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Article

Special Geometric Objects in Generalized Riemannian Spaces

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Abstract: In this paper, we obtained the geometrical objects that are common in different definitions of the generalized Riemannian spaces. These objects are analogies to the Thomas projective parameter and the Weyl projective tensor. After that, we obtained some geometrical objects important for applications in physics.

Keywords: Riemannian space; invariant of Thomas type; invariant of Weyl type; invariant; pressure; energy density; spin tensor

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1. Introduction

In many research articles, books, and monographs, Riemannian and pseudo-Riemannian spaces have been studied. Some of the most significant authors who have developed the theory of these spaces are L. P. Eisenhart [1], N. S. Sinyukov [2], J. Mikeš and his research group [3–6], I. Hinterleitner [7,8], S. E. Stepanov [9], and many others.

An N-dimensional manifold $\mathcal{M}_N$ equipped with the regular symmetric metric tensor $g_{\underline{ij}}$, $g_{\underline{ij}} = g_{\underline{ji}}$ is [2–6,10,11] the (pseudo-)Riemannian space $\overset{g}{\mathbb{R}}_N$, where $\underline{ij}$ denote the symmetrization with respect to indices i and j. The affine connection (Levi–Civita connection) coefficients of the space $\overset{g}{\mathbb{R}}_N$ are the Christoffel symbols $\Gamma^i_{\underline{jk}}$, $\Gamma^i_{\underline{jk}} = \Gamma^i_{\underline{kj}}$, where $\Gamma^i_{\underline{jk}} = g^{\underline{i\alpha}}\Gamma_{\alpha.\underline{jk}}$, $\Gamma_{i.\underline{jk}} = \frac{1}{2}\big(g_{\underline{ji},k} - g_{\underline{jk},i} + g_{\underline{ik},j}\big)$.

One kind of covariant derivative with respect to the symmetric metric tensor $g_{\underline{ij}}$ is

$$a^i_{j|gk} = a^i_{j,k} + \Gamma^i_{\underline{\alpha k}} a^\alpha_j - \Gamma^\alpha_{\underline{jk}} a^i_\alpha, \tag{1}$$

for a tensor a^i_j of the type $(1,1)$, the Christoffel symbols $\Gamma^i_{\underline{jk}}$ and the partial derivative $\partial/\partial x^k$ are denoted by commas.

One Ricci identity [2–6] is founded with respect to the covariant derivative (1), $a^i_{j|gm|gn} - a_{j|gn|gm} = a^\alpha_j \overset{g}{R}{}^i_{\alpha mn} - a^i_\alpha \overset{g}{R}{}^\alpha_{jmn}$. With respect to this identity, the curvature tensor, the Ricci tensor, and the scalar curvature of the associated space $\overset{g}{\mathbb{R}}_N$ are obtained

$$\overset{g}{R}{}^i_{jmn} = \Gamma^i_{\underline{jm},n} - \Gamma^i_{\underline{jn},m} + \Gamma^\alpha_{\underline{jm}}\Gamma^i_{\underline{\alpha n}} - \Gamma^\alpha_{\underline{jn}}\Gamma^i_{\underline{\alpha m}}, \tag{2}$$

$$\overset{g}{R}_{ij} = \overset{g}{R}{}^\alpha_{ij\alpha} = \Gamma^\alpha_{\underline{ij},\alpha} - \Gamma^\alpha_{\underline{i\alpha},j} + \Gamma^\alpha_{\underline{ij}}\Gamma^\beta_{\underline{\alpha\beta}} - \Gamma^\alpha_{\underline{i\beta}}\Gamma^\beta_{\underline{j\alpha}}, \tag{3}$$

$$\overset{g}{R} = g^{\underline{\alpha\beta}}\overset{g}{R}_{\alpha\beta} = g^{\underline{\alpha\beta}}\big(\Gamma^\gamma_{\underline{\alpha\beta},\gamma} - \Gamma^\gamma_{\underline{\alpha\gamma},\beta} + \Gamma^\gamma_{\underline{\alpha\beta}}\Gamma^\delta_{\underline{\gamma\delta}} - \Gamma^\gamma_{\underline{\alpha\delta}}\Gamma^\delta_{\underline{\beta\gamma}}\big). \tag{4}$$

Based on research articles (L. P. Eisenhart, [10,11]), many researchers have studied and developed the theories of generalized Riemannian spaces and special kinds of them. The physical meaning of curvature tensors in the sense of Eisenhart's definition is presented in [12].

The studies about the affine connection spaces with torsion are started by the research of L. P. Eisenhart [13]. An N-dimensional manifold $\mathcal{M}_N$ equipped with the affine connection with torsion ∇, whose coefficients are L^i_{jk}, $L^i_{jk} \not\equiv L^i_{kj}$, for at least one pair of indices (j,k), is the (general) affine connection space $\mathbb{GA}_N$.

The symmetric and antisymmetric parts of the coefficients L^i_{jk} are

$$L^i_{\underline{jk}} = \frac{1}{2}\big(L^i_{jk} + L^i_{kj}\big) \quad \text{and} \quad L^i_{\underset{\vee}{jk}} = \frac{1}{2}\big(L^i_{jk} - L^i_{kj}\big).$$

The tensor $T^i_{jk} = 2L^i_{\underset{\vee}{jk}}$ is the torsion tensor for the space $\mathbb{GA}_N$.

The manifold $\mathcal{M}_N$ equipped with the torsion-free affine connection $\overset{0}{\nabla}$, whose coefficients are $L^i_{\underline{jk}}$, is the associated space $\mathbb{A}_N$ of the space $\mathbb{GA}_N$.

