{\gamma}\bigg((N+1)\Big(L^{\epsilon}_{\underline{\beta\epsilon}|\delta} - \rho^{\epsilon}_{\underline{\beta\epsilon}|\delta} + \rho^{\epsilon}_{\underline{\beta\delta}}\big(L^{\zeta}_{\underline{\epsilon\zeta}} - \rho^{\zeta}_{\underline{\epsilon\zeta}}\big)\Big) + \big(L^{\epsilon}_{\underline{\beta\epsilon}} - \rho^{\epsilon}_{\underline{\beta\epsilon}}\big)\big(L^{\zeta}_{\underline{\delta\zeta}} - \rho^{\zeta}_{\underline{\delta\zeta}}\big)\bigg)\\
&+ \frac{1}{(N+1)^2}\delta^{\alpha}_{\delta}\bigg((N+1)\Big(L^{\epsilon}_{\underline{\beta\epsilon}|\gamma} - \rho^{\epsilon}_{\underline{\beta\epsilon}|\gamma} + \rho^{\epsilon}_{\underline{\beta\gamma}}\big(L^{\zeta}_{\underline{\epsilon\zeta}} - \rho^{\zeta}_{\underline{\epsilon\zeta}}\big)\Big) + \big(L^{\epsilon}_{\underline{\beta\epsilon}} - \rho^{\epsilon}_{\underline{\beta\epsilon}}\big)\big(L^{\zeta}_{\underline{\gamma\zeta}} - \rho^{\zeta}_{\underline{\gamma\zeta}}\big)\bigg).\end{aligned}$$

In the case of $\bar{\rho}^{\alpha}_{\underline{\beta\gamma}}\bar{\rho}^{\delta}_{\underline{\epsilon\zeta}} = \rho^{\alpha}_{\underline{\beta\gamma}}\rho^{\delta}_{\underline{\epsilon\zeta}}$, the basic invariant $\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta}$ given by (29) reduces to

$$\begin{aligned}\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta} ={}& \overset{0}{R}{}^{\alpha}_{\beta\gamma\delta} - \rho^{\alpha}_{\underline{\beta\gamma}|\delta} + \rho^{\alpha}_{\underline{\beta\delta}|\gamma} + \frac{1}{N+1}\delta^{\alpha}_{\beta}\big(\overset{0}{R}_{[\gamma\delta]} + \rho^{\epsilon}_{[\underline{\gamma\epsilon}|\delta]}\big)\\
&- \frac{1}{(N+1)^2}\delta^{\alpha}_{\gamma}\Big((N+1)\big[L^{\epsilon}_{\underline{\beta\epsilon}|\delta} - \rho^{\epsilon}_{\underline{\beta\epsilon}|\delta} + \rho^{\epsilon}_{\underline{\beta\delta}}L^{\zeta}_{\underline{\epsilon\zeta}}\big] + L^{\epsilon}_{\underline{\beta\epsilon}}L^{\zeta}_{\underline{\delta\zeta}} - L^{\epsilon}_{\underline{\beta\epsilon}}\rho^{\zeta}_{\underline{\delta\zeta}} - L^{\epsilon}_{\underline{\delta\epsilon}}\rho^{\zeta}_{\underline{\beta\zeta}}\Big)\\
&+ \frac{1}{(N+1)^2}\delta^{\alpha}_{\delta}\Big((N+1)\big[L^{\epsilon}_{\underline{\beta\epsilon}|\gamma} - \rho^{\epsilon}_{\underline{\beta\epsilon}|\gamma} + \rho^{\epsilon}_{\underline{\beta\gamma}}L^{\zeta}_{\underline{\epsilon\zeta}}\big] + L^{\epsilon}_{\underline{\beta\epsilon}}L^{\zeta}_{\underline{\gamma\zeta}} - L^{\epsilon}_{\underline{\beta\epsilon}}\rho^{\zeta}_{\underline{\gamma\zeta}} - L^{\epsilon}_{\underline{\gamma\epsilon}}\rho^{\zeta}_{\underline{\beta\zeta}}\Big).\end{aligned}$$

After contracting the difference $0 = \overset{0}{\bar{\mathcal{W}}}{}^{\alpha}_{\beta\gamma\delta} - \overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta}$ by α and β, one obtains the trivial equality.

On the other hand, if one contracts the equality $0 = \overset{0}{\bar{\mathcal{W}}}{}^{\alpha}_{\beta\gamma\delta} - \overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta}$, for $\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta}$ and $\overset{0}{\bar{\mathcal{W}}}{}^{\alpha}_{\beta\gamma\delta}$ of the form (29), by α and δ, one obtains

$$\begin{aligned}0 ={}& \big(\overset{0}{\bar{R}}_{\beta\gamma} - \overset{0}{R}_{\beta\gamma}\big) - \big(\bar{\rho}^{\epsilon}_{\underline{\beta[\gamma}\|\epsilon]} - \rho^{\epsilon}_{\underline{\beta[\gamma}|\epsilon]}\big) + \big(\bar{\rho}^{\epsilon}_{\underline{\beta\gamma}}\bar{\rho}^{\zeta}_{\underline{\epsilon\zeta}} - \rho^{\epsilon}_{\underline{\beta\gamma}}\rho^{\zeta}_{\underline{\epsilon\zeta}}\big) - \big(\bar{\rho}^{\epsilon}_{\underline{\beta\zeta}}\bar{\rho}^{\zeta}_{\underline{\gamma\epsilon}} - \rho^{\epsilon}_{\underline{\beta\zeta}}\rho^{\zeta}_{\underline{\gamma\epsilon}}\big)\\
&- \frac{1}{N+1}\big(\overset{0}{\bar{R}}_{[\beta\gamma]} + \bar{\rho}^{\epsilon}_{[\underline{\beta\epsilon}\|\gamma]}\big) - \frac{1}{N+1}\big(\overset{0}{R}_{[\beta\gamma]} + \rho^{\epsilon}_{[\underline{\beta\epsilon}|\gamma]}\big) + (N-1)X_{\beta\gamma},\end{aligned}$$

for the corresponding tensor $X_{\beta\gamma}$. If expressing $X_{\beta\gamma}$ from the last equality, it becomes

$$\begin{aligned}X_{\beta\gamma} ={}& -\frac{N}{N^2-1}\big(\overset{0}{\bar{R}}_{\beta\gamma} - \overset{0}{R}_{\beta\gamma}\big) - \frac{1}{N^2-1}\big(\overset{0}{\bar{R}}_{\gamma\beta} - \overset{0}{R}_{\gamma\beta}\big)\\
&- \frac{1}{N-1}\big(\bar{\rho}^{\epsilon}_{\underline{\beta\gamma}}\bar{\rho}^{\zeta}_{\underline{\epsilon\zeta}} - \rho^{\epsilon}_{\underline{\beta\gamma}}\rho^{\zeta}_{\underline{\epsilon\zeta}}\big) + \frac{1}{N-1}\big(\bar{\rho}^{\epsilon}_{\underline{\beta\zeta}}\bar{\rho}^{\zeta}_{\underline{\gamma\epsilon}} - \rho^{\epsilon}_{\underline{\beta\zeta}}\rho^{\zeta}_{\underline{\gamma\epsilon}}\big)\\
&+ \frac{1}{N-1}\big(\bar{\rho}^{\epsilon}_{\underline{\beta\gamma}\|\epsilon} - \rho^{\epsilon}_{\underline{\beta\gamma}|\epsilon}\big) - \frac{N}{N^2-1}\big(\bar{\rho}^{\epsilon}_{\underline{\beta\epsilon}\|\gamma} - \rho^{\epsilon}_{\underline{\beta\epsilon}|\gamma}\big) - \frac{1}{N^2-1}\big(\bar{\rho}^{\epsilon}_{\underline{\gamma\epsilon}\|\beta} - \rho^{\epsilon}_{\underline{\gamma\epsilon}|\eta}\big).\end{aligned}$$

