



# Isoperiodic families of Poncelet polygons inscribed in a circle and circumscribed about conics from a confocal pencil

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## Abstract

Poncelet polygons inscribed in a circle and circumscribed about conics from a confocal family naturally arise in the analysis of the numerical range and Blaschke products. We examine the behaviour of such polygons when the inscribed conic varies through a confocal pencil and discover cases when each conic from the confocal family is inscribed in an  $n$ -polygon, which is inscribed in the circle, with the same  $n$ . Complete geometric characterization of such cases for  $n \in \{4, 6\}$  is given and proved that this cannot happen for other values of  $n$ . We establish a relationship of such families of Poncelet quadrangles and hexagons to solutions of a Painlevé VI equation.

**Keywords** Poncelet polygons · Elliptic curves · Cayley-type conditions · Isoperiodic confocal families · Painlevé VI equations · Okamoto transformations

**Mathematics Subject Classification** 14H70 · 37J70 · 34M55 · 37A10

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## 1 Introduction

Although Poncelet polygons have been occupying the attention of major mathematicians for more than a bicentennial, from Poncelet, Jacobi, Cayley, via Darboux and Lebesgue, to Griffiths, Harris, Berger, Kozlov, Narasimhan, and Hitchin, to mention a few, it seems that new, interesting and unexpected phenomena and connections related to Poncelet porism are still waiting to be discovered.

The main object of our paper are Poncelet polygons inscribed in a given circle and circumscribed about conics from a confocal family, which is a setting that emerges from the study of the numerical range and Blaschke products, see [5] and [21] and references therein.

In this work, we found that in some cases all such Poncelet polygons have the same number of sides, even as the inscribed conic changes in the confocal pencil. This new phenomenon substantially improves and modifies the understanding and intuition related to injectivity and monotonicity of induced rotational numbers. (More about rotational numbers can be found in e.g. [6, 8, 9, 13, 18].) We give elegant classical geometric characterisations of those families for  $n = 4$  and  $n = 6$  and show that no other natural  $n$  gives rise to such families.

We also discern both analogy and disparity of the obtained results with the case when the circumscribed conic is not a circle but an ellipse which also belongs to the confocal family. That case has been extensively studied in connection with the elliptical billiards and the Great Poncelet Theorem, see for example [1, 4, 6, 7, 10, 14, 19, 20] and references therein.

This paper is organised as follows. In Sect. 2, we review the Poncelet porism and analytic conditions for closure in our setting. In particular, we analyse those conditions for polygons with three, four and five sides.

At the beginning of Sect. 3, we introduce *isoperiodic families*. Each such family consists of a circle and a confocal family of conics, such that each polygon inscribed in the circle and circumscribed about any of the confocal conics is closed with  $n$  sides. Theorems 3.6 and 3.7 give a complete characterisation such isoperiodic families for  $n \in \{4, 6\}$ . Section 4 addresses the natural question of the existence of such families exist for other values of  $n$  and gives the negative answer in Theorem 4.4.

Section 5 provides a further study of Blaschke ellipses. In particular, in Theorem 5.5 we characterize the Blaschke ellipse as the only conic from a confocal pencil, *with both focal points in the interior of the unit disk*, which is 3-Poncelet inscribed in the unit circle. This theorem also answers negatively the following question:

*Is it possible to have a triangle inscribed in a circle and circumscribed about an ellipse intersecting the circle whose foci are within the circle?*

Similarly, in Theorem 5.11 we characterize the Blaschke ellipse as the only conic from a confocal pencil, *with both focal points in the interior of the unit disk*, which is 4-Poncelet inscribed in the unit circle.

In Sect. 6, the obtained isoperiodic confocal families are used for the construction of explicit solutions to a Painlevé VI equation, see Theorem 6.3 for  $n = 4$  and Theorem 6.6 and for  $n = 6$ .