One kind of covariant derivative with respect to the affine connection $\overset{0}{\nabla}$ is [2–6]:

$$a^i_{j|k} = a^i_{j,k} + L^i_{\underline{\alpha k}}a^\alpha_j - L^\alpha_{\underline{jk}}a^i_\alpha.$$

The corresponding Ricci-type identity is $a^i_{j|mn} - a^i_{j|nm} = a^\alpha_j R^i_{\alpha mn} - a^i_\alpha R^\alpha_{jmn}$, where

$$R^i_{jmn} = L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + L^\alpha_{\underline{jm}}L^i_{\underline{\alpha n}} - L^\alpha_{\underline{jn}}L^i_{\underline{\alpha m}}, \tag{5}$$

is the curvature tensor of the space $\mathbb{A}_N$.

The Ricci tensor of the associated space $\mathbb{A}_N$ is

$$R_{ij} = R^\alpha_{ij\alpha} = L^\alpha_{\underline{ij},\alpha} - L^\alpha_{\underline{i\alpha},j} + L^\alpha_{\underline{ij}}L^\beta_{\underline{\alpha\beta}} - L^\alpha_{\underline{i\beta}}L^\beta_{\underline{j\alpha}}. \tag{6}$$

1.1. Generalized Riemannian Spaces

An N-dimensional manifold $\mathcal{M}_N$ equipped with the nonsymmetric metric tensor g_{ij} is [1] the generalized Riemannian space $\mathbb{G}\overset{g}{\mathbb{R}}_N$ (in the Eisenhart's sense).

The symmetric and antisymmetric parts of the metric tensor g_{ij} are

$$g_{\underline{ij}} = \frac{1}{2}\big(g_{ij} + g_{ji}\big) \quad \text{and} \quad g_{\underset{\vee}{ij}} = \frac{1}{2}\big(g_{ij} - g_{ji}\big).$$

We assume that the matrix $\big[g_{\underline{ij}}\big]$ is regular. In this case, g_{ij} is a metric tensor of some Riemannian space, which we denote as $\overset{g}{\mathbb{R}}_N$. Hence, the components $g^{\underline{ij}}$ of the contravariant metric tensor are $\big[g^{\underline{ij}}\big] = \big[g_{\underline{ij}}\big]^{-1}$. For this reason, the equality $g^{\underline{i\alpha}}g_{\underline{j\alpha}} = \delta^i_j$ holds for the

Kronecker δ-symbol δ^i_j. For this reason, the tensors $g_{\underline{ij}}$ and $g^{\underline{ij}}$ are used for lowering and raising the indices in the $\mathbb{G}\overset{g}{\mathbb{R}}_N$ space.

The affine connection coefficients of the $\mathbb{G}\overset{g}{\mathbb{R}}_N$ space are the generalized Christoffell symbols [1]:

$$\Gamma^i_{jk} = \frac{1}{2} g^{\underline{i\alpha}} \big(g_{j\alpha,k} - g_{jk,\alpha} + g_{\alpha k,j} \big).$$

One obtains that the symmetric and antisymmetric parts $\Gamma^i_{\underline{jk}} = \frac{1}{2}\big(\Gamma^i_{jk} + \Gamma^i_{kj}\big)$ and $\Gamma^i_{\underset{\vee}{jk}} = \frac{1}{2}\big(\Gamma^i_{jk} - \Gamma^i_{kj}\big)$ are

$$\Gamma^i_{\underline{jk}} = \frac{1}{2} g^{\underline{i\alpha}} \big(g_{\underline{j\alpha},k} - g_{\underline{jk},\alpha} + g_{\underline{\alpha k},j} \big),$$
$$\Gamma^i_{\underset{\vee}{jk}} = -\frac{1}{2} g^{\underline{i\alpha}} \big(g_{\underset{\vee}{\alpha j},k} + g_{\underset{\vee}{jk},\alpha} - g_{\underset{\vee}{\alpha k},j} \big).$$

The tensor $2\Gamma^i_{\underset{\vee}{jk}}$ is the torsion tensor for the space $\mathbb{G}\overset{g}{\mathbb{R}}_N$.

Motivated by the Einstein Metricity Condition

$$g_{ij\wr k} = g_{ij,k} - \Gamma^\alpha_{ik} g_{\alpha j} - \Gamma^\alpha_{kj} g_{i\alpha} = 0,$$

S. Ivanov and M. Lj. Zlatanović (see [14,15]) obtained the generalized Riemannian space $\mathbb{GR}_N$, whose metric tensor is g_{ij}, but the affine connection coefficients are

$$L^i_{jk} = \Gamma^i_{\underline{jk}} - \frac{1}{2} g^{\underline{i\alpha}} \big(T_{j\alpha k} + T_{k\alpha j} + g_{\underline{k\alpha}\underset{1}{|}j} + g_{\underline{\alpha j}\underset{1}{|}k} - g_{\underline{jk}\underset{1}{|}\alpha} \big) + T^i_{jk}, \tag{7}$$

for $g_{\underline{ij}\underset{1}{|}k} = g_{\underline{ij},k} - L^\alpha_{ik} g_{\underline{\alpha j}} - L^\alpha_{jk} g_{\underline{i\alpha}}$ and the torsion tensor T^i_{jk}, $T^i_{jk} = -T^i_{kj}$.

