Annals of Pure and Applied Logic 175 (2024) 103486

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

journal homepage: www.elsevier.com/locate/apal

Full Length Article

Posets of copies of countable ultrahomogeneous tournaments

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A R T I C L E I N F O

Article history:
Received 15 October 2023
Received in revised form 22 May 2024
Accepted 3 June 2024
Available online 6 June 2024

MSC:
03C15
03C50
03E40
06A06

Keywords:
Ultrahomogeneous tournament
Random tournament
Dense local order
Poset of copies
Sacks forcing
σ-Closed forcing

A B S T R A C T

The *poset of copies* of a relational structure $\mathbb{X}$ is the partial order $\mathbb{P}(\mathbb{X}) := \langle\{Y \subset X : \mathbb{Y} \cong \mathbb{X}\}, \subset\rangle$ and each similarity of such posets (e.g. isomorphism, forcing equivalence = isomorphism of Boolean completions, $\mathbb{B}_{\mathbb{X}} := \mathrm{ro}\,\mathrm{sq}\,\mathbb{P}(\mathbb{X}))$ determines a classification of structures. Here we consider the structures from Lachlan's list of countable ultrahomogeneous tournaments: $\mathbb{Q}$ (the rational line), $\mathbb{S}(2)$ (the circular tournament), and $\mathbb{T}^\infty$ (the countable homogeneous universal tournament); as well as the ultrahomogeneous digraphs $\mathbb{S}(3)$, $\mathbb{Q}[\mathbb{I}_n]$, $\mathbb{S}(2)[\mathbb{I}_n]$ and $\mathbb{T}^\infty[\mathbb{I}_n]$ from Cherlin's list.

If $\mathbb{G}_{\mathrm{Rado}}$ (resp. $\mathbb{Q}_n$) denotes the countable homogeneous universal graph (resp. n-labeled linear order), it turns out that $\mathbb{P}(\mathbb{T}^\infty) \cong \mathbb{P}(\mathbb{G}_{\mathrm{Rado}})$ and that $\mathbb{P}(\mathbb{Q}_n)$ densely embeds in $\mathbb{P}(\mathbb{S}(n))$, for $n \in \{2,3\}$.

Consequently, $\mathbb{B}_{\mathbb{X}} \cong \mathrm{ro}\,(\mathbb{S} * \pi)$, where $\mathbb{S}$ is the poset of perfect subsets of $\mathbb{R}$ and π an $\mathbb{S}$-name such that $1_{\mathbb{S}} \Vdash$ "π is a separative, atomless and σ-closed forcing" (thus $1_{\mathbb{S}} \Vdash$ "$\pi \equiv_{forc} (P(\omega)/\mathrm{Fin})^+$", under CH), whenever $\mathbb{X}$ is a countable structure equimorphic with $\mathbb{Q}$, $\mathbb{Q}_n$, $\mathbb{S}(2)$, $\mathbb{S}(3)$, $\mathbb{Q}[\mathbb{I}_n]$ or $\mathbb{S}(2)[\mathbb{I}_n]$.

Also, $\mathbb{B}_{\mathbb{X}} \cong \mathrm{ro}\,(\mathbb{S} * \pi)$, where $1_{\mathbb{S}} \Vdash$ "π is an ω-distributive forcing", whenever $\mathbb{X}$ is a countable graph containing a copy of $\mathbb{G}_{\mathrm{Rado}}$, or a countable tournament containing a copy of $\mathbb{T}^\infty$, or $\mathbb{X} = \mathbb{T}^\infty[\mathbb{I}_n]$.

1. Introduction

If $\mathbb{X}$ and $\mathbb{Y}$ are relational structures of the same language we will write $\mathbb{X} \hookrightarrow \mathbb{Y}$ iff there is an embedding (isomorphism onto a substructure) $f : \mathbb{X} \to \mathbb{Y}$. By $\mathbb{P}(\mathbb{X})$ we denote the set $\{Y \subset X : \mathbb{Y} \cong \mathbb{X}\}$ of copies of $\mathbb{X}$ inside $\mathbb{X}$; the partial ordering $\langle\mathbb{P}(\mathbb{X}), \subset\rangle$ will be called the *poset of copies of* $\mathbb{X}$ and shortly denoted by $\mathbb{P}(\mathbb{X})$, whenever the context admits.

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https://doi.org/10.1016/j.apal.2024.103486

It is easy to see that the correspondence $\mathbb{X} \mapsto \mathbb{B}_\mathbb{X}$ (where $\mathbb{B}_\mathbb{X}$ is the Boolean completion of the separative quotient of the poset $\mathbb{P}(\mathbb{X})$, $\mathrm{ro}\,\mathrm{sq}\,\mathbb{P}(\mathbb{X})$) extends to a functor from the category of all relational structures and isomorphisms to its subcategory of all homogeneous complete Boolean algebras and, defining two relational structures $\mathbb{X}$ and $\mathbb{Y}$ to be similar iff $\mathbb{B}_\mathbb{X} \cong \mathbb{B}_\mathbb{Y}$, we obtain a coarse classification of relational structures (see [8]). The position of this similarity in the hierarchy of set-theoretical and model-theoretical similarities of structures was investigated in [6,9]; in particular, for relational structures $\mathbb{X}$ and $\mathbb{Y}$ we have:

$$\mathbb{X} \rightleftarrows \mathbb{Y} \Rightarrow \mathbb{P}(\mathbb{X}) \equiv_{forc} \mathbb{P}(\mathbb{Y}) \Leftrightarrow \mathbb{B}_\mathbb{X} \cong \mathbb{B}_\mathbb{Y}, \tag{1}$$

where $\rightleftarrows$ denotes the equimorphism (bi-embedability) relation ($\mathbb{X} \rightleftarrows \mathbb{Y}$ iff $\mathbb{X} \hookrightarrow \mathbb{Y}$ and $\mathbb{Y} \hookrightarrow \mathbb{X}$) and $\equiv_{forc}$ the forcing equivalence of posets. So, the mentioned classification of structures can be explored using the methods of set-theoretic forcing.

In this paper we continue the investigation of countable ultrahomogeneous relational structures in this context. We recall that a relational structure $\mathbb{X}$ is called *ultrahomogeneous* iff every isomorphism between finite substructures of $\mathbb{X}$ extends to an automorphism of $\mathbb{X}$. By (1), a statement concerning the algebra $\mathbb{B}_\mathbb{X}$ adjoined to a countable ultrahomogeneous structure $\mathbb{X}$ holds for all the structures from its equimorphism class. For example, if $\mathbb{Q}$ denotes the rational line, $\langle Q, <_\mathbb{Q}\rangle$, then $\mathbb{B}_\mathbb{Q} \cong \mathbb{B}_\mathbb{X}$, for each countable non-scattered linear order $\mathbb{X}$.

All the definitions and facts concerning ultrahomogeneous structures used in this paper can be found in the survey [15] of Macpherson. By $\mathbb{G}_{\mathrm{Rado}}$ we denote the Rado graph and by $\mathbb{Q}_n$ (for $n \in \mathbb{N}$) the countable ultrahomogeneous n-labeled linear order, that is the structure $\mathbb{Q}_n := \langle Q, <_\mathbb{Q}, A_1, \dots, A_n\rangle$, where $\{A_1, \dots, A_n\}$ is a partition of the set Q such that the sets A_i, $i \le n$, are dense in $\mathbb{Q}$.

In order to state the known results which will be used in this paper, by $\mathbb{S}$ we denote the Sacks perfect set forcing (the set of perfect subsets of $\mathbb{R}$ ordered by the inclusion) and, in order to avoid repetition, we introduce the following notation for two properties of a countable relational structure $\mathbb{X}$:

$\mathcal{P}_1$: $\mathbb{P}(\mathbb{X}) \equiv_{forc} \mathbb{S} * \pi$, for some $\mathbb{S}$-name π for a preorder, where $1_\mathbb{S} \Vdash$ "π is a separative, atomless and σ-closed forcing";

$\mathcal{P}_2$: $\mathbb{P}(\mathbb{X}) \equiv_{forc} \mathbb{S} * \pi$, for some $\mathbb{S}$-name π for a preorder, where $1_\mathbb{S} \Vdash$ "π is an ω-distributive forcing".