Based on this computation, it is proved that is

$$\overset{0}{\bar{W}}{}^{\alpha}_{\beta\gamma\delta}=\overset{0}{W}{}^{\alpha}_{\beta\gamma\delta},$$

for

$$\begin{aligned}\overset{0}{W}{}^{\alpha}_{\beta\gamma\delta}=&\overset{0}{R}{}^{\alpha}_{\beta\gamma\delta}+\frac{1}{N+1}\delta^{\alpha}_{\beta}\big(\overset{0}{R}_{[\gamma\delta]}+\rho^{\epsilon}_{[\underline{\gamma\epsilon}|\delta]}\big)+\frac{N}{N^2-1}\delta^{\alpha}_{[\gamma}\overset{0}{R}_{\beta\delta]}+\frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\overset{0}{R}_{\delta]\beta}\\&-\rho^{\alpha}_{\underline{\beta\gamma}|\delta}+\rho^{\alpha}_{\underline{\beta\delta}|\gamma}+\rho^{\epsilon}_{\underline{\beta\gamma}}\rho^{\alpha}_{\underline{\epsilon\delta}}-\rho^{\epsilon}_{\underline{\beta\delta}}\rho^{\alpha}_{\underline{\epsilon\gamma}}\\&-\frac{1}{N-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\delta]}|\epsilon}+\frac{1}{N-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\delta]}}\rho^{\zeta}_{\underline{\epsilon\zeta}}-\frac{1}{N-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\zeta}}\rho^{\zeta}_{\underline{\delta]\epsilon}}\\&+\frac{N}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\epsilon}|\delta]}+\frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\delta]\epsilon}|\beta},\end{aligned}$$

and the corresponding $\overset{0}{\bar{W}}{}^{\alpha}_{\beta\gamma\delta}$. If $\bar{\rho}^{\alpha}_{\underline{\beta\gamma}}\bar{\rho}^{\delta}_{\underline{\epsilon\zeta}}=\rho^{\alpha}_{\underline{\beta\gamma}}\rho^{\delta}_{\underline{\epsilon\zeta}}$, the geometrical object $\overset{0}{W}{}^{\alpha}_{\beta\gamma\delta}$ reduces to

$$\begin{aligned}\overset{0}{W}{}^{\alpha}_{\beta\gamma\delta}=&\overset{0}{R}{}^{\alpha}_{\beta\gamma\delta}+\frac{1}{N+1}\delta^{\alpha}_{\beta}\big(\overset{0}{R}_{[\gamma\delta]}+\rho^{\epsilon}_{[\underline{\gamma\epsilon}|\delta]}\big)+\frac{N}{N^2-1}\delta^{\alpha}_{[\gamma}\overset{0}{R}_{\beta\delta]}+\frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\overset{0}{R}_{\delta]\beta}\\&-\rho^{\alpha}_{\underline{\beta\gamma}|\delta}+\rho^{\alpha}_{\underline{\beta\delta}|\gamma}-\frac{1}{N-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\delta]}|\epsilon}+\frac{N}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\epsilon}|\delta]}+\frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\delta]\epsilon}|\beta}.\end{aligned}$$

Using simple calculus, one finds that traces $\overset{0}{\mathcal{T}}{}^{\epsilon}_{\underline{\beta\epsilon}}$, $\overset{0}{W}{}^{\epsilon}_{\epsilon\gamma\delta}$, $\overset{0}{W}{}^{\epsilon}_{\beta\epsilon\delta}$, $\overset{0}{W}{}^{\epsilon}_{\beta\gamma\epsilon}$ vanish. That means that no one invariant may be obtained after contracting equalities $0=\overset{0}{\bar{\mathcal{T}}}{}^{\alpha}_{\underline{\beta\gamma}}-\overset{0}{\mathcal{T}}{}^{\alpha}_{\beta\gamma}$ and $0=\overset{0}{\bar{\mathcal{W}}}{}^{\alpha}_{\beta\gamma\delta}-\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta}$ by α and any of the covariant indices.

The following equalities hold

$$\overset{0}{\mathcal{T}}{}^{\alpha}_{\underline{\beta\gamma}}=\overset{0}{\mathcal{T}}{}^{\alpha}_{\underline{\gamma\beta}},\quad\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta}=-\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\delta\gamma},\quad\overset{0}{W}{}^{\alpha}_{\beta\gamma\delta}=-\overset{0}{W}{}^{\alpha}_{\beta\delta\gamma}.\tag{35}$$

Because the invariants $\overset{0}{\mathcal{T}}{}^{\alpha}_{\underline{\beta\gamma}}$ and $\overset{0}{\bar{\mathcal{T}}}{}^{\alpha}_{\underline{\beta\gamma}}$ have the same form, the basic invariant of Thomas type for the mapping f is total. The basic invariant of Weyl type for the mapping f is obtained with respect to the functional combination of the basic invariants of Thomas type. Because this combination does not affect the form of the resulting object, the basic invariant of Weyl type for the mapping f is total. The derived invariant for the mapping f is obtained by contraction of equality $0=\overset{0}{\bar{\mathcal{W}}}{}^{\alpha}_{\beta\gamma\delta}-\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta}$ by α and δ. For this reason, and because the basic invariant of Weyl type for the mapping f is total, the derived invariant for the mapping f is total, too. □

Corollary 1. *The geometrical objects*

$$\overset{0}{\mathcal{T}}g^{\alpha}_{\underline{\beta\gamma}}=\Gamma^{\alpha}_{\underline{\beta\gamma}}-\rho^{\alpha}_{\underline{\beta\gamma}}-\frac{1}{N+1}\Big(\big(\Gamma^{\delta}_{\underline{\gamma\delta}}-\rho^{\delta}_{\underline{\gamma\delta}}\big)\delta^{\alpha}_{\beta}+\big(\Gamma^{\delta}_{\underline{\beta\delta}}-\rho^{\delta}_{\underline{\beta\delta}}\big)\delta^{\alpha}_{\gamma}\Big),\tag{36}$$

$$\begin{aligned}\overset{0}{\mathcal{W}}g^{\alpha}_{\beta\gamma\delta}=&\overset{0}{R}g^{\alpha}_{\beta\gamma\delta}-\rho^{\alpha}_{\underline{\beta\gamma}|g\delta}+\rho^{\alpha}_{\underline{\beta\delta}|g\gamma}+\rho^{\epsilon}_{\underline{\beta\gamma}}\rho^{\alpha}_{\underline{\epsilon\delta}}-\rho^{\epsilon}_{\underline{\beta\delta}}\rho^{\alpha}_{\underline{\epsilon\gamma}}+\frac{1}{N+1}\delta^{\alpha}_{\beta}\rho^{\epsilon}_{[\underline{\gamma\epsilon}|g\delta]}\\&-\frac{1}{(N+1)^2}\delta^{\alpha}_{\gamma}\left((N+1)\Big(\Gamma^{\epsilon}_{\underline{\beta\epsilon}|g\delta}-\rho^{\epsilon}_{\underline{\beta\epsilon}|g\delta}+\rho^{\epsilon}_{\underline{\beta\delta}}\big(\Gamma^{\zeta}_{\underline{\epsilon\zeta}}-\rho^{\zeta}_{\underline{\epsilon\zeta}}\big)\Big)+\big(\Gamma^{\epsilon}_{\underline{\beta\epsilon}}-\rho^{\epsilon}_{\underline{\beta\epsilon}}\big)\big(\Gamma^{\zeta}_{\underline{\delta\zeta}}-\rho^{\zeta}_{\underline{\delta\zeta}}\big)\right)\\&+\frac{1}{(N+1)^2}\delta^{\alpha}_{\delta}\left((N+1)\Big(\Gamma^{\epsilon}_{\underline{\beta\epsilon}|g\gamma}-\rho^{\epsilon}_{\underline{\beta\epsilon}|g\gamma}+\rho^{\epsilon}_{\underline{\beta\gamma}}\big(\Gamma^{\zeta}_{\underline{\epsilon\zeta}}-\rho^{\zeta}_{\underline{\epsilon\zeta}}\big)\Big)+\big(\Gamma^{\epsilon}_{\underline{\beta\epsilon}}-\rho^{\epsilon}_{\underline{\beta\epsilon}}\big)\big(\Gamma^{\zeta}_{\underline{\gamma\zeta}}-\rho^{\zeta}_{\underline{\gamma\zeta}}\big)\right),\end{aligned}\tag{37}$$

are basic invariants for mapping $f:\mathbb{R}_N\to\bar{\mathbb{R}}_N$ *determined by*

$$\bar{\Gamma}^{\alpha}_{\underline{\beta\gamma}}=\Gamma^{\alpha}_{\underline{\beta\gamma}}+\psi_{\gamma}\delta^{\alpha}_{\beta}+\psi_{\beta}\delta^{\alpha}_{\gamma}+\bar{\rho}^{\alpha}_{\underline{\beta\gamma}}-\rho^{\alpha}_{\underline{\beta\gamma}}. \tag{38}$$