The curvature tensor and the Ricci tensor of the associated space $\mathbb{R}_N$ are given by ((5) and (6)). The scalar curvature of the associated space $\mathbb{R}_N$ is

$$R = g^{\underline{\alpha\beta}} \big(L^\gamma_{\underline{\alpha\beta},\gamma} - L^\gamma_{\underline{\alpha\gamma},\beta} + L^\gamma_{\underline{\alpha\beta}} L^\delta_{\underline{\gamma\delta}} - L^\gamma_{\underline{\alpha\delta}} L^\delta_{\underline{\beta\gamma}} \big), \tag{8}$$

for the corresponding affine connection coefficients $L^i_{\underline{jk}}$.

The $\mathbb{GR}_N$ space obtained and used in [14,15] is a special kind of affine connection space $\mathbb{GA}_N$ in Eisenhart's sense [13].

1.2. Mappings of Space $\mathbb{A}_N$

Invariants of different mappings are significant objects in mathematical research. Unlike in the theory of fixed points, where the existence of an object whose value does not change under the action of a function is noted [16,17], in differential geometry, specific geometric objects are determined that do not change under the action of different mappings [2–6,18,19].

The generalized Riemannian space $\mathbb{G}\overset{g}{\mathbb{R}}_N$ in the Eisenhart's sense [1] is the special case of the affine connection space $\mathbb{GA}_N$ (see [13]).

A diffeomorphism $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$, in which the affine connection $\overset{0}{\nabla}$ of the space $\mathbb{A}_N$ transforms to the affine connection $\overset{0}{\overline{\nabla}}$ of the space $\overline{\mathbb{A}}_N$ is *the mapping* of the space $\mathbb{A}_N$.

If the mapping f transforms the affine connection coefficients $L^i_{\underline{jk}}$ which correspond to the affine connection $\overset{0}{\nabla}$ of the space $\mathbb{A}_N$ to the affine connection coefficients $\overline{L}^i_{\underline{jk}}$ of the affine connection $\overset{0}{\overline{\nabla}}$ of the space $\overline{\mathbb{A}}_N$, the tensor

$$P^i_{\underline{jk}} = \overline{L}^i_{\underline{jk}} - L^i_{\underline{jk}}, \tag{9}$$

is the deformation tensor for the mapping f.

After adding a symmetric tensor $\pi^i_{\underline{jk}}$, $\pi^i_{\underline{jk}} = \pi^i_{\underline{kj}}$ of the type $(1,2)$ to the affine connection coefficient $L^i_{\underline{jk}}$, i.e., $L^i_{\underline{jk}} \to L^i_{\underline{jk}} + \pi^i_{\underline{jk}}$, one obtains the geometrical objects $\tilde{L}^i_{\underline{jk}}$ which are the coefficients of the corresponding (unique) affine connection $\overset{0}{\tilde{\nabla}}$. For this reason, any deformation tensor $P^i_{\underline{jk}}$ generates unique mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$.

Geodesic Mappings of Space $\mathbb{A}_N$

A curve $\ell = \big(\ell^i(t)\big)$ in the space $\mathbb{A}_N$ is a curve that satisfies the following system of partial differential equations [2–6]

$$\frac{\partial^2 \ell^i}{\partial t^2} + L^i_{\underline{\alpha\beta}} \frac{d\ell^\alpha}{dt} \frac{d\ell^\beta}{dt} = \rho \frac{d\ell^i}{dt},$$

for a scalar function ρ.

A mapping $f : \mathbb{A}_N \to \bar{\mathbb{A}}_N$, which any geodesic line of space $\mathbb{A}_N$ transmits to a geodesic line of the space $\bar{\mathbb{A}}_N$, is the geodesic mapping [2–6].

The basic equation of geodesic mapping f is

$$\bar{L}^i_{\underline{jk}} = L^i_{\underline{jk}} + \psi_j \delta^i_k + \psi_k \delta^i_j, \tag{10}$$

for a 1-form ψ_j.

N. S. Sinyukov [2] and J. Mikeš with his research group [3–6] contracted the equality (10) by i and k, expressed the 1-form ψ_j as $\psi_j = \dfrac{1}{N+1}\big(\bar{L}^\alpha_{\underline{j\alpha}} - L^\alpha_{\underline{j\alpha}}\big)$, substituted this expression into the basic Equation (12), and obtained that it is $\bar{T}g{}^i_{jk} = Tg{}^i_{jk}$ for

$$T^i_{jk} = L^i_{\underline{jk}} - \frac{1}{N+1}\big(L^\alpha_{\underline{j\alpha}} \delta^i_k + L^\alpha_{\underline{k\alpha}} \delta^i_j\big), \tag{11}$$

and the corresponding $\bar{T}^i_{\underline{jk}}$. The geometric object T^i_{jk} is the Thomas Projective parameter initially obtained by T. Thomas [20].