We recall that dense embeddings between posets preserve forcing equivalence. Thus, if $\mathbb{X}$ and $\mathbb{Y}$ are relational structures and the poset $\mathbb{P}(\mathbb{X})$ densely embeds into $\mathbb{P}(\mathbb{Y})$, then $\mathbb{X}$ has property $\mathcal{P}_1$ iff $\mathbb{Y}$ does (and similarly for $\mathcal{P}_2$). This argument will be used in several places in the text.

Fact 1.1. *Let* $\mathrm{sh}(\mathbb{S})$ *denote the size of the continuum in the Sacks extension (the cardinal* κ *such that* $1_\mathbb{S} \Vdash \mathfrak{c} = \check{\kappa}$*) and let* $\mathbb{X}$ *be a countable relational structure.*

(a) $\mathcal{P}_1$ *implies* $\mathcal{P}_2$*;*

(b) If $\mathcal{P}_1$ *is true and* $\mathrm{sh}(\mathbb{S}) = \aleph_1$*, then* $1_\mathbb{S} \Vdash$ *"*$\pi \equiv_{forc} (P(\omega)/\mathrm{Fin})^+$*";*

(c) CH and, more generally, the equality $\mathfrak{b} = \aleph_1$ *implies that* $\mathrm{sh}(\mathbb{S}) = \aleph_1$*.*

Proof. Since each σ-closed forcing is ω-distributive (a) is true. It is a folklore fact that under CH each separative, atomless and σ-closed forcing of size $\mathfrak{c}$ is forcing equivalent to $(P(\omega)/\mathrm{Fin})^+$. In the Sacks extension $V_\mathbb{S}[G]$ we have $|\pi_G| = \mathfrak{c} = \aleph_1$, because in π_G we can construct a copy of the binary tree ${}^{<\omega}2$ (since π_G is atomless) and take a lower bound for each of its ($\mathfrak{c}$-many) branches (since π_G is σ-closed). Thus (b) is true. For (c) see [17]. □

Theorem 1.2. *(a) Each countable linear order containing a copy of* $\mathbb{Q}$ *has property* $\mathcal{P}_1$ *[10].*

(b) Each countable n-labeled linear order containing a copy of $\mathbb{Q}_n$ *has property* $\mathcal{P}_1$ *[13].*

(c) Each countable graph containing a copy of $\mathbb{G}_{\text{Rado}}$ *has property* $\mathcal{P}_2$ *[4,11,12].*[1]

The aim of this paper is to complete the picture for countable ultrahomogeneous tournaments. We recall Lachlan's classification of these structures [14]: Each countable ultrahomogeneous tournament is isomorphic to one of the following:

- $\mathbb{Q}$, the rational line,
- $\mathbb{S}(2)$, the dense local order (the circular tournament),
- $\mathbb{T}^\infty$, the countable random (i.e. homogeneous universal) tournament.

In Sections 2 and 3 we show that $\mathbb{T}^\infty$ has $\mathcal{P}_2$ and that $\mathbb{S}(2)$ has $\mathcal{P}_1$ and in Section 4 we obtain similar results for infinitely many ultrahomogeneous digraphs from Cherlin's list [1]: $\mathbb{S}(3)$, $\mathbb{T}[I_n]$ and $I_n[\mathbb{T}]$, where $\mathbb{T} \in \{\mathbb{Q}, \mathbb{T}^\infty, \mathbb{S}(2)\}$ and $n \in \mathbb{N}$. More precisely, the main results of the paper are the following.

- $\mathbb{P}(\mathbb{T}^\infty) \cong \mathbb{P}(\mathbb{G}_{\text{Rado}})$ and, hence, $\mathbb{B}_{\mathbb{T}^\infty} \cong \mathbb{B}_{\mathbb{G}_{\text{Rado}}}$.
  Each countable tournament $\mathbb{X}$ containing a copy of $\mathbb{T}^\infty$ has property $\mathcal{P}_2$.
- $\mathbb{P}(\mathbb{Q}_2)$ densely embeds in $\mathbb{P}(\mathbb{S}(2))$ and, hence, $\mathbb{B}_{\mathbb{S}(2)} \cong \mathbb{B}_{\mathbb{Q}_2}$.
  Each countable tournament $\mathbb{X}$ equimorphic with $\mathbb{S}(2)$ has property $\mathcal{P}_1$.
- $\mathbb{P}(\mathbb{Q}_3)$ densely embeds in $\mathbb{P}(\mathbb{S}(3))$ and, hence, $\mathbb{B}_{\mathbb{S}(3)} \cong \mathbb{B}_{\mathbb{Q}_3}$.
  Each countable digraph $\mathbb{X}$ equimorphic with $\mathbb{S}(3)$ has property $\mathcal{P}_1$.

The following elementary fact will be used in the sequel.

Fact 1.3. *Let* $\mathbb{X} = \langle X, \rho \rangle$ *be a countable ultrahomogeneous relational structure of a finite language. Then*
(a) The theory $\text{Th}(\mathbb{X})$ *is* ω*-categorical and admits quantifier elimination;*
(b) $\mathbb{P}(\mathbb{X})$ *is equal to the set of domains of elementary substructures of* $\mathbb{X}$*.*

Proof. For (a) see [5], p. 350. If $A \in \mathbb{P}(\mathbb{X})$, then $\mathbb{A} \models \text{Th}(\mathbb{X})$ and, since by (a) $\text{Th}(\mathbb{X})$ is model complete, $\mathbb{A} \prec \mathbb{X}$. Conversely, if $\mathbb{A} = \langle A, \rho \upharpoonright A \rangle \prec \mathbb{X}$, then $\mathbb{A} \equiv \mathbb{X}$ and, since $\text{Th}(\mathbb{X})$ is ω-categorical, $\mathbb{A} \cong \mathbb{X}$, that is, $A \in \mathbb{P}(\mathbb{X})$. □

2. The random tournament

The Rado graph If $\langle G, \sim \rangle$ is a graph and $K \subset H \in [G]^{<\omega}$, let us define

$$G_K^H := \Big\{ v \in G \setminus H : \forall k \in K \, (v \sim k) \ \wedge \ \forall h \in H \setminus K \, (v \not\sim h) \Big\}.$$

(Clearly, $G_\emptyset^\emptyset = G$.) The *Rado graph*, $\mathbb{G}_{\text{Rado}}$, [16] (the Erdős-Rényi graph [2], the countable random graph) is the unique (up to isomorphism) countable homogeneous universal[2] graph and the Fraïssé limit of the amalgamation class of all finite graphs; see [3], where a proof of the following fact can be found.

Fact 2.1. *For a countable graph* $\mathbb{G} = \langle G, \sim \rangle$ *the following is equivalent*
(g1) $\mathbb{G} \cong \mathbb{G}_{\text{Rado}}$,

[1] In [11] and [12] it was proved that $\mathbb{P}(\mathbb{G}_{\text{Rado}}) \equiv_{forc} \mathbb{P} * \pi$, where $\mathbb{P}$ is a poset which adds a generic real, has the 2-localization property (and, hence, the Sacks property) has the $\aleph_0$-covering property (thus preserves ω_1) and does not produce splitting reals and π is a $\mathbb{P}$-name for a preorder such that $1_{\mathbb{P}} \Vdash$ "π is an ω-distributive forcing". The forcing equivalence $\mathbb{P}(\mathbb{G}_{\text{Rado}}) \equiv_{forc} \mathbb{S} * \pi$ from $\mathcal{P}_2$ was proved in [4].

[2] We recall that a countable graph (resp. tournament) is called *(countably) universal* iff it contains a copy of each countable graph (resp. tournament).