If $\bar{\rho}^{\alpha}_{\underline{\beta\gamma}}\bar{\rho}^{\delta}_{\underline{\epsilon\zeta}}=\rho^{\alpha}_{\underline{\beta\gamma}}\rho^{\delta}_{\underline{\epsilon\zeta}}$, *the invariant* $\overset{0}{\mathcal{W}}g^{\alpha}_{\beta\gamma\delta}$ *reduces to*

$$\begin{aligned}\overset{0}{\mathcal{W}}g^{\alpha}_{\beta\gamma\delta}=&\overset{0}{R}g^{\alpha}_{\beta\gamma\delta}-\rho^{\alpha}_{\underline{\beta\gamma}|g\delta}+\rho^{\alpha}_{\underline{\beta\delta}|g\gamma}+\frac{1}{N+1}\delta^{\alpha}_{\beta}\rho^{\epsilon}_{[\underline{\gamma\epsilon}|g\delta]}\\&-\frac{1}{(N+1)^2}\delta^{\alpha}_{\gamma}\Big((N+1)\big(\Gamma^{\epsilon}_{\underline{\beta\epsilon}|g\delta}-\rho^{\epsilon}_{\underline{\beta\epsilon}|g\delta}+\rho^{\epsilon}_{\underline{\beta\delta}}\Gamma^{\zeta}_{\underline{\epsilon\zeta}}\big)+\Gamma^{\epsilon}_{\underline{\beta\epsilon}}\Gamma^{\zeta}_{\underline{\delta\zeta}}-\Gamma^{\epsilon}_{\underline{\beta\epsilon}}\rho^{\zeta}_{\underline{\delta\zeta}}-\Gamma^{\epsilon}_{\underline{\delta\epsilon}}\rho^{\zeta}_{\underline{\beta\zeta}}\Big)\\&+\frac{1}{(N+1)^2}\delta^{\alpha}_{\delta}\Big((N+1)\big(\Gamma^{\epsilon}_{\underline{\beta\epsilon}|g\gamma}-\rho^{\epsilon}_{\underline{\beta\epsilon}|g\gamma}+\rho^{\epsilon}_{\underline{\beta\gamma}}\Gamma^{\zeta}_{\underline{\epsilon\zeta}}\big)+\Gamma^{\epsilon}_{\underline{\beta\epsilon}}\Gamma^{\zeta}_{\underline{\gamma\zeta}}-\Gamma^{\epsilon}_{\underline{\beta\epsilon}}\rho^{\zeta}_{\underline{\gamma\zeta}}-\Gamma^{\epsilon}_{\underline{\gamma\epsilon}}\rho^{\zeta}_{\underline{\beta\zeta}}\Big).\end{aligned}$$

The derived invariant of Weyl type for the mapping f *is*

$$\begin{aligned}\overset{0}{W}g^{\alpha}_{\beta\gamma\delta}=&\overset{0}{R}g^{\alpha}_{\beta\gamma\delta}+\frac{1}{N+1}\delta^{\alpha}_{\beta}\rho^{\epsilon}_{[\underline{\gamma\epsilon}|g\delta]}+\frac{1}{N-1}\big(\overset{0}{R}g_{\beta\delta}\delta^{\alpha}_{\gamma}-\overset{0}{R}g_{\beta\gamma}\delta^{\alpha}_{\delta}\big)\\&-\rho^{\alpha}_{\underline{\beta\gamma}|g\delta}+\rho^{\alpha}_{\underline{\beta\delta}|g\gamma}+\rho^{\epsilon}_{\underline{\beta\gamma}}\rho^{\alpha}_{\underline{\epsilon\delta}}-\rho^{\epsilon}_{\underline{\beta\delta}}\rho^{\alpha}_{\underline{\epsilon\gamma}}\\&-\frac{1}{N-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\delta}]|g\epsilon}+\frac{1}{N-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\delta}]}\rho^{\zeta}_{\underline{\epsilon\zeta}}-\frac{1}{N-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\zeta}}\rho^{\zeta}_{\underline{\delta]\epsilon}}\\&+\frac{N}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\epsilon}|g\delta]}+\frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\delta]\epsilon}|g\beta},\end{aligned} \tag{39}$$

which reduces to

$$\begin{aligned}\overset{0}{W}g^{\alpha}_{\beta\gamma\delta}=&\overset{0}{R}g^{\alpha}_{\beta\gamma\delta}+\frac{1}{N+1}\delta^{\alpha}_{\beta}\rho^{\epsilon}_{[\underline{\beta\epsilon}|g\gamma]}+\frac{1}{N-1}\big(\overset{0}{R}g_{\beta\delta}\delta^{\alpha}_{\gamma}-\overset{0}{R}g_{\beta\gamma}\delta^{\alpha}_{\delta}\big)\\&-\rho^{\alpha}_{\underline{\beta\gamma}|g\delta}+\rho^{\alpha}_{\underline{\beta\delta}|g\gamma}-\frac{1}{N-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\delta}]|g\epsilon}+\frac{N}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\epsilon}|g\delta]}+\frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\delta]\epsilon}|g\beta},\end{aligned} \tag{40}$$

in the case of $\bar{\rho}^{\alpha}_{\underline{\beta\gamma}}\bar{\rho}^{\delta}_{\underline{\epsilon\zeta}}=\rho^{\alpha}_{\underline{\beta\gamma}}\rho^{\delta}_{\underline{\epsilon\zeta}}$.

The invariants for mapping f *listed in this corollary are total.* □

3. Invariants for Second Type Almost Geodesic Mappings of Space $\mathbb{A}_N$

The next theorem will be proved below.

Theorem 2. *Let* $f:\mathbb{A}_N\to\bar{\mathbb{A}}_N$ *be a second type almost geodesic mapping of a symmetric affine connection space* $\mathbb{A}_N$.

The geometrical objects

$$\mathcal{AG}_2\overset{0}{\mathcal{T}}{}^{\alpha}_{\underline{\beta\gamma}}=L^{\alpha}_{\underline{\beta\gamma}}-\rho^{\alpha}_{\underline{\beta\gamma}}-\frac{1}{N+1}\big[\big(L^{\delta}_{\underline{\gamma\delta}}-\rho^{\delta}_{\underline{\gamma\delta}}\big)\delta^{\alpha}_{\beta}+\big(L^{\delta}_{\underline{\beta\delta}}-\rho^{\delta}_{\underline{\beta\delta}}\big)\delta^{\alpha}_{\gamma}\big], \tag{41}$$

$$\begin{aligned}\mathcal{AG}_2\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta}=&\overset{0}{R}{}^{\alpha}_{\beta\gamma\delta}-\rho^{\alpha}_{\underline{\beta\gamma}|\delta}+\rho^{\alpha}_{\underline{\beta\delta}|\gamma}+\frac{1}{N+1}\delta^{\alpha}_{\beta}\big(\overset{0}{R}_{[\gamma\delta]}+\rho^{\epsilon}_{[\underline{\gamma\epsilon}|\delta]}\big)\\&-\frac{1}{(N+1)^2}\delta^{\alpha}_{\gamma}\Big((N+1)\big(L^{\epsilon}_{\underline{\beta\epsilon}|\delta}-\rho^{\epsilon}_{\underline{\beta\epsilon}|\delta}+\rho^{\epsilon}_{\underline{\beta\delta}}L^{\zeta}_{\underline{\epsilon\zeta}}\big)+L^{\epsilon}_{\underline{\beta\epsilon}}L^{\zeta}_{\underline{\delta\zeta}}-L^{\epsilon}_{\underline{\beta\epsilon}}\rho^{\zeta}_{\underline{\delta\zeta}}-L^{\epsilon}_{\underline{\delta\epsilon}}\rho^{\zeta}_{\underline{\beta\zeta}}\Big)\\&+\frac{1}{(N+1)^2}\delta^{\alpha}_{\delta}\Big((N+1)\big(L^{\epsilon}_{\underline{\beta\epsilon}|\gamma}-\rho^{\epsilon}_{\underline{\beta\epsilon}|\gamma}+\rho^{\epsilon}_{\underline{\beta\gamma}}L^{\zeta}_{\underline{\epsilon\zeta}}\big)+L^{\epsilon}_{\underline{\beta\epsilon}}L^{\zeta}_{\underline{\gamma\zeta}}-L^{\epsilon}_{\underline{\beta\epsilon}}\rho^{\zeta}_{\underline{\gamma\zeta}}-L^{\epsilon}_{\underline{\gamma\epsilon}}\rho^{\zeta}_{\underline{\beta\zeta}}\Big),\end{aligned} \tag{42}$$

for $\rho^{\alpha}_{\underline{\beta\gamma}}=-\sigma_{\gamma}F^{\alpha}_{\beta}-\sigma_{\beta}F^{\alpha}_{\gamma}$, *are the basic invariants of Thomas and Weyl type for the mapping* f.