After that, N. S. Sinyukov [2] and J. Mikeš with his collaborators [3–6] applied H. Weyl's methodology [21] to obtain invariant from the transformation of curvature tensor R^i_{jmn} caused by the basic Equation (10):

$$\bar{R}^i_{jmn} = R^i_{jmn} + \big(\psi_{j|n} - \psi_j \psi_n\big)\delta^i_m - \big(\psi_{j|m} - \psi_j \psi_m\big)\delta^i_n + \big(\psi_{m|n} - \psi_{n|m}\big)\delta^i_j. \tag{12}$$

They contracted the relation (12) by i and j, and obtained that it is $\psi_{m|n} - \psi_{n|m} = R_{mn} - \bar{R}_{mn}$. The contraction of relation (12) by i and n gave

$$\psi_{j|m} - \psi_j \psi_m = \left(\frac{N}{N^2-1} R_{jm} + \frac{1}{N^2-1} R_{mj}\right) - \left(\frac{N}{N^2-1} \bar{R}_{jm} + \frac{1}{N^2-1} \bar{R}_{mj}\right). \tag{13}$$

When substituting the expression (13) into the Equation (12), they obtained the equality $\bar{W}^i_{jmn} = W^i_{jmn}$ for

$$W^i_{jmn} = R^i_{jmn} + \frac{1}{N+1}\delta^i_j(R_{mn} - R_{nm}) \\ + \frac{1}{N^2-1}\Big((NR_{jn} + R_{nj})\delta^i_n - (NR_{jm} + R_{mj})\delta^i_n\Big), \tag{14}$$

and the corresponding $\bar{W}^i_{jmn}$.

All of the traces $W^\alpha_{jm\alpha}$, $W^\alpha_{j\alpha n}$, $W^\alpha_{\alpha mn}$ vanish. For this reason, it is not possible to use the Weyl projective tensor to obtain an invariant for the geodesic mapping that is a linear monic polynomial of Ricci tensor R_{ij}.

The last presented methodology was used for obtaining invariants of mappings defined on a nonsymmetric affine connection space $\mathbb{GA}_N$. Many authors have obtained significant results in these generalizations. Some of them are M. S. Stanković [22–24], M. Lj. Zlatanović [22–26], S. M. Minčić [23], M. S. Najdanović [27], and many others.

Preferred Methodology for Obtaining Invariants of Mappings

Motivated by the basic Equation (10) for geodesic mapping $f : \mathbb{A}_N \to \bar{\mathbb{A}}_N$, with substituted $\psi_j = \frac{1}{N+1}(\bar{L}^\alpha_{\underline{j\alpha}} - L^\alpha_{\underline{j\alpha}})$,

$$\bar{L}^i_{\underline{jk}} = L^i_{\underline{jk}} + \frac{1}{N+1}(\bar{L}^\alpha_{\underline{j\alpha}}\delta^i_k + \bar{L}^\alpha_{\underline{k\alpha}}\delta^i_j) - \frac{1}{N+1}(L^\alpha_{\underline{j\alpha}}\delta^i_k + L^\alpha_{\underline{k\alpha}}\delta^i_j), \tag{10a}$$

the methodology for obtaining invariants of a mapping $F : \mathbb{A}_N \to \bar{\mathbb{A}}_N$ is developed in the following way [19]:

- The deformation tensor $\bar{L}^i_{\underline{jk}} - L^i_{\underline{jk}}$ is expressed as

$$\bar{L}^i_{jk} - L^i_{jk} = \bar{\omega}^i_{\underline{jk}} - \omega^i_{\underline{jk}}, \tag{15}$$

  for geometrical objects $\omega^i_{\underline{jk}} = \omega_{\underline{kj}} \in \mathbb{R}_N$, $\bar{\omega}^i_{\underline{jk}} = \bar{\omega}^i_{\underline{kj}} \in \bar{\mathbb{R}}_N$.
- In the next step, it was concluded that $\bar{L}^i_{\underline{jk}} - \bar{\omega}^i_{\underline{jk}} = L^i_{\underline{jk}} - \omega^i_{\underline{jk}}$. In this way, it was proved that the geometrical object $\mathcal{T}^i_{\underline{jk}} = L^i_{\underline{jk}} - \omega^i_{\underline{jk}}$ is an invariant for the mapping F. The geometrical object $\mathcal{T}^i_{\underline{jk}}$ is the associated basic invariant of Thomas type for the mapping F.
- In the next, based on the equality

$$\bar{T}^i_{\underline{jm},n} - \bar{T}^i_{\underline{jn},m} + \bar{T}^\alpha_{\underline{jm}}\bar{T}^i_{\underline{\alpha n}} - \bar{T}^\alpha_{\underline{jn}}\bar{T}^i_{\underline{\alpha m}} = T^i_{\underline{jm},n} - T^i_{\underline{jn},m} + T^\alpha_{\underline{jm}}T^i_{\underline{\alpha n}} - T^\alpha_{\underline{jn}}T^i_{\underline{\alpha m}},$$

  the next invariant for mapping F is obtained:

$$\mathcal{W}^i_{jmn} = R^i_{jmn} - \omega^i_{\underline{jm}|n} + \omega^i_{\underline{jn}|m} + \omega^\alpha_{\underline{jm}}\omega^i_{\underline{\alpha n}} - \omega^\alpha_{\underline{jn}}\omega^i_{\underline{\alpha m}}. \tag{16}$$

  The invariant $\mathcal{W}^i_{jmn}$ is the associated basic invariant of the Weyl type for the mapping F.
- After contracting the difference $\bar{\mathcal{W}}^i_{jmn} - \mathcal{W}^i_{jmn} = 0$, another invariant W^i_{jmn} for the mapping F was obtained.
- The trace $\mathcal{W}^\alpha_{ij\alpha}$ is a linear monic function of the Ricci tensor, unlike the trace $W^\alpha_{ij\alpha}$.

By using this methodology, we proved that two invariants with respect to the transformation of curvature tensor R^i_{jmn} may be obtained [28]. The trace $\mathcal{W}^\alpha_{ij\alpha}$ of the first of these two invariants is a monic linear polynomial of Ricci tensor R_{ij}.