(g2) $G^H_K \neq \emptyset$, *whenever* $K \subset H \in [G]^{<\omega}$,
(g3) $|G^H_K| = \omega$, *whenever* $K \subset H \in [G]^{<\omega}$.

The random tournament If $\langle T, \to \rangle$ is a tournament, and $K \subset H \in [T]^{<\omega}$, let

$$T^H_K := \Big\{ v \in T \setminus H : \forall k \in K \; (k \to v) \; \wedge \; \forall h \in H \setminus K \; (v \to h) \Big\}.$$

(Clearly, $T^\emptyset_\emptyset = T$.) The *random tournament*, $\mathbb{T}^\infty$, is the unique (up to isomorphism) countable homogeneous universal tournament and the Fraïssé limit of the amalgamation class of all finite tournaments (see [3]).

Fact 2.2. *For a countable tournament* $\mathbb{T} = \langle T, \to \rangle$ *the following is equivalent*
(t1) $\mathbb{T} \cong \mathbb{T}^\infty$,
(t2) $T^H_K \neq \emptyset$, *whenever* $K \subset H \in [T]^{<\omega}$,
(t3) $|T^H_K| = \omega$, *whenever* $K \subset H \in [T]^{<\omega}$.

Proof. (t1) $\Rightarrow$ (t2). Let $\mathbb{T} = \langle T, \to \rangle \cong \mathbb{T}^\infty$, $K \subset H \in [T]^{<\omega}$ and $p \notin H$. Then $\mathbb{T}_0 := \langle H \cup \{p\}, \rho \rangle$, where

$$\rho = (\to \restriction H) \cup \{\langle k, p \rangle : k \in K\} \cup \{\langle p, h \rangle : h \in H \setminus K\},$$

is a finite tournament and, since the age of $\mathbb{T}$ is the class of all finite tournaments, there is an embedding $f : \mathbb{T}_0 \hookrightarrow \mathbb{T}$. Now the restriction $\varphi := f^{-1} \restriction f[H]$ is a finite partial isomorphism of $\mathbb{T}$ which maps $f[H]$ onto H and, by the ultrahomogeneity of $\mathbb{T}$ there is $F \in \mathrm{Aut}(\mathbb{T})$ such that $\varphi \subset F$. Let $v := F(f(p))$. For $k \in K$ we have $\langle k, p \rangle \in \rho$ and, hence, $\langle f(k), f(p) \rangle \in \to$, which implies $\langle F(f(k)), F(f(p)) \rangle \in \to$. Since $F(f(k)) = \varphi(f(k)) = f^{-1}(f(k)) = k$, we have $\langle k, v \rangle \in \to$. Similarly, $\langle v, h \rangle \in \to$, for all $h \in H \setminus K$, and, thus, $v \in T^H_K$.

(t2) $\Rightarrow$ (t3). Suppose that (t2) is true and that $T^H_K = \{v_1, \dots, v_n\}$. Then, by (t2) there is $v \in T_K^{H \cup \{v_1, \dots, v_n\}}$ and, hence, $v \in T^H_K$ and $v \notin H \cup \{v_1, \dots, v_n\}$, which is a contradiction.

(t3) $\Rightarrow$ (t1). Assuming (t3) we show first that for each $n \in \mathbb{N}$ each finite tournament $\mathbb{A}$ of size n embeds in $\mathbb{T}$. For $n = 1$ the statement is obviously true. Suppose that it is true for n and that $\mathbb{A} = \langle A, \rho \rangle$ is a tournament, where $A = \{a_1, \dots a_{n+1}\}$. Then for $\mathbb{A}_0 = \langle A_0, \rho \restriction A_0 \rangle$, where $A_0 = \{a_1, \dots a_n\}$, there is an embedding $f : \mathbb{A}_0 \hookrightarrow \mathbb{T}$ and if $K := \{i \leq n : \langle a_i, a_{n+1} \rangle \in \rho\}$, by (t3) there is $v \in T$ such that $f(a_i) \to v$, for each $i \in K$, and $v \to f(a_i)$, for all $i \in \{1, \dots, n\} \setminus K$. Thus $f[A_0] \cup \{v\}$ is a copy of $\mathbb{A}$ in $\mathbb{T}$.

Now we show that $\mathbb{T}$ has the 1-extension property. Let $\varphi : H \to T$ be a finite partial isomorphism, $v \in T \setminus H$ and $K := \{k \in H : k \to v\}$. By (t3) there is $w \in T$ such that $\varphi(k) \to w$, for all $k \in K$ and $w \to \varphi(h)$, for all $h \in H \setminus K$. Thus $\varphi \cup \{\langle v, w \rangle\}$ is a finite partial isomorphism of $\mathbb{T}$. $\square$

Theorem 2.3. $\mathbb{P}(\mathbb{T}^\infty) \cong \mathbb{P}(\mathbb{G}_{\mathrm{Rado}})$ *and, hence,* $\mathbb{B}_{\mathbb{T}^\infty} \cong \mathbb{B}_{\mathbb{G}_{\mathrm{Rado}}}$.
Each countable tournament containing a copy of $\mathbb{T}^\infty$ *has property* $\mathcal{P}_2$.

Proof. W.l.o.g., we suppose that $\mathbb{G}_{\mathrm{Rado}} = \langle \omega, \sim \rangle$ and define a binary relation $\to$ on the set ω in the following way: for $m, n \in \omega$ let

$$m \to n \Leftrightarrow (m < n \wedge m \sim n) \vee (m > n \wedge m \not\sim n). \tag{2}$$

Since the relations $\sim$ and $\sim^c := \omega^2 \setminus \sim$ are symmetric, by (2) we have

$$\to = (< \cap \sim) \cup (<^{-1} \cap \sim^c) \quad \text{and} \quad \to^{-1} = (<^{-1} \cap \sim) \cup (< \cap \sim^c) \tag{3}$$

Now we have: $\to \cap \Delta_\omega = \emptyset$ so the relation $\to$ is irreflexive, $\to \cap \to^{-1} = \emptyset$, and $\to$ is asymmetric and $\to \cup \to^{-1} = < \cup <^{-1} = \omega^2 \setminus \Delta_\omega$; thus the structure $\mathbb{T} := \langle \omega, \to \rangle$ is a tournament.

For a proof that $\mathbb{T} \cong \mathbb{T}^\infty$, we check (t2) of Fact 2.2. If $K \subset H \in [\omega]^{<\omega}$, then, by (g3) of Fact 2.1, there is $v \in G^H_K$ such that $v > h$, for all $h \in H$. Now, if $k \in K$, then $k < v$ and $k \sim v$ so, by (2), $k \to v$. If $h \in H \setminus K$, then $v > h$ and $v \not\sim h$ so, by (2) again, $v \to h$. Thus $v \in T^H_K \neq \emptyset$, (t2) is true and $\mathbb{T} \cong \mathbb{T}^\infty$ indeed.

In order to prove that

$$\mathbb{P}(\langle \omega, \sim \rangle) = \mathbb{P}(\langle \omega, \to \rangle) \tag{4}$$

we take first $A \in \mathbb{P}(\langle \omega, \sim \rangle)$ and show that the countable tournament $\langle A, \to\restriction A\rangle$ satisfies (t2) of Fact 2.2. So, if $K \subset H \in [A]^{<\omega}$, then, since $\langle A, \sim\restriction A\rangle \cong \mathbb{G}_{\mathrm{Rado}}$, by (g3) of Fact 2.1, there is $v \in A^H_K$ such that $v > h$, for all $h \in H$. Now, if $k \in K$, then $k < v$ and $k \sim v$ so, by (2), $k \to v$. If $h \in H \setminus K$, then $v > h$ and $v \not\sim h$ so, by (2) again, $v \to h$. Thus $\langle A, \to\restriction A\rangle$ satisfies (t2) and $\langle A, \to\restriction A\rangle \cong \mathbb{T}^\infty$, which means that $A \in \mathbb{P}(\langle \omega, \to \rangle)$.