The geometrical object

$$\begin{aligned}\mathcal{AG}_2\overset{0}{W}{}^{\alpha}_{\beta\gamma\delta}=&\overset{0}{R}{}^{\alpha}_{\beta\gamma\delta}+\frac{1}{N+1}\delta^{\alpha}_{\beta}\big(\overset{0}{R}_{[\gamma\delta]}+\rho^{\epsilon}_{[\underline{\gamma\epsilon}|\delta]}\big)+\frac{N}{N^2-1}\delta^{\alpha}_{[\gamma}\overset{0}{R}_{\beta\delta]}+\frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\overset{0}{R}_{\delta]\beta}\\&-\rho^{\alpha}_{\underline{\beta\gamma}|\delta}+\rho^{\alpha}_{\underline{\beta\delta}|\gamma}-\frac{1}{N-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\delta}]|\epsilon}+\frac{N}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\beta\epsilon}|\delta]}+\frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\rho^{\epsilon}_{\underline{\delta]\epsilon}|\beta},\end{aligned} \tag{43}$$

for $\rho^{\alpha}_{\underline{\beta\gamma}}$ *and the corresponding covariant derivatives as in the basic invariants* (41) and (42), *is the derived invariant for mapping* f.

The invariant $\mathcal{AG}_2\overset{0}{\mathcal{T}}{}^{\alpha}_{\underline{\beta\gamma}}$ *for mapping* f *is total.*

The invariants $\mathcal{AG}_2\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta}$ *and* $\mathcal{AG}_2\overset{0}{W}{}^{\alpha}_{\beta\gamma\delta}$ *for mapping* f *are valued. They are total if and only if the mapping* f *has the property of reciprocity.*

Proof. It is appropriate to assume that geometrical objects $\psi_\gamma\delta^\alpha_\beta + \psi_\beta\delta^\alpha_\gamma$ and $\sigma_\gamma F^\alpha_\beta + \sigma_\beta F^\alpha_\gamma$ in the first of Equation (16) are linearly independent. Otherwise, this mapping reduces to the geodesic one.

Because the geometrical objects $\psi_\gamma\delta^\alpha_\beta + \psi_\beta\delta^\alpha_\gamma$ and $\tilde{\rho}^{\alpha}_{\underline{\beta\gamma}} = \sigma_\gamma F^\alpha_\beta + \sigma_\beta F^\alpha_\gamma$ are linearly independent, we obtain the tensor $\hat{\tilde{\rho}}$ of type $(1,2)$, whose components are $\tilde{\tilde{\rho}}^{\alpha}_{\underline{\beta\gamma}} = -\tilde{\rho}^{\alpha}_{\underline{\beta\gamma}}$.

From the first of the basic equations from (16), compared with (27), we obtain

$$\rho^{\alpha}_{\underline{\beta\gamma}} = -\tilde{\rho}^{\alpha}_{\beta\gamma} = -\sigma_\gamma F^\alpha_\beta - \sigma_\beta F^\alpha_\gamma.$$

Based on the second of basic Equation (16), the covariant derivative $\left(\sigma_\beta F^\alpha_\gamma\right)_{|\delta}$ is

$$\begin{aligned}\left(\sigma_\beta F^\alpha_\gamma\right)_{|\delta} = {}& \sigma_{\beta|\delta}F^\alpha_\gamma - \sigma_\beta F^\alpha_{\delta|\gamma} - \sigma_\beta\sigma_\delta F^\epsilon_\gamma F^\alpha_\epsilon - \sigma_\beta\sigma_\gamma F^\epsilon_\delta F^\alpha_\epsilon \\ &+ \sigma_\beta\nu_\gamma\delta^\alpha_\delta + \sigma_\beta\nu_\delta\delta^\alpha_\gamma + \sigma_\beta\mu_\gamma F^\alpha_\delta + \sigma_\beta\mu_\delta F^\alpha_\gamma.\end{aligned}$$

From this expression, one obtains the following

$$\begin{aligned}\rho^{\alpha}_{\underline{\beta\gamma}|\delta} = {}& -\sigma_{\beta|\delta}F^\alpha_\gamma + \sigma_\beta F^\alpha_{\delta|\gamma} + \sigma_\beta\sigma_\delta F^\epsilon_\gamma F^\alpha_\epsilon + 2\sigma_\beta\sigma_\gamma F^\epsilon_\delta F^\alpha_\epsilon \\ &- \sigma_{\gamma|\delta}F^\alpha_\beta + \sigma_\gamma F^\alpha_{\delta|\beta} + \sigma_\gamma\sigma_\delta F^\epsilon_\beta F^\alpha_\epsilon \\ &- \sigma_\beta\nu_\gamma\delta^\alpha_\delta - \sigma_\beta\nu_\delta\delta^\alpha_\gamma - \sigma_\beta\mu_\gamma F^\alpha_\delta - \sigma_\beta\mu_\delta F^\alpha_\gamma \\ &- \sigma_\gamma\nu_\beta\delta^\alpha_\delta - \sigma_\gamma\nu_\delta\delta^\alpha_\beta - \sigma_\gamma\mu_\beta F^\alpha_\delta - \sigma_\gamma\mu_\delta F^\alpha_\beta.\end{aligned}$$

Finally, the next equations hold

$$\begin{aligned}-\rho^{\alpha}_{\underline{\beta\gamma}|\delta} + \rho^{\alpha}_{\underline{\beta\delta}|\gamma} = {}& -\sigma_{\beta|[\gamma}F^\alpha_{\delta]} + \sigma_\beta F^\alpha_{[\gamma|\delta]} + \sigma_\beta\sigma_{[\gamma}F^\epsilon_{\delta]}F^\alpha_\epsilon - \sigma_{[\gamma|\delta]}F^\alpha_\beta + \sigma_{[\gamma}F^\alpha_{\delta]|\beta} \\ &+ \delta^\alpha_{[\gamma}\sigma_{\delta]}\sigma_\beta - \sigma_{[\gamma}\nu_{\delta]}\delta^\alpha_\beta - \sigma_{[\gamma}\mu_\beta F^\alpha_{\delta]} - \sigma_{[\gamma}\mu_{\delta]}F^\alpha_\beta, \\ \rho^{\epsilon}_{\underline{\beta\epsilon}|\delta} = {}& -\sigma_{\beta|\epsilon}F^\epsilon_\delta + \sigma_{\beta|\delta}F + \sigma_\beta F_\delta - \sigma_\beta F^\epsilon_{\delta|\epsilon} + \sigma_\beta\sigma_\epsilon F^\zeta_\delta F^\epsilon_\zeta - \sigma_\beta\sigma_\delta F^\zeta_\epsilon F^\epsilon_\zeta \\ &- \sigma_{\epsilon|\delta}F^\epsilon_\beta + \sigma_{\delta|\epsilon}F^\epsilon_\beta + \sigma_\epsilon F^\epsilon_{\delta|\beta} - \sigma_\delta F_\beta \\ &+ (N-1)\sigma_\beta\sigma_\delta - \sigma_{[\beta}\nu_{\delta]} - \sigma_\epsilon\mu_\beta F^\epsilon_\delta + \sigma_\delta\mu_\beta F - \sigma_\epsilon\mu_\delta F^\epsilon_\beta + \sigma_\delta\mu_\epsilon F^\epsilon_\beta, \\ \rho^{\epsilon}_{[\underline{\beta\epsilon}|\delta]} = {}& -2\sigma_{[\beta|\epsilon}F^\epsilon_{\delta]} + \sigma_{[\beta|\delta]}F + 2\sigma_{[\beta}F_{\delta]} - \sigma_{[\beta}F^\epsilon_{\delta]|\epsilon} + \sigma_{[\beta}\sigma_\epsilon F^\zeta_{\delta]}F^\epsilon_\zeta \\ &+ \sigma_{\epsilon|[\beta}F^\epsilon_{\delta]} - \sigma_\epsilon F^\epsilon_{[\beta|\delta]} - 2\sigma_{[\beta}\nu_{\delta]} - \sigma_{[\beta}\mu_{\delta]}F - \sigma_{[\beta}\mu_\epsilon F^\epsilon_{\delta]},\end{aligned}$$

for $F = F^\alpha_\alpha$ and $F_\beta = F_{|\beta}$.