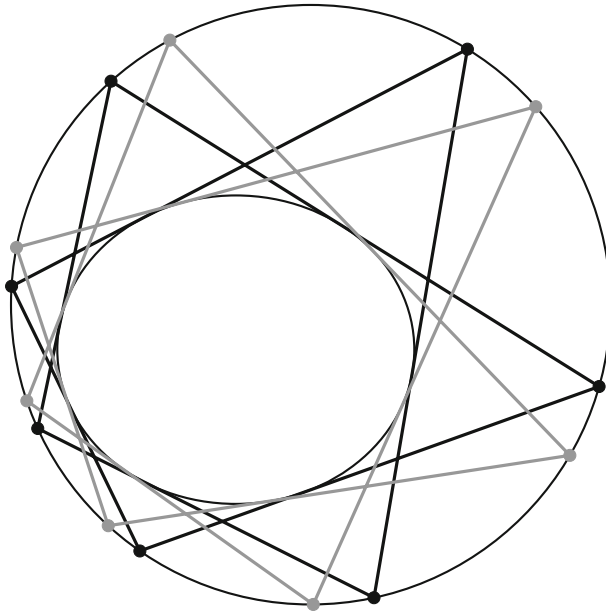


Fig. 1 Two heptagons inscribed in a circle and circumscribed about an ellipse

### 2 Poncelet porism and Cayley’s conditions

Given two smooth conics in a general position in the plane, the Poncelet porism states that existence of a closed polygonal line inscribed in one of those conics and circumscribed about the other one will imply the existence of infinitely many such polygons, which will all have the same number of sides, see Fig. 1.

Moreover, there will be a polygon with  $n$  sides inscribed in conic  $\Gamma$  and circumscribed about conic  $\mathcal{C}$  if and only if:

$$\begin{aligned}
 C_2 = 0, \quad & \begin{vmatrix} C_2 & C_3 \\ C_3 & C_4 \end{vmatrix} = 0, \quad \begin{vmatrix} C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \\ C_4 & C_5 & C_6 \end{vmatrix} = 0, \dots \quad \text{for } n = 3, 5, 7, \dots \\
 C_3 = 0, \quad & \begin{vmatrix} C_3 & C_4 \\ C_4 & C_5 \end{vmatrix} = 0, \quad \begin{vmatrix} C_3 & C_4 & C_5 \\ C_4 & C_5 & C_6 \\ C_5 & C_6 & C_7 \end{vmatrix} = 0, \dots \quad \text{for } n = 4, 6, 8, \dots,
 \end{aligned}$$

where  $\sqrt{\det(\mathcal{C} + \lambda\Gamma)} = C_0 + C_1\lambda + C_2\lambda^2 + C_3\lambda^3 + \dots$  is the Taylor expansion about  $\lambda = 0$ , and we used letters  $\mathcal{C}$  and  $\Gamma$  to denote the matrices of those two conics in the projective plane. This is also equivalent to the divisor relation  $n(P_0 - P_\infty) \sim 0$  on the elliptic curve  $\mu^2 = \det(\mathcal{C} + \lambda\Gamma)$ . The analytic condition for periodicity was derived by Cayley [2], see also [6, 10, 15] and references therein.

**Remark 2.1** Two non-degenerate conics are in a general position when their complexifications intersect at four distinct points. In that case, the cubic curve  $\mu^2 = \det(\mathcal{C} + \lambda\Gamma)$  is smooth. The conics have a point of tangency if and only if the polynomial  $\det(\mathcal{C} + \lambda\Gamma)$  has a multiple root, meaning that the curve has a singular point.

**Definition 2.2** If there is a polygon with  $n$  sides inscribed in  $\Gamma$  and circumscribed about  $C$ , we will say that  $\Gamma$  and  $C$  are an  $n$ -Poncelet pair or that  $C$  is  $n$ -Poncelet inscribed in  $\Gamma$ .

In this work, we will consider the case when  $\Gamma$  is a circle:

$$\Gamma: (x - x_0)^2 + (y - y_0)^2 = 1$$

and  $C = C(t)$  a conic from the following confocal family:

$$C(t) : \frac{x^2}{a-t} + \frac{y^2}{b-t} = 1, \quad a > b > 0.$$

As above, we will use the same letters to denote the matrices corresponding to those two conics:

$$\Gamma = \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ -x_0 & -y_0 & x_0^2 + y_0^2 - 1 \end{pmatrix}, \quad C(t) = \begin{pmatrix} \frac{1}{a-t} & 0 & 0 \\ 0 & \frac{1}{b-t} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We denote:

$$\begin{aligned} D_t(\lambda) &= (a-t)(b-t) \det(C(t) + \lambda\Gamma) \\ &= -(a-t)(b-t)\lambda^3 + \lambda^2 ((a-t)(y_0^2 - 1) + (b-t)(x_0^2 - 1)) \\ &\quad - (a-t)(b-t) + \lambda(x_0^2 + y_0^2 - 1 - (a-t) - (b-t)) - 1, \end{aligned} \tag{2.1}$$

and the Taylor series around  $\lambda = 0$ :

$$\sqrt{D_t(\lambda)} = C_0(t) + C_1(t)\lambda + C_2(t)\lambda^2 + C_3(t)\lambda^3 + \dots$$