Conversely, we take $A \in \mathbb{P}(\langle \omega, \to \rangle)$ and show that the graph $\langle A, \sim\restriction A\rangle$ satisfies (g2) of Fact 2.1. So, if $K \subset H \in [A]^{<\omega}$, then, since $\langle A, \to\restriction A\rangle \cong \mathbb{T}^\infty$, by (t3) of Fact 2.2, there is $v \in A^H_K$ such that $v > h$, for all $h \in H$. If $k \in K$, then $k < v$ and $k \to v$ so, by (2), $k \sim v$, that is $v \sim k$. If $h \in H \setminus K$, then $v > h$ and $v \to h$ so, by (2) again, $v \not\sim h$. Thus $\langle A, \sim\restriction A\rangle$ satisfies (g2) and $\langle A, \sim\restriction A\rangle \cong \mathbb{G}_{\mathrm{Rado}}$, which means that $A \in \mathbb{P}(\langle \omega, \sim \rangle)$.

So, since $\langle \omega, \to \rangle \cong \mathbb{T}^\infty$ by (4) we have $\mathbb{P}(\mathbb{T}^\infty) \cong \mathbb{P}(\mathbb{G}_{\mathrm{Rado}})$ and, hence, $\mathbb{P}(\mathbb{T}^\infty) \equiv_{forc} \mathbb{P}(\mathbb{G}_{\mathrm{Rado}})$. If $\mathbb{X}$ is a countable tournament and $\mathbb{T}^\infty \hookrightarrow \mathbb{X}$, then, by the universality of $\mathbb{T}^\infty$, $\mathbb{X} \hookrightarrow \mathbb{T}^\infty$, so $\mathbb{X} \rightleftarrows \mathbb{T}^\infty$ and, hence, $\mathbb{P}(\mathbb{X}) \equiv_{forc} \mathbb{P}(\mathbb{T}^\infty) \equiv_{forc} \mathbb{P}(\mathbb{G}_{\mathrm{Rado}})$ which, together with Theorem 1.2(c) implies that $\mathbb{X}$ has property $\mathcal{P}_2$. □

3. The dense local order

The countable homogeneous universal n-labeled linear order For $n \in \mathbb{N}$ let $L_n = \langle R, \alpha_1, \dots, \alpha_n \rangle$ be a relational language, where $\mathrm{ar}(R) = 2$ and $\mathrm{ar}(\alpha_i) = 1$, for $i \leq n$. We recall that the L_n-structures of the form $\mathbb{X} = \langle X, <, A_1, \dots, A_n \rangle$, where $<$ is a linear order on the set X and $\{A_1, \dots, A_n\}$ a partition of X, are called *n-labeled linear orders*. Since the L_n-structure $\mathbb{Q}_n$ is ultrahomogeneous, the L_n-theory $\mathcal{T}_n$ saying that an L_n-structure $\mathbb{X} = \langle X, <, A_1, \dots, A_n \rangle$ is a model of $\mathcal{T}_n$ iff $\langle X, < \rangle$ is a dense linear order without end-points and $\{A_1, \dots, A_n\}$ a partition of X into dense subsets of $\langle X, < \rangle$ is ω-categorical. Consequently we have $D \in \mathbb{P}(\mathbb{Q}_n)$ iff $\langle D, <_{\mathbb{Q}}\restriction D, A_1 \cap D, \dots, A_n \cap D \rangle \models \mathcal{T}_n$, that is

Fact 3.1. *$D \in \mathbb{P}(\mathbb{Q}_n)$ if and only if $\langle D, <_{\mathbb{Q}}\restriction D\rangle$ is dense linear order without end points and the sets $A_i \cap D$, for $i \in \{1, \dots, n\}$, are its dense subsets.*

The dense local order $\mathbb{S}(2)$ If $q_1, q_2 \in Q$ and $q_1 \neq q_2$, then, since $q_1 - q_2 \neq k\pi$, for all $k \in \mathbb{Z}$, $e^{q_1 i}$ and $e^{q_2 i}$ are different and non-antipodal points of the unit circle $S^1 := \{e^{ti} : t \in [0, 2\pi)\}$ in the complex plane and $S = \{e^{qi} : q \in Q\}$ is a dense subset of S^1. The *dense local order* is the tournament $\mathbb{S}(2) = \langle S, \to \rangle$, where

$$e^{q_1 i} \to e^{q_2 i} \quad \Leftrightarrow \quad q_2 - q_1 \in \bigcup_{k \in \mathbb{Z}} (2k\pi, 2k\pi + \pi), \tag{5}$$

which means that the shorter oriented path from $e^{q_1 i}$ to $e^{q_2 i}$ is the anticlockwise oriented one. In order to simplify notation let $L_2 =: \langle R, \alpha, \beta \rangle$.

Clearly, $\{A, B\}$ is a partition of the set S into the left and right part, where

$$A := \Big\{ e^{qi} : q \in \bigcup_{k \in \mathbb{Z}} (\tfrac{\pi}{2} + 2k\pi, \tfrac{3\pi}{2} + 2k\pi) \cap Q \Big\} \text{ and}$$
$$B := \Big\{ e^{qi} : q \in \bigcup_{k \in \mathbb{Z}} (\tfrac{3\pi}{2} + 2k\pi, \tfrac{5\pi}{2} + 2k\pi) \cap Q \Big\}.$$

So $\langle S, \to, A, B \rangle$ is an L_2-structure and the L_2-formula

$$\lambda(u,v) := \Big[\Big((\alpha(u)\wedge\alpha(v))\vee(\beta(u)\wedge\beta(v))\Big)\wedge R(u,v)\Big] \\ \vee\ \Big[\Big((\alpha(u)\wedge\beta(v))\vee(\beta(u)\wedge\alpha(v))\Big)\wedge R(v,u)\Big] \tag{6}$$

defines the tournament relation $\rho := \{\langle x,y\rangle \in S\times S : \langle S,\to,A,B\rangle \models \lambda[x,y]\}$ on the set S, which preserves $\to$ between the elements of the same part, and reverses $\to$ between the elements of different parts, namely,

$$\rho = \Big[\to\cap\Big((A\times A)\cup(B\times B)\Big)\Big]\cup\Big[\to^{-1}\cap\Big((A\times B)\cup(B\times A)\Big)\Big]. \tag{7}$$

Note that, by the comment above (7), the initial relation $\to$ is defined by the formula λ in the L_2-structure $\mathbb{Y} := \langle S,\rho,A,B\rangle$, that is

$$\forall x,y\in S\ \ \Big(x\to y \Leftrightarrow \langle S,\rho,A,B\rangle\models\lambda[x,y]\Big). \tag{8}$$

It is easy to see that $\langle S,\rho\rangle$ is a dense linear order without end points and that A and B are its dense subsets (see [14], p. 434), which means that the L_2-structure $\mathbb{Y} = \langle S,\rho,A,B\rangle$ is a model of $\mathcal{T}_2$ and, since $\mathcal{T}_2$ is an ω-categorical theory, $\mathbb{Y}\cong\mathbb{Q}_2$.

For $x,y\in S^1$, let $x^\frown y$ denote the set of elements of S belonging to the shorter arc determined by x and y and let $a(x)$ denote the antipodal point of x.

Theorem 3.2. *$\mathbb{P}(\mathbb{Q}_2)$ densely embeds in $\mathbb{P}(\mathbb{S}(2))$ and, hence, $\mathbb{B}_{\mathbb{S}(2)}\cong\mathbb{B}_{\mathbb{Q}_2}$.*
Each countable tournament equimorphic with $\mathbb{S}(2)$ has property $\mathcal{P}_1$.