In the case of almost geodesic mapping f, it holds $\bar{\rho}^{\alpha}_{\underline{\beta\gamma}}\bar{\rho}^{\delta}_{\underline{\epsilon\zeta}} = \rho^{\alpha}_{\underline{\beta\gamma}}\rho^{\delta}_{\underline{\epsilon\zeta}}$, which completes the proof of this theorem. □

Invariants for π_2*-Mappings of Space* $\mathbb{R}_N$

A mapping $f : \mathbb{R}_N \to \bar{\mathbb{R}}_N$ determined with basic equations

$$\begin{cases}\bar{\Gamma}^{\alpha}_{\underline{\beta\gamma}} = \Gamma^{\alpha}_{\underline{\beta\gamma}} + \psi_\gamma\delta^\alpha_\beta + \psi_\beta\delta^\alpha_\gamma + 2\sigma_\gamma F^\alpha_\beta + 2\sigma_\beta F^\alpha_\gamma, \\ F^\alpha_{\beta|_g\gamma} + F^\alpha_{\gamma|_g\beta} + 2\sigma_\beta F^\delta_\gamma F^\alpha_\delta + 2\sigma_\gamma F^\delta_\beta F^\alpha_\delta = \nu_\gamma\delta^\alpha_\beta + \nu_\beta\delta^\alpha_\gamma + \mu_\gamma F^\alpha_\beta + \mu_\beta F^\alpha_\gamma,\end{cases} \tag{44}$$

for 1-forms ψ_β, σ_β, μ_β, ν_β, and affinor F^α_β, is the second type almost geodesic mapping of Riemannian space $\mathbb{R}_N$.

The almost geodesic mapping f has the property of reciprocity if its inverse mapping, $f^{-1} : \bar{\mathbb{R}}_N \to \mathbb{R}_N$, is an almost geodesic mapping of second type and affinor F^α_β is an invariant for this mapping.

The necessary and sufficient condition for almost geodesic mapping f of space $\mathbb{R}_N$ to have the property of reciprocity is given by (17).

Analogously as above, the validity of the next theorem will be confirmed.

Theorem 3. *Let $f : \mathbb{R}_N \to \bar{\mathbb{R}}_N$ be a second type almost geodesic mapping of a Riemannian space $\mathbb{R}_N$.*

The geometrical objects

$$\mathcal{AG}_2\overset{0}{\mathcal{T}}{}^{g}{}^{\alpha}_{\underline{\beta\gamma}} = \Gamma^\alpha_{\underline{\beta\gamma}} - \rho^\alpha_{\underline{\beta\gamma}} - \frac{1}{N+1}\big[(\Gamma^\delta_{\underline{\gamma\delta}} - \rho^\delta_{\underline{\gamma\delta}})\delta^\alpha_\beta + (\Gamma^\delta_{\underline{\beta\delta}} - \rho^\delta_{\underline{\beta\delta}})\delta^\alpha_\gamma\big], \tag{45}$$

$$\begin{aligned}\mathcal{AG}_2\overset{0}{\mathcal{W}}{}^{g}{}^{\alpha}_{\beta\gamma\delta} = {} & R^{g}{}^{\alpha}_{\beta\gamma\delta} - \rho^\alpha_{\underline{\beta\gamma}|^g\delta} + \rho^\alpha_{\underline{\beta\delta}|^g\gamma} + \frac{1}{N+1}\delta^\alpha_\beta\big(R^g{}_{[\gamma\delta]} + \rho^\epsilon_{[\underline{\gamma\epsilon}|^g\delta]}\big) \\ & - \frac{1}{(N+1)^2}\delta^\alpha_\gamma\Big((N+1)\big(\Gamma^\epsilon_{\underline{\beta\epsilon}|^g\delta} - \rho^\epsilon_{\underline{\beta\epsilon}|^g\delta} + \rho^\epsilon_{\underline{\beta\delta}}\Gamma^\zeta_{\underline{\epsilon\zeta}}\big) + \Gamma^\epsilon_{\underline{\beta\epsilon}}\Gamma^\zeta_{\underline{\delta\zeta}} - \Gamma^\epsilon_{\underline{\beta\epsilon}}\rho^\zeta_{\underline{\delta\zeta}} - \Gamma^\epsilon_{\underline{\delta\epsilon}}\rho^\zeta_{\underline{\beta\zeta}}\Big) \\ & + \frac{1}{(N+1)^2}\delta^\alpha_\delta\Big((N+1)\big(\Gamma^\epsilon_{\underline{\beta\epsilon}|^g\gamma} - \rho^\epsilon_{\underline{\beta\epsilon}|^g\gamma} + \rho^\epsilon_{\underline{\beta\gamma}}\Gamma^\zeta_{\underline{\epsilon\zeta}}\big) + \Gamma^\epsilon_{\underline{\beta\epsilon}}\Gamma^\zeta_{\underline{\gamma\zeta}} - \Gamma^\epsilon_{\underline{\beta\epsilon}}\rho^\zeta_{\underline{\gamma\zeta}} - \Gamma^\epsilon_{\underline{\gamma\epsilon}}\rho^\zeta_{\underline{\beta\zeta}}\Big),\end{aligned} \tag{46}$$

for $\rho^\alpha_{\underline{\beta\gamma}} == -\sigma_\gamma F^\alpha_\beta - \sigma_\beta F^\alpha_\gamma$ are the basic invariants of Thomas and Weyl type for the mapping f.

The geometrical object

$$\begin{aligned}\mathcal{AG}_2\overset{0}{W}{}^{g}{}^{\alpha}_{\beta\gamma\delta} = {} & R^{g}{}^{\alpha}_{\beta\gamma\delta} + \frac{1}{N+1}\delta^\alpha_\beta\rho^\epsilon_{[\underline{\gamma\epsilon}|^g\delta]} + \frac{1}{N-1}\big(\delta^\alpha_\gamma R^g{}_{\beta\delta} + \delta^\alpha_\delta R^g{}_{\beta\gamma}\big) \\ & - \rho^\alpha_{\underline{\beta\gamma}|^g\delta} + \rho^\alpha_{\underline{\beta\delta}|^g\gamma} - \frac{1}{N-1}\delta^\alpha_{[\gamma}\rho^\epsilon_{\underline{\beta\delta}]|^g\epsilon} + \frac{N}{N^2-1}\delta^\alpha_{[\gamma}\rho^\epsilon_{\underline{\beta\epsilon}|^g\delta]} + \frac{1}{N^2-1}\delta^\alpha_{[\gamma}\rho^\epsilon_{\underline{\delta]\epsilon}|^g\beta},\end{aligned} \tag{47}$$

for $\rho^\alpha_{\underline{\beta\gamma}}$ and the corresponding covariant derivatives as in the basic invariants (45) and (46)*, is the derived invariant for mapping f.*

The invariant $\mathcal{AG}_2\overset{0}{\mathcal{T}}{}^{g}{}^{\alpha}_{\underline{\beta\gamma}}$ for mapping f is total.

The invariants $\mathcal{AG}_2\overset{0}{\mathcal{W}}{}^{g}{}^{\alpha}_{\beta\gamma\delta}$ and $\mathcal{AG}_2\overset{0}{W}{}^{g}{}^{\alpha}_{\beta\gamma\delta}$ for mapping f are valued. They are totalled if and only if the mapping f has the property of reciprocity. □

Proof. Let $f : \mathbb{R}_N \to \bar{\mathbb{R}}_N$ be an almost geodesic mapping of the second type. The basic equations of this mapping are given by (44).