In this paper, we focused on the associated invariants of Thomas and Weyl type of the third class for a special mapping. These invariants are (see the Equations (2.6, 2.9) in [19]):

$$\mathcal{T}^i_{(3).jk} = L^i_{\underline{jk}} + \frac{1}{2}P^i_{\underline{jk}}, \tag{17}$$

$$\mathcal{W}^i_{(3).jmn} = R^i_{jmn} + \frac{1}{2}P^i_{\underline{jm}|n} - \frac{1}{2}P^i_{\underline{jn}|m}, \tag{18}$$

where R^i_{jmn} is the curvature tensor of the associated space $\mathbb{A}_N$ and $P^i_{\underline{jk}} = \frac{1}{2}(P^i_{jk} + P^i_{kj})$.

1.3. Variations and Variational Derivatives

Let $f(x)$ be a continuously differentiable function defined on the interval $[a, b]$, $a \ll x \ll b$, and let $F[x, y, z]$ be a function of three variables. The expression

$$J[f] = \int_a^b F[x, f(x), f'(x)]dx,$$

where $f(x)$ ranges over the set of all continuously differentiable functions defined on the interval $[a, b]$, is a functional [29].

The variational (or functional) derivative $\delta J/\delta f$ of the operator $J[f]$ is [29,30]

$$\int \frac{\delta J}{\delta f}\phi(x)dx = \lim_{\varepsilon\to 0}\frac{J[f+\varepsilon\phi] - J[f]}{\varepsilon} = \left[\frac{d}{d\varepsilon}J[f+\varepsilon\varphi]\right]_{\varepsilon=0},$$

where ϕ is an arbitrary function.

For a scalar $\mathcal{L} = \mathcal{L}[f]$ in four-dimensional space and the corresponding operator $S = \int d^4x\mathcal{L}$, it satisfies the equalities

$$\frac{\delta S}{\delta f} = \int d^4x\frac{\delta\mathcal{L}}{\delta f}\delta f \quad \text{and} \quad \frac{\delta S_1 S_2}{\delta f} = \frac{\delta S_1}{\delta f}S_2 + S_1\frac{\delta S_2}{\delta f}.$$

In particular, it holds the equality $\dfrac{\delta \overset{g}{R}}{\delta g^{\underline{ij}}} = \overset{g}{R}_{ij}$.

1.4. Motivation

The Einstein–Hilbert action that corresponds to the symmetric metric tensor $g_{\underline{ij}}$ is [31]

$$\overset{g}{S} = \int d^4x\sqrt{|g|}(\overset{g}{R} - 2\Lambda + \mathcal{L}_M),$$

for a term $\mathcal{L}_M$ describing any matter fields appearing in the theory, the metric determinant $g = \det[g_{\underline{ij}}]$ and the cosmological constant $\Lambda = 1.1056 \times 10^{-52}m^{-2}$.

The Einstein's equations of motion are

$$\overset{g}{R}_{ij} - \frac{1}{2}\overset{g}{R}g_{\underline{ij}} + \frac{1}{2}\Lambda g_{\underline{ij}} = \overset{g}{T}_{ij}, \tag{19}$$

where $\overset{g}{T}_{ij}$ is the energy–momentum tensor.

In [32], the energy–momentum tensor $\overset{g}{T}_{ij}$ is expressed as

$$\overset{g}{T}_{ij} = \overset{g}{\rho}u_iu_j + q_iu_j + q_ju_i - (\overset{g}{p}h_{ij} + \pi_{ij}),$$

for the energy density $\overset{g}{\rho}$, the pressure $\overset{g}{p}$, the 4-velocity u_i, $u_\alpha u^\alpha = 1$, the 1-form q_i such that $u_\alpha q^\alpha = 0$, the trace-free tensor π_{ij} of type $(0,2)$ which, together with the 4-velocity u_i, satisfies the equality $\pi_{i\alpha} u^\alpha = 0$ and the tensor $h_{ij} = g_{\underline{ij}} - u_i u_j$.

The next equalities are satisfied [32]:

$$\overset{g}{\rho} = \overset{g}{T}_{\alpha\beta} u^\alpha u^\beta, \quad \text{and} \quad \overset{g}{p} = -\frac{1}{3}\overset{g}{T}{}^\alpha_\alpha + \frac{1}{3}\overset{g}{T}_{\alpha\beta} u^\alpha u^\beta.$$

The following equalities are satisfied [12,33]

$$\overset{g}{p} = \frac{1}{3}\overset{g}{R}_{\alpha\beta} u^\alpha u^\beta + \frac{1}{6}\overset{g}{R} - \Lambda \quad \text{and} \quad \overset{g}{\rho} = \overset{g}{R}_{\alpha\beta} u^\alpha u^\beta - \frac{1}{2}\overset{g}{R} + \Lambda, \tag{20}$$

in the reference system $u^i = g^{\underline{i\alpha}} u_\alpha$, such as

$$\overset{g}{p}_1 = \frac{1}{3}\overset{g}{R}_{11} + \frac{1}{6}\overset{g}{R} - \Lambda \quad \text{and} \quad \overset{g}{\rho}_1 = \overset{g}{R}_{11} - \frac{1}{2}\overset{g}{R} + \Lambda, \tag{21}$$

in the comoving reference system $u^i = g^{\underline{i\alpha}} u_\alpha = \delta^i_1$.