Proof. Since $\mathbb{Y} := \langle S,\rho,A,B\rangle\cong\mathbb{Q}_2$ we have $\mathbb{P}(\mathbb{Q}_2)\cong\mathbb{P}(\mathbb{Y})$ and we show that $\mathbb{P}(\mathbb{Y})$ is a dense subset of $\mathbb{P}(\mathbb{S}(2))$. First we prove that $\mathbb{P}(\mathbb{Y})\subset\mathbb{P}(\mathbb{S}(2))$. So, if $D\in\mathbb{P}(\mathbb{Y})$, then there is an isomorphism

$$F:\langle S,\rho,A,B\rangle \to_{iso} \langle D,\rho\restriction D, A\cap D, B\cap D\rangle \tag{9}$$

and in order to prove that $D\in\mathbb{P}(\mathbb{S}(2))$ it remains to be shown that the mapping $F:\langle S,\to\rangle\to\langle D,\to\restriction D\rangle$ is an isomorphism.

Now for $x,y\in S$ we have: $x\to y$ iff (by (8)) $\langle S,\rho,A,B\rangle\models\lambda[x,y]$ iff (by (9)) $\langle D,\rho\restriction D,A\cap D,B\cap D\rangle\models\lambda[F(x),F(y)]$ iff (since λ is a Σ_0-formula and, thus (D,S)-absolute) $\langle S,\rho,A,B\rangle\models\lambda[F(x),F(y)]$ iff (by (8)) $F(x)\to F(y)$. Thus $F:\langle S,\to\rangle\to\langle D,\to\restriction D\rangle$ is an isomorphism, $D\in\mathbb{P}(\mathbb{S}(2))$ and we have proved that $\mathbb{P}(\mathbb{Y})\subset\mathbb{P}(\mathbb{S}(2))$.

Claim 3.3. *If $D\in\mathbb{P}(\mathbb{S}(2))$, then $\langle D,\rho\restriction D\rangle$ is a dense linear order with at most one end point and $A_1 := A\cap D$ and $B_1 := B\cap D$ are its dense subsets.*

Proof. By Fact 1.3(b), $D\in\mathbb{P}(\mathbb{S}(2))$ implies that $\mathbb{D} := \langle D,\to\restriction D\rangle$ is an elementary substructure of $\mathbb{S}(2)$. So, by the Tarski-Vaught theorem, in particular, for each formula $\theta(u,v,w)$ of the language $L_b = \langle R\rangle$, where R is a binary relational symbol, we have:

$$\forall x,y\in D\ \Big(\exists s\in S\ \ \mathbb{S}(2)\models\theta[x,y,s]\ \Rightarrow\exists z\in D\ \ \mathbb{D}\models\theta[x,y,z]\Big). \tag{10}$$

Now $\langle D,\rho\restriction D\rangle$ is a linear order and we prove that A_1 is its dense subset, that is

$$\forall x,y\in D\ \ \Big(x\rho y\Rightarrow\exists z\in A_1\ \ x\rho z\rho y\Big). \tag{11}$$

So, let $x,y\in D$ and $x\rho y$. Then, since ρ is a strict linear order, $\neg y\rho x$.

If $x, y \in A_1$, then by (7) we have $x \to y$. Since for $s \in x^\frown y$ we have $x \to s \to y$, by (10) there is $z \in D$ such that $x \to z \to y$. Now $z \in B_1$ would imply that $y\rho z\rho x$ and, hence, $y\rho x$, which is false. Thus $z \in A_1$ and, by (7), $x\rho z\rho y$.

If $x, y \in B_1$, then by (7) we have $x \to y$. Since for $s \in a(x)^\frown a(y)$ we have $y \to s \to x$, by (10) there is $z \in D$ such that $y \to z \to x$. $z \in B_1$ would imply that $y\rho z\rho x$ and, hence, $y\rho x$, which is false. Thus $z \in A_1$ and, by (7), $x\rho z\rho y$.

If $x \in A_1$ and $y \in B_1$, then by (7) we have $y \to x$. Since for $s \in x^\frown a(y)$ we have $y \to s$ and $x \to s$, by (10) there is $z \in D$ such that $y \to z$ and $x \to z$. Assuming that $z \in B_1$ we would have $y\rho z\rho x$ and, hence, $y\rho x$, which is false. Thus $z \in A_1$ and, by (7), $x\rho z\rho y$.

If $x \in B_1$ and $y \in A_1$, then by (7) we have $y \to x$ Since for $s \in a(x)^\frown y$ we have $s \to x$ and $s \to y$, by (10) there is $z \in D$ such that $z \to x$ and $z \to y$. Assuming that $z \in B_1$ we would have $y\rho z\rho x$, and, hence, $y\rho x$, which is false. Thus $z \in A_1$ and, by (7), $x\rho z\rho y$.

So A_1 is a dense subset of $\langle D, \rho \restriction D\rangle$ and the proof for B_1 is similar. This implies that $\langle D, \rho \restriction D\rangle$ is a dense linear order.

Suppose that there are $x = \min_{\langle D,\rho\restriction D\rangle} D$ and $y = \max_{\langle D,\rho\restriction D\rangle} D$. Then

$$\forall z \in D \setminus \{x, y\} \;\; x\rho z\rho y. \tag{12}$$

If $x \to y$, then, since $x\rho y$, by (7) we have $x, y \in A$ or $x, y \in B$; say $x, y \in A$. For $s \in a(y)^\frown x$ we have $s \to x$ and $s \to y$ and, by (10), there is $z \in D$ such that $z \to x$ and $z \to y$. Now, by (12) we have $x\rho z$ and $z\rho y$ and, since $z \to x$ and $z \to y$, by (7) we obtain $z \in B$ and $z \in A$, which is impossible. If $x, y \in B$ we obtain a contradiction in the same way.

If $y \to x$, then, since $x\rho y$, by (7) x and y are in different elements of the partition $\{A, B\}$; say $x \in A$ and $y \in B$. For $s \in y^\frown x$ we have $y \to s \to x$ and, by (10), there is $z \in D$ such that $y \to z \to x$. By (12) and (7), from $A \ni x\rho z$ and $z \to x$ it follows that $z \in B$ and from $z\rho y \in B$ and $y \to z$ it follows that $z \in A$ and we have a contradiction. If $y \in A$ and $x \in B$ we obtain a contradiction in the same way. □

Now we prove that $\mathbb{P}(\mathbb{Y})$ is a dense suborder of $\mathbb{P}(\mathbb{S}(2))$. If $D \in \mathbb{P}(\mathbb{S}(2))$, then, by Claim 3.3, $\langle D, \rho \restriction D\rangle$ is a dense linear order and $A_1 := D \cap A$ and $B_1 := D \cap B$ are its dense subsets. Let D' be the set obtained from D by deleting its end point, if it exists. Then $\langle D', \rho \restriction D'\rangle$ is a dense linear order without end points, $A_1' := D' \cap A$ and $B_1' := D' \cap B$ are its dense and disjoint subsets and, hence $\mathbb{D}' := \langle D', \rho \restriction D', A_1', B_1'\rangle \models \mathcal{T}_2$, which, since the theory $\mathcal{T}_2$ is ω-categorical, implies that $\mathbb{D}' \cong \mathbb{Y}$; so $D' \in \mathbb{P}(\mathbb{Y})$ and, clearly, $D' \subset D$. Thus $\mathbb{P}(\mathbb{Y})$ is dense in $\mathbb{P}(\mathbb{S}(2))$ and, hence, $\mathbb{P}(\mathbb{S}(2)) \equiv_{forc} \mathbb{P}(\mathbb{Y}) \cong \mathbb{P}(\mathbb{Q}_2)$ so $\mathbb{P}(\mathbb{S}(2)) \equiv_{forc} \mathbb{P}(\mathbb{Q}_2)$.