From the first of these equations, we recognize that is $\rho^\alpha_{\underline{\beta\gamma}} = -\sigma_\gamma F^\alpha_\beta - \sigma_\beta F^\alpha_\gamma$. For the geometrical object $\rho^\alpha_{\underline{\beta\gamma}}$ given in this way, the following equations hold:

$$\begin{aligned}-\rho^\alpha_{\underline{\beta\gamma}|^g\delta} + \rho^\alpha_{\underline{\beta\delta}|^g\gamma} = {} & -\sigma_{\beta|^g[\gamma}F^\alpha_{\delta]} + \sigma_\beta F^\alpha_{[\gamma|^g\delta]} + \sigma_\beta\sigma_{[\gamma}F^\epsilon_{\delta]}F^\alpha_\epsilon - \sigma_{[\gamma|^g\delta]}F^\alpha_\beta + \sigma_{[\gamma}F^\alpha_{\delta]|\beta} \\ & + \delta^\alpha_{[\gamma}\sigma_{\delta]}\sigma_\beta - \sigma_{[\gamma}\nu_{\delta]}\delta^\alpha_\beta - \sigma_{[\gamma}\mu_\beta F^\alpha_{\delta]} - \sigma_{[\gamma}\mu_{\delta]}F^\alpha_\beta,\end{aligned} \tag{48}$$

$$\begin{aligned}\rho^\epsilon_{\underline{\beta\epsilon}|^g\delta} = {} & -\sigma_{\beta|^g\epsilon}F^\epsilon_\delta + \sigma_{\beta|^g\delta}F + \sigma_\beta F_\delta - \sigma_\beta F^\epsilon_{\delta|^g\epsilon} + \sigma_\beta\sigma_\epsilon F^\zeta_\delta F^\epsilon_\zeta - \sigma_\beta\sigma_\delta F^\zeta_\epsilon F^\epsilon_\zeta \\ & - \sigma_{\epsilon|^g\delta}F^\epsilon_\beta + \sigma_{\delta|^g\epsilon}F^\epsilon_\beta + \sigma_\epsilon F^\epsilon_{\delta|^g\beta} - \sigma_\delta F_\beta \\ & + (N-1)\sigma_\beta\sigma_\delta - \sigma_{[\beta}\nu_{\delta]} - \sigma_\epsilon\mu_\beta F^\epsilon_\delta + \sigma_\delta\mu_\beta F - \sigma_\epsilon\mu_\delta F^\epsilon_\beta + \sigma_\delta\mu_\epsilon F^\epsilon_\beta,\end{aligned} \tag{49}$$

$$\begin{aligned}\rho^\epsilon_{[\underline{\beta\epsilon}|^g\delta]} = {} & -2\sigma_{[\beta|^g\epsilon}F^\epsilon_{\delta]} + \sigma_{[\beta|^g\delta]}F + 2\sigma_{[\beta}F_{\delta]} - \sigma_{[\beta}F^\epsilon_{\delta]|^g\epsilon} + \sigma_{[\beta}\sigma_\epsilon F^\zeta_{\delta]}F^\epsilon_\zeta \\ & + \sigma_{\epsilon|^g[\beta}F^\epsilon_{\delta]} - \sigma_\epsilon F^\epsilon_{[\beta|^g\delta]} - 2\sigma_{[\beta}\nu_{\delta]} - \sigma_{[\beta}\mu_{\delta]}F - \sigma_{[\beta}\mu_\epsilon F^\epsilon_{\delta]},\end{aligned} \tag{50}$$

for $F = F^\alpha_\alpha$ and $F_\beta = F_{|^g\beta}$.

After substituting the expressions (48)–(50) into the Equations (36), (37), (39) and (40), the proof of this theorem is completed. □

Theorem 4. *The geometrical objects* $\mathcal{GA}_2\overset{0}{\mathcal{T}}{}^{\alpha}_{\underline{\beta\gamma}}$ *and* $\mathcal{GA}_2\overset{0}{\mathcal{T}}{}^{g\,\alpha}_{\underline{\beta\gamma}}$ *given by* (41) and (45) *satisfy the following equation*

$$\mathcal{GA}_2\overset{0}{\mathcal{T}}{}^{\alpha}_{\underline{\beta\gamma}} = \mathcal{GA}_2\overset{0}{\mathcal{T}}{}^{g\,\alpha}_{\underline{\beta\gamma}} + \mathcal{P}^{\alpha}_{\underline{\beta\gamma}} + \frac{1}{N+1}\big(\mathcal{P}^{\delta}_{\underline{\gamma\delta}}\delta^{\alpha}_{\beta} + \mathcal{P}^{\delta}_{\underline{\beta\delta}}\delta^{\alpha}_{\gamma}\big). \tag{51}$$

The geometrical objects $\mathcal{GA}_2\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta}$ *and* $\mathcal{GA}_2\overset{0}{\mathcal{W}}{}^{g\,\alpha}_{\beta\gamma\delta}$ *given by* (42) and (46) *are correlated as*

$$\begin{aligned}\mathcal{AG}_2\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta} = {} & \mathcal{AG}_2\overset{0}{\mathcal{W}}{}^{g\,\alpha}_{\beta\gamma\delta} + \mathcal{P}^{\alpha}_{\underline{\beta\gamma}|^g\delta} - \mathcal{P}^{\alpha}_{\underline{\beta\delta}|^g\gamma} + \mathcal{P}^{\epsilon}_{\underline{\beta\gamma}}\mathcal{P}^{\alpha}_{\underline{\epsilon\delta}} - \mathcal{P}^{\epsilon}_{\underline{\beta\delta}}\mathcal{P}^{\alpha}_{\underline{\epsilon\gamma}} \\ & + \mathcal{P}^{\alpha}_{\underline{\epsilon[\gamma}}\rho^{\epsilon}_{\underline{\beta\delta}]} - \mathcal{P}^{\epsilon}_{\underline{\beta[\gamma}}\rho^{\alpha}_{\underline{\epsilon\delta}]} - \frac{1}{N+1}\delta^{\alpha}_{\beta}\mathcal{P}^{\epsilon}_{[\underline{\gamma\epsilon}|^g\delta]} \\ & - \frac{1}{(N+1)^2}\delta^{\alpha}_{\gamma}\Big((N+1)\big(\mathcal{P}^{\epsilon}_{\underline{\beta\epsilon}|^g\delta} - \mathcal{P}^{\epsilon}_{\underline{\beta\delta}}\Gamma^{\zeta}_{\underline{\epsilon\zeta}} + \rho^{\epsilon}_{\underline{\beta\delta}}\mathcal{P}^{\zeta}_{\underline{\epsilon\zeta}}\big) \\ & \qquad + \Gamma^{\epsilon}_{\underline{\beta\epsilon}}\mathcal{P}^{\zeta}_{\underline{\delta\zeta}} + \Gamma^{\epsilon}_{\underline{\delta\epsilon}}\mathcal{P}^{\zeta}_{\underline{\beta\zeta}} + \mathcal{P}^{\epsilon}_{\underline{\beta\epsilon}}\mathcal{P}^{\zeta}_{\underline{\delta\zeta}} - \mathcal{P}^{\epsilon}_{\underline{\beta\epsilon}}\rho^{\zeta}_{\underline{\delta\zeta}} - \mathcal{P}^{\epsilon}_{\underline{\delta\epsilon}}\rho^{\zeta}_{\underline{\beta\zeta}}\Big) \\ & + \frac{1}{(N+1)^2}\delta^{\alpha}_{\delta}\Big((N+1)\big(\mathcal{P}^{\epsilon}_{\underline{\beta\epsilon}|^g\gamma} - \mathcal{P}^{\epsilon}_{\underline{\beta\gamma}}\Gamma^{\zeta}_{\underline{\epsilon\zeta}} + \rho^{\epsilon}_{\underline{\beta\gamma}}\mathcal{P}^{\zeta}_{\underline{\epsilon\zeta}}\big) \\ & \qquad + \Gamma^{\epsilon}_{\underline{\beta\epsilon}}\mathcal{P}^{\zeta}_{\underline{\gamma\zeta}} + \Gamma^{\epsilon}_{\underline{\gamma\epsilon}}\mathcal{P}^{\zeta}_{\underline{\beta\zeta}} + \mathcal{P}^{\epsilon}_{\underline{\beta\epsilon}}\mathcal{P}^{\zeta}_{\underline{\gamma\zeta}} - \mathcal{P}^{\epsilon}_{\underline{\beta\epsilon}}\rho^{\zeta}_{\underline{\gamma\zeta}} - \mathcal{P}^{\epsilon}_{\underline{\gamma\epsilon}}\rho^{\zeta}_{\underline{\beta\zeta}}\Big).\end{aligned} \tag{52}$$