2. Main Results

With respect to Equation (7), we conclude the existence of the unique mapping $f : \mathbb{GR}_N \to \overset{g}{\mathbb{GR}}_N$ whose deformation tensor is

$$P^i_{jk} = \frac{1}{2} g^{\underline{i\alpha}} \big(T_{j\alpha k} + T_{k\alpha j} + g_{\underline{k\alpha}\underset{1}{|}j} + g_{\underline{j\alpha}\underset{1}{|}k} - g_{\underline{jk}\underset{1}{|}\alpha} \big) - T^i_{jk}.$$

In this section, we realize the next purposes of this paper: (1) To obtain the associated invariants of Thomas and Weyl type of the third class for the mapping $f : \mathbb{GR}_N \to \overset{g}{\mathbb{GR}}_N$, whose deformation tensor is given by (9); (2) To study the transformation rules of the pressure p and the energy density ρ with respect to the mapping $f : \mathbb{GR}_4 \to \overset{g}{\mathbb{GR}}_4$.

2.1. Invariants

After symmetrizing the Equation (7) by j and k, one obtains

$$\Gamma^i_{\underline{jk}} = L^i_{\underline{jk}} + \frac{1}{2} g^{\underline{i\alpha}} \Big(T_{j\alpha k} + T_{k\alpha j} + g_{\underline{k\alpha}\underset{1}{|}j} + g_{\underline{\alpha j}\underset{1}{|}k} - g_{\underline{jk}\underset{1}{|}\alpha} \Big). \tag{22}$$

From the last equation, after using the equalities

$$\Gamma^i_{\underline{jk}} - L^i_{\underline{jk}} = -\frac{1}{2}\overline{P}^i_{\underline{jk}} - \big(-\frac{1}{2} P^i_{\underline{jk}} \big) = \overline{\omega}^i_{(3).jk} - \omega^i_{(3).jk},$$

one obtains

$$\omega^i_{(3).jk} = -\frac{1}{4} g^{\underline{i\alpha}} \big(T_{j\alpha k} + T_{k\alpha j} + g_{\underline{k\alpha}\underset{1}{|}j} + g_{\underline{\alpha j}\underset{1}{|}k} - g_{\underline{jk}\underset{1}{|}\alpha} \big). \tag{23}$$

Based on $L^i_{jk} = L^i_{\underline{jk}} + \frac{1}{2} T^i_{jk}$, we conclude that $g_{\underline{ij}\underset{1}{|}k} = g_{\underline{ij}|k} - \frac{1}{2} T^\alpha_{ik} g_{\underline{\alpha j}} - \frac{1}{2} T^\alpha_{jk} g_{\underline{i\alpha}}$ such as

$$g^{\underline{i\alpha}} \big(g_{\underline{k\alpha}\underset{1}{|}j} + g_{\underline{j\alpha}\underset{1}{|}k} - g_{\underline{jk}\underset{1}{|}\alpha} \big) = g^{\underline{i\alpha}} \big(g_{\underline{k\alpha}|j} + g_{\underline{j\alpha}|k} - g_{\underline{jk}|\alpha} \big) - g^{\underline{i\alpha}} \big(T_{k\alpha j} + T_{j\alpha k} \big).$$

Hence, the geometrical object $\omega^i_{(3).jk}$ given by (23) reduces to

$$\omega^i_{(3).jk} = -\frac{1}{4} g^{\underline{i\alpha}} \big(g_{\underline{k\alpha}|j} + g_{\underline{j\alpha}|k} - g_{\underline{jk}|\alpha} \big). \tag{23a}$$

After substituting the expression (23a) into the Equations (17) and (18) multiplied by $\frac{4}{3}$, one obtains

$$\widetilde{\mathcal{T}}^i_{(3).jk} = L^i_{\underline{jk}} + \frac{1}{4}g^{\underline{i\alpha}}(g_{\underline{k\alpha}|j} + g_{\underline{j\alpha}|k} - g_{\underline{jk}|\alpha}), \tag{24}$$

$$\begin{aligned}\widetilde{W}^i_{(3).jmn} = R^i_{jmn} &- \frac{1}{3}g^{\underline{i\alpha}}(R_{j\alpha mn} - g_{[\underline{m\alpha}|jn]} + g_{j[\underline{m}|\alpha n]}) \\ &- \frac{1}{3}(g^{\underline{i\alpha}}_{|[m}g_{\underline{n}]\alpha|j} + g^{\underline{i\alpha}}_{|[m}g_{\underline{\alpha j}|n]} - g^{\underline{i\alpha}}_{|[m}g_{\underline{jn}]|\alpha}).\end{aligned} \tag{25}$$

The next theorem holds.

Theorem 1. *Let $f : \mathbb{GR}_N \to \overset{g}{\mathbb{GR}}_N$ be the mapping which transforms the generalized Riemannian space $\mathbb{GR}_N$ in the sense of Ivanov and Zlatanović's definition [14] of the generalized Riemannian space $\overset{g}{\mathbb{GR}}_N$ in the sense of Eisenhart's definition [1]. The geometrical objects $\widetilde{\mathcal{T}}^i_{(3).jk}$ and $\widetilde{W}^i_{(3).jmn}$, given by (24), (25), are the associated basic invariants of the Thomas and Weyl type of the third class for the mapping f.*

2.2. Physical Examples

In this part of the paper, we compare the pressures, energy densities, and state parameters generated by the spaces $\overset{g}{\mathbb{GR}}_4$ and $\mathbb{GR}_4$. We also assume that the equality $\frac{\delta R}{\delta g^{\underline{ij}}} = R_{ij}$ holds for the contravariant metric tensor $g^{\underline{ij}}$ obtained from the metric tensor $g_{\underline{ij}}$.