The second statement follows from the first, Theorem 1.2(b) and (1). □

4. The digraphs $\mathbb{S}(3)$, $\mathbb{T}[\mathbb{I}_n]$ and $\mathbb{I}_n[\mathbb{T}]$

The digraph $\mathbb{S}(3)$ Again we consider the subset $S =: \{e^{qi} : q \in Q\}$ of the unit circle S^1 in the complex plane. If $r : S^1 \to S^1$ is the rotation given by $r(e^{ti}) = e^{(t+\frac{2\pi}{3})i}$ and $x = e^{qi} \in S$, then $r(x), r^2(x) \notin S$, where $r^2(x) := r(r(x))$, and the points x, $r(x)$ and $r^2(x)$ are vertices of a equilateral triangle. If $L = \langle R\rangle$ is the language with one binary relational symbol, R, it is clear that the L_b-structure $\mathbb{S}(3) := \langle S, \to\rangle$, where $\to$ is the binary relation on S defined by

$$e^{q_1 i} \to e^{q_2 i} \;\;\Leftrightarrow\;\; q_2 - q_1 \in \textstyle\bigcup_{k\in\mathbb{Z}} (2k\pi, 2k\pi + \frac{2\pi}{3}), \tag{13}$$

is a digraph; in fact we have $x \to y$ iff $y \in x^\frown r(x)$, where for non-antipodal points $s, t \in S^1$ by $s^\frown t$ we denote the set of elements of S belonging to the shorter arc of S^1 determined by s and t. The digraph $\mathbb{S}(3)$ is not a tournament; namely the L_b-formula $\theta(u, v) := u \neq v \wedge \neg R(u, v) \wedge \neg R(v, u)$ defines the incomparability relation, $\|$, in $\mathbb{S}(3)$: for $x, y \in S$,

$$x \parallel y \Leftrightarrow x \neq y \wedge \neg x \to y \wedge \neg y \to x$$

and we have $x \parallel y$ iff $y \in r(x)^\frown r^2(x)$. In addition, $y \to x$ iff $y \in r^2(x)^\frown x$; so, $\{\Delta_S, \to, \to^{-1}, \parallel\}$ is a partition of the set $S \times S$, where $\Delta_S = \{\langle x, x\rangle : x \in S\}$ is the diagonal of S. $\mathbb{S}(3)$ is one of continuum may ultrahomogeneous digraphs [1].

For convenience, let $L_3 = \langle R, \alpha, \beta, \gamma\rangle$, where $\operatorname{ar}(\alpha) = \operatorname{ar}(\beta) = \operatorname{ar}(\gamma) = 1$. It is evident that $\{A, B, C\}$ is a partition of the set S, where

$$A := \Big\{e^{qi} : q \in \textstyle\bigcup_{k\in\mathbb{Z}}(\frac{3\pi}{6} + 2k\pi, \frac{7\pi}{6} + 2k\pi) \cap Q\Big\},$$
$$B := \Big\{e^{qi} : q \in \textstyle\bigcup_{k\in\mathbb{Z}}(\frac{7\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi) \cap Q\Big\},$$
$$C := \Big\{e^{qi} : q \in \textstyle\bigcup_{k\in\mathbb{Z}}(\frac{11\pi}{6} + 2k\pi, \frac{15\pi}{6} + 2k\pi) \cap Q\Big\},$$

and, clearly,

$$\langle A, \to\restriction A\rangle \cong \langle B, \to\restriction B\rangle \cong \langle C, \to\restriction C\rangle \cong \mathbb{Q}, \tag{14}$$

$$\Big((A \times C) \cup (C \times B) \cup (B \times A)\Big) \cap \to\ = \emptyset \quad \text{and} \tag{15}$$

$$\Big((C \times A) \cup (B \times C) \cup (A \times B)\Big) \cap \to^{-1}\ = \emptyset. \tag{16}$$

Now, $\langle S, \to, A, B, C\rangle$ is an L_3-structure, the L_3-formula

$$\begin{aligned}\lambda(u,v) := & \Big[\Big((\alpha(u)\wedge\alpha(v)) \vee (\beta(u)\wedge\beta(v)) \vee (\gamma(u)\wedge\gamma(v))\Big) \wedge R(u,v)\Big] \\ \vee & \Big[\Big((\alpha(u)\wedge\gamma(v)) \vee (\gamma(u)\wedge\beta(v)) \vee (\beta(u)\wedge\alpha(v))\Big) \wedge R(v,u)\Big] \\ \vee & \Big[\Big((\gamma(u)\wedge\alpha(v)) \vee (\beta(u)\wedge\gamma(v)) \vee (\alpha(u)\wedge\beta(v))\Big) \wedge \ \theta(u,v)\Big]\end{aligned}$$

defines a new binary relation τ on S

$$\begin{aligned}\tau = & \Big[\Big((A \times A) \cup (B \times B) \cup (C \times C)\Big) \cap \to\Big] \\ \cup & \Big[\Big((A \times C) \cup (C \times B) \cup (B \times A)\Big) \cap \to^{-1}\Big] \\ \cup & \Big[\Big((C \times A) \cup (B \times C) \cup (A \times B)\Big) \cap \parallel\Big]\end{aligned} \tag{17}$$

and $\langle S, \tau, A, B, C\rangle$ is an L_3-structure as well. By (17) we have

$$\begin{aligned}\tau^{-1} = & \Big[\Big((A \times A) \cup (B \times B) \cup (C \times C)\Big) \cap \to^{-1}\Big] \\ \cup & \Big[\Big((C \times A) \cup (B \times C) \cup (A \times B)\Big) \cap \to\Big] \\ \cup & \Big[\Big((A \times C) \cup (C \times B) \cup (B \times A)\Big) \cap \parallel\Big].\end{aligned} \tag{18}$$

For completeness we include a proof of the following well-known fact.

Fact 4.1. *(a)* $\langle S, \tau, A, B, C\rangle \cong \mathbb{Q}_3$*;*

(b) $\langle S, \to, A, B, C\rangle$ *and* $\langle S, \tau, A, B, C\rangle$ *are* Σ_0*-bi-definable* L_3*-structures.*

Proof. (a) Since $\to$, $\to^{-1}$ and $\|$ are irreflexive and pairwise disjoint binary relations on S, by (17) the relation τ is irreflexive and, by (17) and (18), $\tau \cap \tau^{-1} = \emptyset$; so the relation τ is asymmetric; so $\langle S, \tau \rangle$ is a digraph. In addition, by (14) - (18) we have $\tau \cup \tau^{-1} = (S \times S) \setminus \Delta_S$, which means that $\langle S, \tau \rangle$ is a tournament.

Suppose that the relation τ is not transitive. Then $x \tau y \tau z \tau x$, for some $x, y, z \in S$, and, by (14), x, y and z are not in the same of the sets A, B and C.

Suppose that two of these points belong to one of these sets, say $x, y \in A$, which implies that $x \to y$. If $z \in B$, then, by (17), $y \parallel z$ and $x \to z$ and, hence $y, z \in x^\frown r(x)$, which implies that $y \not\parallel z$ and we have a contradiction. If $z \in C$, then, by (17), $z \parallel x$ and $z \to y$ and, hence $x, z \in r^2(y)^\frown y$, which implies that $x \not\parallel z$ and we have a contradiction. In a similar way we show that whenever two of the points belong to one of the elements of the partition we obtain a contradiction.

Thus x, y and z are in different elements of the partition and by (17) we have: if $\langle x, y, z \rangle \in (A \times C \times B) \cup (B \times A \times C) \cup (C \times B \times A)$, then $x \to z \to y \to x$ so $\{x, y, z\}$ is a copy of the oriented triangle, C_3, in $\mathbb{S}(3)$, which is impossible; if $\langle x, y, z \rangle \in (A \times B \times C) \cup (B \times C \times A) \cup (C \times A \times B)$, then $x \parallel z \parallel y \parallel x$ and $\{x, y, z\}$ is a copy of the empty digraph, E_3, in $\mathbb{S}(3)$, which is impossible again.

A proof that A, B and C are dense sets in the linear order $\langle S, \tau \rangle$ follows from the proof of Claim 4.3 (take $D = S$). Suppose that $m = \min S$ and, say $m \in A$; but by (17) and (14) we have $\langle A, \tau \restriction A \rangle = \langle A, \to\restriction A \rangle \cong \mathbb{Q}$ and this is impossible. So $\langle S, \tau \rangle$ is a dense linear order without end points, $\langle S, \tau, A, B, C \rangle \models \mathcal{T}_3$ and, hence, $\langle S, \tau, A, B, C \rangle \cong \mathbb{Q}_3$.