The next equation holds

$$\begin{aligned}\mathcal{AG}_2\overset{0}{W}{}^{\alpha}_{\beta\gamma\delta} = {} & \mathcal{AG}_2\overset{0}{W}{}^{g\,\alpha}_{\beta\gamma\delta} + \mathcal{P}^{\alpha}_{\underline{\beta\gamma}|^g\delta} - \mathcal{P}^{\alpha}_{\underline{\beta\delta}|^g\gamma} + \mathcal{P}^{\epsilon}_{\underline{\beta\gamma}}\mathcal{P}^{\alpha}_{\underline{\beta\delta}} - \mathcal{P}^{\epsilon}_{\underline{\beta\delta}}\mathcal{P}^{\alpha}_{\underline{\epsilon\gamma}} + \mathcal{P}^{\alpha}_{\underline{\epsilon[\gamma}}\rho^{\epsilon}_{\underline{\beta\delta}]} - \mathcal{P}^{\epsilon}_{\underline{\beta[\gamma}}\rho^{\alpha}_{\underline{\epsilon\delta}]} \\ & - \frac{1}{N+1}\delta^{\alpha}_{\beta}\mathcal{P}^{\epsilon}_{[\underline{\gamma\epsilon}|^g\delta]} - \frac{1}{N-1}\Big(\delta^{\alpha}_{[\gamma}\mathcal{P}^{\epsilon}_{\underline{\zeta\epsilon}}\rho^{\zeta}_{\underline{\beta\delta}]} - \delta^{\alpha}_{[\gamma}\mathcal{P}^{\zeta}_{\underline{\beta\epsilon}}\rho^{\epsilon}_{\underline{\zeta\delta}]} - \delta^{\alpha}_{[\gamma}\mathcal{P}^{\zeta}_{\underline{\delta]\epsilon}}\rho^{\epsilon}_{\underline{\beta\zeta}}\Big) \\ & + \frac{1}{N-1}\Big(\delta^{\alpha}_{[\gamma}\mathcal{P}^{\epsilon}_{\underline{\beta\delta}]|^g\epsilon} - \delta^{\alpha}_{[\gamma}\mathcal{P}^{\epsilon}_{\underline{\beta\epsilon}|^g\delta]} + \delta^{\alpha}_{[\gamma}\mathcal{P}^{\epsilon}_{\underline{\beta\delta}]}\big(\mathcal{P}^{\zeta}_{\underline{\epsilon\zeta}} - \rho^{\zeta}_{\underline{\epsilon\zeta}}\big) - \delta^{\alpha}_{[\gamma}\mathcal{P}^{\epsilon}_{\underline{\beta\zeta}}\mathcal{P}^{\zeta}_{\underline{\delta]\epsilon}}\Big) \\ & + \frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\mathcal{P}^{\epsilon}_{\underline{\beta\epsilon}|^g\delta]} - \frac{1}{N^2-1}\delta^{\alpha}_{[\gamma}\mathcal{P}^{\epsilon}_{\underline{\delta]\epsilon}|^g\beta}\end{aligned} \tag{53}$$

for the geometrical objects $\mathcal{AG}_2\overset{0}{W}{}^{\alpha}_{\beta\gamma\delta}$ *and* $\mathcal{AG}_2\overset{0}{W}{}^{g\,\alpha}_{\beta\gamma\delta}$ *given by* (43) and (47).

Proof. The difference of $L^{\alpha}_{\underline{\beta\gamma}} - \Gamma^{\alpha}_{\underline{\beta\gamma}}$ in the common reference system is the tensor, $\mathcal{P}^{\alpha}_{\underline{\beta\gamma}} = L^{\alpha}_{\underline{\beta\gamma}} - \Gamma^{\alpha}_{\underline{\beta\gamma}}$. Hence, the next equations hold:

$$\rho^{\alpha}_{\underline{\beta\gamma}|\delta} - \rho^{\alpha}_{\underline{\beta\gamma}|^g\delta} = \mathcal{P}^{\alpha}_{\underline{\epsilon\delta}}\rho^{\epsilon}_{\underline{\beta\gamma}} - \mathcal{P}^{\epsilon}_{\underline{\beta\delta}}\rho^{\alpha}_{\underline{\epsilon\gamma}} - \mathcal{P}^{\epsilon}_{\underline{\gamma\delta}}\rho^{\alpha}_{\underline{\beta\epsilon}}, \tag{54}$$

$$\rho^{\epsilon}_{\underline{\beta\delta}|\epsilon} - \rho^{\epsilon}_{\underline{\beta\delta}|^g\epsilon} = \mathcal{P}^{\epsilon}_{\underline{\zeta\epsilon}}\rho^{\zeta}_{\underline{\beta\delta}} - \mathcal{P}^{\zeta}_{\underline{\beta\epsilon}}\rho^{\epsilon}_{\underline{\zeta\delta}} - \mathcal{P}^{\zeta}_{\underline{\delta\epsilon}}\rho^{\epsilon}_{\underline{\beta\zeta}}, \tag{55}$$

$$\rho^{\epsilon}_{\underline{\beta\epsilon}|\delta} - \rho^{\epsilon}_{\underline{\beta\epsilon}|^g\delta} = -\mathcal{P}^{\epsilon}_{\underline{\beta\delta}}\rho^{\zeta}_{\underline{\epsilon\zeta}}, \tag{56}$$

$$\rho^{\epsilon}_{[\underline{\beta\epsilon}|\delta]} - \rho^{\epsilon}_{[\underline{\beta\epsilon}|^g\delta]} = 0, \tag{57}$$

$$L^{\epsilon}_{\underline{\beta\epsilon}|\delta} - \Gamma^{\epsilon}_{\underline{\beta\epsilon}|^g\delta} = \mathcal{P}^{\epsilon}_{\underline{\beta\epsilon}|^g\delta} - 2\mathcal{P}^{\epsilon}_{\underline{\beta\delta}}\Gamma^{\zeta}_{\underline{\epsilon\zeta}}, \tag{58}$$

$$L^{\epsilon}_{\underline{\beta\epsilon}}L^{\zeta}_{\underline{\delta\zeta}} - \Gamma^{\epsilon}_{\underline{\beta\epsilon}}\Gamma^{\zeta}_{\underline{\delta\zeta}} = \Gamma^{\epsilon}_{\underline{\beta\epsilon}}\mathcal{P}^{\zeta}_{\underline{\delta\zeta}} + \Gamma^{\epsilon}_{\underline{\delta\epsilon}}\mathcal{P}^{\zeta}_{\underline{\beta\zeta}} + \mathcal{P}^{\epsilon}_{\underline{\beta\epsilon}}\mathcal{P}^{\zeta}_{\underline{\delta\zeta}}, \tag{59}$$

$$\overset{0}{R}{}^{\alpha}_{\beta\gamma\delta} - \overset{0}{R}{}^{g\,\alpha}_{\beta\gamma\delta} = \mathcal{P}^{\alpha}_{\underline{\beta\gamma}|^g\delta} - \mathcal{P}^{\alpha}_{\underline{\beta\delta}|^g\gamma} + \mathcal{P}^{\epsilon}_{\underline{\beta\gamma}}\mathcal{P}^{\alpha}_{\underline{\epsilon\delta}} - \mathcal{P}^{\epsilon}_{\underline{\beta\delta}}\mathcal{P}^{\alpha}_{\underline{\epsilon\gamma}}, \tag{60}$$

$$\overset{0}{R}_{\beta\gamma} - \overset{0}{R}{}^{g}{}_{\beta\gamma} = \mathcal{P}^{\epsilon}_{\underline{\beta\gamma}|^g\epsilon} - \mathcal{P}^{\epsilon}_{\underline{\beta\epsilon}|^g\gamma} + \mathcal{P}^{\epsilon}_{\underline{\beta\gamma}}\mathcal{P}^{\zeta}_{\underline{\epsilon\zeta}} - \mathcal{P}^{\epsilon}_{\underline{\beta\zeta}}\mathcal{P}^{\zeta}_{\underline{\gamma\epsilon}}, \tag{61}$$

$$\overset{0}{R}_{[\beta\gamma]} - \overset{0}{R}{}^{g}{}_{[\beta\gamma]} = -\mathcal{P}^{\epsilon}_{[\underline{\beta\epsilon}|^g\gamma]}. \tag{62}$$

With respect to the expressions (54)–(62) substituted in differences $\overset{0}{\mathcal{W}}{}^{\alpha}_{\beta\gamma\delta} - \overset{0}{\mathcal{W}}{}^{g}{}^{\alpha}_{\beta\gamma\delta}$, and $\overset{0}{W}{}^{\alpha}_{\beta\gamma\delta} - \overset{0}{W}{}^{g}{}^{\alpha}_{\beta\gamma\delta}$, together with the difference $L^{\alpha}_{\underline{\beta\gamma}} - \Gamma^{\alpha}_{\underline{\beta\gamma}} = \mathcal{P}^{\alpha}_{\underline{\beta\gamma}}$, the proof of this theorem is completed. □

4. Conclusions

In this research, we obtained different invariants for almost geodesic mappings of the second type defined on symmetric affine connection spaces and on Riemannian spaces, as well. The achieved results are as follows:

The results about invariants presented in [17] were reviewed. Through this review, the general formula of invariants for mappings of symmetric affine connection spaces was accentuated.