Let us consider the Einstein–Hilbert action

$$S = \int d^4x\sqrt{|g|}(R - 2\Lambda + \mathcal{L}_M), \tag{26}$$

for the scalar curvature of the associated Riemannian space $\mathbb{R}_4$ in the sense of the definition from [14,15].

As in [12], after varying the Einstein–Hilbert action (26) by $g^{\underline{ij}}$ we obtain

$$R_{ij} - \frac{1}{2}Rg_{\underline{ij}} + \frac{1}{2}\Lambda g_{\underline{ij}} = T_{ij}.$$

In a reference system $u^i = g^{\underline{i\alpha}}u_\alpha$, the pressure and the energy density are

$$p = \frac{1}{3}R_{\alpha\beta}u^\alpha u^\beta + \frac{1}{6}R - \Lambda, \tag{27}$$

$$\rho = R_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}R + \Lambda, \tag{28}$$

for the 4-velocity (u^1, u^2, u^3, u^4).

In the comoving reference system $u^i = g^{\underline{i\alpha}}u_\alpha = \delta^i_1$, the pressure and the energy densitygiven by (27), (28) reduce to

$$p_1 = \frac{1}{3}R_{11} + \frac{1}{6}R - \Lambda, \tag{29}$$

$$\rho_1 = R_{11} - \frac{1}{2}R + \Lambda. \tag{30}$$

With respect to Equations (22) and (23a), we get

$$\Gamma^i_{\underline{jk}} = L^i_{\underline{jk}} + \frac{1}{2}g^{\underline{i\alpha}}(g_{\underline{k\alpha}|j} + g_{\underline{j\alpha}|k} - g_{\underline{jk}|\alpha}) \equiv L^i_{\underline{jk}} - 2\omega^i_{(3).jk}, \tag{22a}$$

for the tensor $\omega^i_{(3).jk}$ given by (23a).

After substituting the expression (22a) in Equations (3), (4), (6), and (8), one obtains

$$\overset{g}{R}_{ij} = R_{ij} - 2\big(\omega^{\alpha}_{(3).ij|\alpha} - \omega^{\alpha}_{(3).i\alpha|j}\big) + 4\big(\omega^{\alpha}_{(3).ij}\omega^{\beta}_{(3).\alpha\beta} - \omega^{\alpha}_{(3).i\beta}\omega^{\beta}_{(3).j\alpha}\big), \tag{31}$$

$$\overset{g}{R} = R - 2g^{\underline{\alpha\beta}}\big(\omega^{\gamma}_{(3).\alpha\beta|\gamma} - \omega^{\gamma}_{(3).\alpha\gamma|\beta}\big) + 4g^{\underline{\alpha\beta}}\big(\omega^{\gamma}_{(3).\alpha\beta}\omega^{\delta}_{(3).\gamma\delta} - \omega^{\gamma}_{(3).\alpha\delta}\omega^{\delta}_{(3).\beta\gamma}\big). \tag{32}$$

As we concluded above, the symmetric part of the deformation tensor for the mapping $f : \mathbb{GR}_4 \to \overset{g}{\mathbb{GR}}_4$ is $P^{i}_{\underline{jk}} = -2\omega^{i}_{(3).jk}$. Hence, the geometrical object $\omega^{i}_{(3).jk}\omega^{p}_{(3).qr}$ is an invariant for the mapping f. The contravariant symmetric metric tensor $g^{\underline{ij}}$ is also an invariant for the mapping f.

If one substitutes Equations (31) and (32) in the expressions (19)–(21), (27)–(30), one will complete the proof for the next theorem.

Theorem 2. *The mapping $f : \mathbb{GR}_4 \to \overset{g}{\mathbb{GR}}_4$ transforms the energy–momentum tensor T_{ij} to the energy–momentum tensor $\overset{g}{T}_{ij}$ by the rule*

$$\overset{g}{T}_{ij} = T_{ij} - 2\omega^{\alpha}_{(3).ij|\alpha} + 2\omega^{\alpha}_{(3).i\alpha|j} + g^{\underline{\alpha\beta}}\big(\omega^{\gamma}_{(3).\alpha\beta|\gamma} - \omega^{\gamma}_{(3).\alpha\gamma|\beta}\big)g_{\underline{ij}}.$$

The following equalities $\mathbf{E_1} - \mathbf{E_4}$ *are equivalent*

$$\mathbf{E_1}: \ T_{ij} = \overset{g}{T}_{ij}, \qquad \mathbf{E_2}: \ \omega^{\alpha}_{(3).ij|\alpha} - \omega^{\alpha}_{(3).i\alpha|j} = \frac{1}{2}g^{\underline{\alpha\beta}}\big(\omega^{\gamma}_{(3).\alpha\beta|\gamma} - \omega^{\gamma}_{(3).\alpha\gamma|\beta}\big)g_{\underline{ij}},$$

$$\mathbf{E_3}: \ \omega^{\alpha}_{(3).ij|\alpha} = \omega^{\alpha}_{(3).i\alpha|j}, \qquad \mathbf{E_4}: \ g^{\underline{\alpha\beta}}\big(\omega^{\gamma}_{(3).\alpha\beta|\gamma} - \omega^{\gamma}_{(3).\alpha\gamma|\beta}\big) = 0.$$

The pressures p and $\overset{g}{p}$ obtained with respect to the spaces $\mathbb{GR}_4$ and $\overset{g}{\mathbb{GR}}_4$ satisfy the equation

$$\overset{g}{p} = p - \frac{1}{3}\big(\omega^{\gamma}_{(3).\alpha\beta|\gamma} - \omega^{\gamma}_{(3).\alpha\gamma|\beta}\big)\cdot\big(2u^{\alpha}u^{\beta} + g^{\underline{\alpha\beta}}\big). \tag{33}$$