(b) First, $\tau = \{\langle x, y \rangle \in S \times S : \langle S, \to, A, B, C \rangle \models \lambda[x, y]\}$ and we show that $\to = \{\langle x, y \rangle \in S \times S : \langle S, \tau, A, B, C \rangle \models \mu[x, y]\}$, where $\mu(u, v)$ is the L_3-formula

$$\mu(u,v) := \Big[\Big((\alpha(u) \wedge \alpha(v)) \vee (\beta(u) \wedge \beta(v)) \vee (\gamma(u) \wedge \gamma(v))\Big) \wedge\ R(u,v)\Big]$$
$$\vee\ \Big[\Big((\gamma(u) \wedge \alpha(v)) \vee (\beta(u) \wedge \gamma(v)) \vee (\alpha(u) \wedge \beta(v))\Big) \wedge \neg R(u,v)\Big],$$

that is, defining $U := A^2 \cup B^2 \cup C^2$, $V := (C \times A) \cup (B \times C) \cup (A \times B)$ and $W := (A \times C) \cup (C \times B) \cup (B \times A)$ we prove that

$$\to = (U \cap \tau) \cup (V \setminus \tau). \tag{19}$$

By (15) we have $\to = (U \cap \to) \cup (V \cap \to)$ and, by (17), $U \cap \tau = U \cap \to$. By (17) and (16) we have $V \setminus \tau = V \setminus \parallel = V \cap (\to \cup \to^{-1}) = V \cap \to$ so (19) is true. Since the formulas λ and μ are quantifier free, statement (b) is proved. $\square$

Theorem 4.2. *$\mathbb{P}(\mathbb{Q}_3)$ densely embeds in $\mathbb{P}(\mathbb{S}(3))$ and, hence, $\mathbb{B}_{\mathbb{S}(3)} \cong \mathbb{B}_{\mathbb{Q}_3}$.*
Each countable digraph equimorphic with $\mathbb{S}(3)$ has property $\mathcal{P}_1$.

Proof. Let $\mathbb{Y} := \langle S, \tau, A, B, C \rangle$. By Fact 4.1(a) we have $\mathbb{P}(\mathbb{Q}_3) \cong \mathbb{P}(\mathbb{Y})$ so it is sufficient to show that $\mathbb{P}(\mathbb{Y})$ is a dense subset of $\mathbb{P}(\mathbb{S}(3))$. We prove first that $\mathbb{P}(\mathbb{Y}) \subset \mathbb{P}(\mathbb{S}(3))$. So, if $D \in \mathbb{P}(\mathbb{Y})$, then there is an isomorphism

$$F : \langle S, \tau, A, B, C \rangle \to_{iso} \langle D, \tau \restriction D, A \cap D, B \cap D, C \cap D \rangle \tag{20}$$

and in order to prove that $D \in \mathbb{P}(\mathbb{S}(3))$ it remains to be shown that the mapping $F : \langle S, \to \rangle \to \langle D, \to\restriction D \rangle$ is an isomorphism. By Fact 4.1(b), the relation $\to$ is defined by the L_3-formula μ in the structure $\mathbb{Y}$, that is

$$\forall x, y \in S\ \Big(x \to y\ \Leftrightarrow\ \langle S, \tau, A, B, C \rangle \models \mu[x, y]\Big). \tag{21}$$

Now for $x, y \in S$ we have: $x \to y$ iff (by (21)) $\langle S, \tau, A, B, C\rangle \models \mu[x, y]$ iff (by (20)) $\langle D, \tau \restriction D, A\cap D, B\cap D, C\cap D\rangle \models \mu[F(x), F(y)]$ iff (since μ is a Σ_0-formula and, thus, (D, S)-absolute) $\langle S, \tau, A, B, C\rangle \models \mu[F(x), F(y)]$ iff (by (21)) $F(x) \to F(y)$. Thus $F : \langle S, \to\rangle \to \langle D, \to\restriction D\rangle$ is an isomorphism, $D \in \mathbb{P}(\mathbb{S}(3))$ and $\mathbb{P}(\mathbb{Y}) \subset \mathbb{P}(\mathbb{S}(3))$ indeed.

Claim 4.3. *If $D \in \mathbb{P}(\mathbb{S}(3))$, then $\langle D, \tau \restriction D\rangle$ is a dense linear order and the sets $A_1 := A \cap D$, $B_1 := B \cap D$ and $C_1 := C \cap D$ are dense in $\langle D, \tau \restriction D\rangle$.*

Proof. By Fact 1.3(b), if $D \in \mathbb{P}(\mathbb{S}(3))$, then $\mathbb{D} := \langle D, \to\restriction D\rangle \prec \mathbb{S}(3)$. So, by the Tarski-Vaught theorem, for each L_b-formula $\theta(u, v, w)$ we have:

$$\forall x, y \in D \ \Big(\exists s \in S \ \ \mathbb{S}(3) \models \theta[x, y, s] \ \Rightarrow \exists z \in D \ \ \mathbb{D} \models \theta[x, y, z]\Big). \tag{22}$$

By Fact 4.1(a) $\langle D, \tau \restriction D\rangle$ is a linear order and we prove that A_1 is its dense subset. So, assuming that $x, y \in D$ and $x\tau y$ we will find a $z \in A_1$ such that $x\tau z\tau y$.

If $x, y \in A_1$, then by (17) we have $x \to y$. Since for $s \in x^\frown y$ we have $x \to s \to y$, by (22), there is $z \in D$ such that $x \to z \to y$. Since $x, y \in A$, by (15) we have $z \notin B \cup C$, which implies that $z \in A_1$. Thus, by (17) we have $x\tau z\tau y$.

If $x, y \in B_1$, then by (17) we have $x \to y$. Since for $s \in r^2(x)^\frown r^2(y)$ we have $s \to x$ and $y \parallel s$, by (22) there is $z \in D$ such that $z \to x$ and $y \parallel z$. Since $x \in B$, by (15) we have $z \notin C$, and assuming that $z \in B$ we would have $y \not\parallel z$ (because $\langle B, \to\restriction B\rangle$ is a linear order). Thus $z \in A_1$ and, by (17), $\langle x, z\rangle \in (B\times A)\cap \to^{-1}\subset \tau$ and $\langle z, y\rangle \in (A \times B)\cap \parallel\subset \tau$. So we have $x\tau z\tau y$.

If $x, y \in C_1$, then by (17) we have $x \to y$. Since for $s \in r(x)^\frown r(y)$ we have $x \parallel s$ and $y \to s$, by (22) there is $z \in D$ such that $x \parallel z$ and $y \to z$. Since $y \in C$, by (15) we have $z \notin B$, and assuming that $z \in C$ we would have $x \not\parallel z$ (because $\langle C, \to\restriction C\rangle$ is a linear order). Thus $z \in A_1$ and, by (17), $\langle x, z\rangle \in (C \times A)\cap \parallel\subset \tau$ and $\langle z, y\rangle \in (A \times C)\cap \to^{-1}\subset \tau$. So we have $x\tau z\tau y$.

If $x \in A_1$, $y \in B_1$, then by (17) we have $x \parallel y$. Since for $s \in x^\frown r^2(y)$ we have $x \to s$ and $s \parallel y$, by (22) there is $z \in D$ such that $x \to z$ and $z \parallel y$. Since $x \in A$, by (15) we have $z \notin C$, and assuming that $z \in B$ we would have $z \not\parallel y$ (because $\langle B, \to\restriction B\rangle$ is a linear order). Thus $z \in A_1$ and, by (17), $\langle x, z\rangle \in (A \times A)\cap \to\subset \tau$ and $\langle z, y\rangle \in (A \times B)\cap \parallel\subset \tau$. So we have $x\tau z\tau y$.