The review of results obtained in [17] was completed with the formula of invariants with respect to mappings whose deformation tensor is expressed in the form (27). Together with this formula, the definition of two types of invariants was reviewed [20].

One invariant of Thomas type and two invariants of Weyl type were obtained through a review of results from [17,18]. As the main result of this research, one invariant of Thomas type (the basic one) and two invariants of Weyl type (the basic and the derived ones) for second type almost geodesic mappings of the type π_2 were obtained.

The obtained invariants of Weyl type for second almost geodesic mappings were totalled if and only if the mapping had the property of reciprocity. Otherwise, these mappings were valued. The invariants of Thomas type for second type almost geodesic mappings were totalled. It was the last result achieved in this research.

Using the difference $\mathcal{AG}_2\mathcal{W}^{g}{}^{\alpha}_{\mu\nu\alpha} - \mathcal{AG}_2\bar{\mathcal{W}}^{g}{}^{\alpha}_{\mu\nu\alpha} = 0$, the variation of Einstein tensor $E_{\mu\nu} = R^{g}{}_{\mu\mu} - \frac{1}{2}g_{\underline{\mu\nu}}R^{g}$, $\delta E_{\mu\nu} = \bar{E}_{\mu\nu} - E_{\mu\nu}$, under the second type almost geodesic mapping $f : \mathbb{R}_4 \to \bar{\mathbb{R}}_4$ could be obtained.

In this study, the transformation rules of self dual affine connections and the corresponding transformation rules of affine connection coefficients and the corresponding curvature tensors under second type almost geodesic mappings were analysed. In future research, the analysis of invariants for second type almost geodesic mappings equipped with the affine connection ∇ and the corresponding dual affine connection ∇^* defined in [27,28] are going to be studied.

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References

1. Eisenhart, L.P. *Riemannian Geometry*; Princeton University Press: Princeton, NJ, USA, 1949.
2. Eisenhart, L.P. *Non-Riemannian Geometry*; American Mathematical Society: New York, NY, USA, 1927.
3. J. Mikeš, J.; Stepanova, E.; Vanžurová, A.; Bácsó, S.; Berezovski, V.E.; Chepurna, O.; Chodorová, M.; Chudá, H.; Gavrilchenko, M.L.; Haddad, M. *Differential Geometry of Special Mappings*, 2nd ed.; Palacký University Press: Olomouc, Czech Republic, 2019.

4. Mikeš, O.J.; Pokorna, O.; Starko, P.G. On Almost Geodesic Mappings $\pi_2(e)$ onto Riemannian Spaces. *Rend. Circ. Mat. Palermo* **2004**, *72*, 151–157.
5. Sinyukov, N.S. *Geodesic Mappings of Riemannian Spaces*; Nauka: Moscow, Russia, 1979.
6. Berezovski, V.; Cherevko, Y.; Hinterleitner, I.; Peška, P. Geodesic Mappings of Spaces with Affine Connections onto Generalized Symmetric and Ricci-Symmetric Spaces. *Mathematics* **2020**, *8*, 1560. [CrossRef]
7. Berezovskii, V.; Guseva, N.I.; Mikeš, J. Geodesic Mappings of Equiaffine and Ricci Symmetric Spaces. *Math. Notes* **2021**, *110*, 293–296. [CrossRef]
8. Berezovski, V.; Cherevko, I.; Hinterleitner, I.; Mikeš, J. Canonical Almost Geodesic Mappings of the First Type onto Generalized Ricci Symmetric Spaces. *Filomat* **2022**, *36*, 1089–1097. [CrossRef]
9. Sinyukov, N.S. Almost geodesic mapping of affine-connected and Riemannian spaces. *Itogi Nauk. Tekhniki. Seriya Probl. Geom. Tr. Geom. Semin.* **1982**, *13*, 3–26.
10. Berezovski, V.; Mikeš, J. On a Classification of Almost Geodesic Mappings of Affine Connection Spaces. *Acta Univ. Palacki. Olomuc. Fac. Rerum Nat. Math.* **1996**, *35*, 21–24.
11. Berezovski, V.; Mikeš, J. Almost Geodesic Mappings of Spaces with Affine Connection. *J. Math. Sci.* **2015**, *207*, 389–409. [CrossRef]
12. Berezovski, V.; Basco, S.; Cherevko, Y.; Mikeš, J. Canonical Almost Geodesic Mappings $\pi_2(e)$, $e = \pm 1$, of Spaces with Affine Connection onto m-Symmetric Spaces. *Miskolc Math. Notes* **2023**, *24*, 93–104. [CrossRef]
13. Ryparova, L.; Mikeš, J.; Peška, P. Almost Geodesic Curves and Geodesic Mappings. *J. Soviet Math.* **2023**, *221*, 93–103. (In Russian) [CrossRef]
14. Mikeš, J.; Berezovski, V.E.; Stepanova, E.; Chudá, H. Geodesic Mappings and Their Generalizations. *J. Math. Sci.* **2016**, *217*, 607–623. [CrossRef]
15. Mikeš, J.; Stepanova, E.; Vanžurova, A.; Bácsó, S.; Berezovski, V.E.; Chepurna, O.; Chodorová, M.; Chudá, H.; Gavrilchenko, M.L.; Haddad, M. *Differential Geometry of Special Mappings*; Palacky University: Olomouc, Czech Republic, 2015.
16. Mikeš, J.; Vanžurová, A.; Hinterleitner, I. *Geodesic Mappings and Some Generalizations*; Palacky University: Olomouc, Czechia, 2009.
17. Vesić, N.O. Basic invariants of geometric mappings. *Miskolc Math. Notes* **2020**, *21*, 473–487. [CrossRef]
18. Vesić, N.O.; Milenković, V.M.; Stanković, M.S. Two Invariants for Geometric Mappings. *Axioms* **2020**, *11*, 239. [CrossRef]
19. Simjanović, D.J. Tensor Calculus at Symmetric and Non-Symmetric Affine Connection Spaces with Application in the Fields of Linear Programming and Design of Fuzzy Controllers. Ph.D. Thesis, Faculty of Electronic Engineering, Niš, Serbia, 2024.
20. Simjanović, D.J.; Vesić, N.O. Novel invariants for almost geodesic mappings of the third type. *Miskolc Math. Notes* **2021**, *22*, 961–975. [CrossRef]
21. Peška, P.; Jukl, M.; Mikeš, J. Tensor Decompositions and Their Properties. *Mathematics* **2023**, *11*, 3638. [CrossRef]
22. Pistruil, M.I.; Kurbatova, I.N. On quasi-geodesic mappings of specialpseudo-Riemannian spaces. *Proc. Int. Geom. Center* **2022**, *15*, 121–139.
23. Rovenski, V.; Mikeš, J.; Stepanov, S. The Scalar Curvature of a Riemannian Almost Paracomplex Manifold and Its Conformal Transformations. *Mathematics* **2021**, *9*, 1379. [CrossRef]
24. Dodelson, S. *Modern Cosmology*; Fermi National Accelerator Laboratory, University of Chicago: Chicago, IL, USA, 2003.
25. Blau, M. *Lecture Notes on General Relativity*; Albert Einstein Center for Fundamental Physics, Universität Bern: Bern, Switzerland, 2015.
26. Bach, R. Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungstensorbegriffs. *Math. Z.* **1921**, *9*, 110–135. [CrossRef]
27. Amari, S.; Nagaoka, H. Method of Information Geometry. In *AMS Monograph*; Oxford University Press: Oxford, UK, 2000.
28. Nielsen, F. An Elementary Introduction to Information Geometry. *Entropy* **2020**, *22*, 1100. [CrossRef] [PubMed]