In the comoving reference system $u^i = g^{\underline{i\alpha}}u_\alpha = \delta^i_1$, the Equation (33) reduces to

$$\overset{g}{p}_1 = p_1 - \frac{2}{3}\big(\omega^{\alpha}_{(3).11|\alpha} - \omega^{\alpha}_{(3).1\alpha|1}\big) - \frac{1}{3}g^{\underline{\alpha\beta}}\big(\omega^{\gamma}_{(3).\alpha\beta|\gamma} - \omega^{\gamma}_{(3).\alpha\gamma|\beta}\big).$$

The pressure p is an invariant for the mapping $f : \mathbb{GR}_4 \to \overset{g}{\mathbb{GR}}_4$ if and only if

$$0 = \big(\omega^{\gamma}_{(3).\alpha\beta|\gamma} - \omega^{\gamma}_{(3).\alpha\gamma|\beta}\big)\cdot\big(2u^{\alpha}u^{\beta} + g^{\underline{\alpha\beta}}\big). \tag{34}$$

In the comoving reference system $u^i = g^{\underline{i\alpha}}u_\alpha = \delta^i_1$, the condition (34) reduces to

$$\omega^{\alpha}_{(3).11|\alpha} - \omega^{\alpha}_{(3).1\alpha|1} = -\frac{1}{2}g^{\underline{\alpha\beta}}\big(\omega^{\gamma}_{(3).\alpha\beta|\gamma} - \omega^{\gamma}_{(3).\alpha\gamma|\beta}\big).$$

The energy densities ρ and $\overset{g}{\rho}$ obtained with respect to the spaces $\mathbb{GR}_4$ and $\overset{g}{\mathbb{GR}}_4$ satisfy the equation

$$\overset{g}{\rho} = \rho - \big(\omega^{\gamma}_{(3).\alpha\beta|\gamma} - \omega^{\gamma}_{(3).\alpha\gamma|\beta}\big)\cdot\big(2u^{\alpha}u^{\beta} - g^{\underline{\alpha\beta}}\big).$$

In the comoving reference system $u^i = g^{\underline{i\alpha}}u_\alpha = \delta^i_1$, the Equation (33) reduces to

$$\overset{g}{\rho}_1 = \rho_1 - 2\big(\omega^{\alpha}_{(3).11|\alpha} - \omega^{\alpha}_{(3).1\alpha|1}\big) + g^{\underline{\alpha\beta}}\big(\omega^{\gamma}_{(3).\alpha\beta|\gamma} - \omega^{\gamma}_{(3).\alpha\gamma|\beta}\big).$$

The energy density ρ is an invariant for the mapping $f : \mathbb{GR}_4 \to \overset{g}{\mathbb{GR}}_4$ if and only if

$$0 = \big(\omega^{\gamma}_{(3).\alpha\beta|\gamma} - \omega^{\gamma}_{(3).\alpha\gamma|\beta}\big) \cdot \big(2u^{\alpha}u^{\beta} - g^{\underline{\alpha\beta}}\big). \tag{35}$$

In the comoving reference system $u^i = g^{\underline{i\alpha}}u_\alpha = \delta^i_1$, the condition (35) reduces to

$$\omega^{\alpha}_{(3).11|\alpha} - \omega^{\alpha}_{(3).1\alpha|1} = \frac{1}{2}g^{\underline{\alpha\beta}}\big(\omega^{\gamma}_{(3).\alpha\beta|\gamma} - \omega^{\gamma}_{(3).\alpha\gamma|\beta}\big).$$

The geometrical object $\omega^i_{(3).jk}$ used in this theorem is given by (23a).

2.3. Contorsion and Spin Tensors

The covariant contorsion tensor of space $\mathbb{GR}_N$ is

$$K_{ijk} = \frac{1}{2}g^{\underline{i\alpha}}\big(L_{\alpha.\underset{\vee}{jk}} - L_{j.\underset{\vee}{\alpha k}} + L_{k.\underset{\vee}{\alpha j}}\big).$$

The corresponding spin tensor is [34]

$$\sigma^i_{jk} = \frac{1}{\kappa}g^{\underline{i\alpha}}\big(L_{i.\underset{\vee}{j\alpha}} + \delta^i_j L^{\alpha}_{\underset{\vee}{k\alpha}} - \delta^i_k L^{\alpha}_{\underset{\vee}{j\alpha}}\big). \tag{36}$$

After lowering the index i in (36), we obtain the covariant spin tensor

$$\sigma_{ijk} = \frac{1}{\kappa}g^{\underline{\alpha\beta}}\big(g_{\underline{k\alpha}}L_{i.\underset{\vee}{j\beta}} + g_{\underline{ik}}L_{j.\underset{\vee}{\alpha\beta}} - g_{\underline{jk}}L_{k.\underset{\vee}{\alpha\beta}}\big) = \frac{1}{\kappa}L_{i.\underset{\vee}{jk}}.$$

3. Conclusions

In this paper, we connected different definitions of generalized Riemannian spaces through their corresponding mapping.

In Section 2.1, we obtained the associated invariants of Thomas and Weyl type for this mapping. The Purpose 1 of this paper is realized in this section.

In Section 2.2, we analyzed some physical terms and their changes with respect to transformation from one to another definition of the generalized Riemannian space. We obtained the necessary and sufficient conditions for these terms to be invariant under this transformation.

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