If $x \in A_1$, $y \in C_1$, then by (17) we have $y \to x$. Since for $s \in x^\frown r(y)$ we have $x \to s$ and $y \to s$, by (22) there is $z \in D$ such that $x \to z$ and $y \to z$. Since $x \in A$, by (15) we have $z \notin C$; since $y \in C$, by (15) we have $z \notin B$, Thus $z \in A_1$ and, by (17), $\langle x, z\rangle \in (A \times A)\cap \to\subset \tau$ and $\langle z, y\rangle \in (A \times C)\cap \to^{-1}\subset \tau$. So, $x\tau z\tau y$.

If $x \in B_1$, $y \in C_1$, then by (17) we have $x \parallel y$. Since for $s \in r^2(x)^\frown r(y)$ we have $s \to x$ and $y \to s$, by (22) there is $z \in D$ such that $z \to x$ and $y \to z$. Since $x \in B$, by (15) we have $z \notin C$; since $y \in C$, by (15) we have $z \notin B$, Thus $z \in A_1$ and, by (17), $\langle x, z\rangle \in (B \times A)\cap \to^{-1}\subset \tau$ and $\langle z, y\rangle \in (A \times C)\cap \to^{-1}\subset \tau$. Thus, $x\tau z\tau y$.

If $x \in B_1$, $y \in A_1$, then by (17) we have $y \to x$. Since for $s \in r^2(x)^\frown y$ we have $s \to x$ and $s \to y$, by (22) there is $z \in D$ such that $z \to x$ and $z \to y$. Since $x \in B$, by (15) we have $z \notin C$, and since $y \in A$, by (15) we have $z \notin B$. Thus $z \in A_1$ and, by (17), $\langle x, z\rangle \in (B \times A)\cap \to^{-1}\subset \tau$ and $\langle z, y\rangle \in (A \times A)\cap \to\subset \tau$. So, $x\tau z\tau y$.

If $x \in C_1$, $y \in A_1$, then by (17) we have $x \parallel y$. Since for $s \in r(x)^\frown y$ we have $x \parallel s$ and $s \to y$, by (22) there is $z \in D$ such that $x \parallel z$ and $z \to y$. Since $y \in A$, by (15) we have $z \notin B$; and assuming that $z \in C$ we would have $x \not\parallel z$ (because $\langle C, \to\restriction C\rangle$ is a linear order). Thus $z \in A_1$ and, by (17), $\langle x, z\rangle \in (C \times A)\cap \parallel\subset \tau$ and $\langle z, y\rangle \in (A \times A)\cap \to\subset \tau$. So we have $x\tau z\tau y$.

If $x \in C_1$, $y \in B_1$, then by (17) we have $y \to x$. Since for $s \in r(x)^\frown r^2(y)$ we have $x \parallel s$ and $y \parallel s$, by (22) there is $z \in D$ such that $x \parallel z$ and $y \parallel z$. Since $\langle C, \to\restriction C\rangle$ and $\langle B, \to\restriction B\rangle$ are linear orders, assuming that $z \in C$ (resp. $z \in B$) we would have $x \not\parallel z$ (resp. $y \not\parallel z$). Thus $z \in A_1$ and, by (17), $\langle x, z\rangle \in (C \times A)\cap \parallel\subset \tau$ and $\langle z, y\rangle \in (A \times B)\cap \parallel\subset \tau$. So we have $x\tau z\tau y$.

Proofs that B_1 and C_1 are dense sets in the linear order $\langle D, \tau\rangle$ are similar. □

Now, if $D \in \mathbb{P}(\mathbb{S}(3))$, then, by Claim 4.3, $\langle D, \tau \upharpoonright D\rangle$ is a dense linear order and $A \cap D$, $B \cap D$ and $C \cap D$ are dense sets in $\langle D, \tau\rangle$. Let D' be the set obtained from D by deleting its end points, if they exist. Then $\langle D', \tau \upharpoonright D'\rangle$ is a dense linear order without end points and $\{A \cap D', B \cap D', C \cap D'\}$ is a partition of D' into three dense subsets of $\langle D', \tau \upharpoonright D'\rangle$. Thus $\mathbb{D}' := \langle D', \tau \upharpoonright D', A \cap D', B \cap D', C \cap D'\rangle$ is a substructure of $\mathbb{Y}$ and $\mathbb{D}' \models \mathcal{T}_3$, which, since the theory $\mathcal{T}_3$ is ω-categorical and, by Fact 4.1(a), $\mathbb{Y} \models \mathcal{T}_3$, implies that $\mathbb{D}' \cong \mathbb{Y}$. So $D' \in \mathbb{P}(\mathbb{Y})$, $D' \subset D$ and $\mathbb{P}(\mathbb{Y})$ is a dense suborder of $\mathbb{P}(\mathbb{S}(3))$ indeed. Thus $\mathbb{P}(\mathbb{S}(3)) \equiv_{forc} \mathbb{P}(\mathbb{Y}) \cong \mathbb{P}(\mathbb{Q}_3)$ and, hence, $\mathbb{P}(\mathbb{S}(3)) \equiv_{forc} \mathbb{P}(\mathbb{Q}_3)$.

The second statement follows from the first, Theorem 1.2(b) and (1). □

Wreath products $\mathbb{T}[\mathbb{I}_n]$ *and* $\mathbb{I}_n[\mathbb{T}]$. One subclass of the class of all ultrahomogeneous digraphs (Cherlin's list [1]) is described as follows. Let $\mathbb{T}$ be an ultrahomogeneous tournament (thus $\mathbb{T} \in \{\mathbb{Q}, \mathbb{T}^\infty, \mathbb{S}(2)\}$) and, for an integer $n \geq 2$, let $\mathbb{I}_n$ denote the digraph with n vertices and with no arrows. Then the digraphs

- $\mathbb{T}[\mathbb{I}_n]$ (obtained by replacement of each point of $\mathbb{T}$ by a copy of $\mathbb{I}_n$) and
- $\mathbb{I}_n[\mathbb{T}]$ (obtained by replacement of each point of $\mathbb{I}_n$ by a copy of $\mathbb{T}$)

are ultrahomogeneous, the L_b-formula $\varphi(u, v) := \neg R(u, v) \wedge \neg R(v, u)$ defines the "unrelatedness" binary relation $\sim$ on the domain and, hence, all automorphisms preserve $\sim$.

It is easy to see that all embeddings of $\mathbb{T}[\mathbb{I}_n] = \bigcup_{t \in T} I_n^t$ preserve the relation $\sim$ as well and hence, $\mathbb{P}(\mathbb{T}[\mathbb{I}_n]) = \{\bigcup_{t \in A} I_n^t : A \in \mathbb{P}(\mathbb{T})\} \cong \mathbb{P}(\mathbb{T})$. So, the digraphs $\mathbb{Q}[\mathbb{I}_n]$ and $\mathbb{S}(2)[\mathbb{I}_n]$ have property $\mathcal{P}_1$ while $\mathbb{T}^\infty[\mathbb{I}_n]$ has $\mathcal{P}_2$.

On the other hand, the digraphs $\mathbb{I}_n[\mathbb{T}]$ are disconnected and, by Theorem 5.2 of [7], $\mathbb{P}(\mathbb{I}_n[\mathbb{T}]) \cong \mathbb{P}(\mathbb{T})^n$. Thus, for example, the poset $\mathbb{P}(\mathbb{I}_n[\mathbb{S}(2)]) \equiv_{forc} (\mathbb{S} * \pi)^n$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgement

The authors would like to thank the referee for devoted and careful reading and constructive suggestions.

Both authors are supported by the Science Fund of the Republic of Serbia, Program IDEAS, Grant No. 7750027: Set-theoretic, model-theoretic and Ramsey-theoretic phenomena in mathematical structures: similarity and diversity–SMART. In addition, the second author is partially supported by grants from NSERC (455916) and CNRS (IMJ-PRG UMR7586).

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