Notice that the coefficients  $C_k(t)$  are polynomial in  $t$ , as well as in  $a, b, x_0^2, y_0^2$ . Thus, the Cayley conditions:

$$\begin{aligned} p_n(t) &:= \det \begin{pmatrix} C_2(t) & C_3(t) & \dots & C_{m+1}(t) \\ C_3(t) & C_4(t) & \dots & C_{m+2}(t) \\ \dots & \dots & \dots & \dots \\ C_{m+1}(t) & C_{m+2}(t) & \dots & C_{2m}(t) \end{pmatrix} = 0, \quad \text{for } n = 2m + 1, \\ p_n(t) &:= \det \begin{pmatrix} C_3(t) & C_4(t) & \dots & C_{m+2}(t) \\ C_4(t) & C_5(t) & \dots & C_{m+3}(t) \\ \dots & \dots & \dots & \dots \\ C_{m+2}(t) & C_{m+3}(t) & \dots & C_{2m+1}(t) \end{pmatrix} = 0, \quad \text{for } n = 2m + 2. \end{aligned} \tag{2.2}$$

define polynomials in  $t$ , denoted as  $p_n(t)$ , which are also polynomials in  $a, b, x_0^2, y_0^2$ .

Let us denote the corresponding family of curves as:

$$\mathcal{E}_t : \mu^2 = D_t(\lambda). \tag{2.3}$$

**Remark 2.3** The curve  $\mathcal{E}_t$  is elliptic whenever conic  $C(t)$  is non-degenerate and with no points of tangency with  $\Gamma$ , i.e. whenever  $t \notin \{a, b\}$  and all roots of  $D_t(\lambda)$  are simple. Thus, the curve  $\mathcal{E}_t$  is smooth for all but finitely many values of  $t \in \mathbf{C}$ .

Notice that, for  $t \in \{a, b\}$ , the curve  $\mathcal{E}_t$  is not cubic: its degree will be equal 2.

**Proposition 2.4** Circle  $\Gamma$  and conic  $C(t)$  are an  $n$ -Poncelet pair if and only if

$$p_n(t) = 0,$$

for smooth and non-smooth cubics  $\mathcal{E}_t$ , i.e. for any  $t \notin \{a, b\}$ , where  $p_n(t)$  is defined in (2.2).

**Proof** If  $\mathcal{E}_t$  is smooth, this is just a straightforward Cayley’s condition. Otherwise,  $D_t(\lambda)$  is a cubic polynomial with a double root, which is equivalent to conics  $\Gamma$  and  $\mathcal{C}(t)$  having at least one point of tangency. In this case, the conclusion follows from the analysis in [14] (see also [11]).  $\square$

Now, we are going to expand the above Cayley’s conditions for a few small values of  $n$ .

**Corollary 2.5** *Conic  $\mathcal{C}(t)$  is 3-Poncelet inscribed in  $\Gamma$  if and only if  $C_2(t) = 0$ :*

$$4t + 1 - 2(a + b) + (a - b)^2 - 2x_0^2(1 + a - b) - 2y_0^2(1 - a + b) + (x_0^2 + y_0^2)^2 = 0.$$

That equation has the unique solution in  $t$ , which we will denote as  $t_3$ :

$$t_3 = -\frac{1}{4} \left( 1 - 2(a + b) + (a - b)^2 - 2x_0^2(1 + a - b) - 2y_0^2(1 - a + b) + (x_0^2 + y_0^2)^2 \right). \tag{2.4}$$

**Corollary 2.6** *Conic  $\mathcal{C}(t)$  is 4-Poncelet inscribed in  $\Gamma$  if and only if  $C_3(t) = 0$ , i.e:*

$$\alpha_4 + \beta_4 t = 0,$$

where:

$$\begin{aligned} \alpha_4 &= 8ab + 4(x_0^2 + y_0^2 - a - b - 1)t_3, \\ \beta_4 &= 8t_3 + 4(1 - a - b - x_0^2 - y_0^2), \end{aligned}$$

and  $t_3$  is given by (2.4).

**Corollary 2.7** *Conic  $\mathcal{C}(t)$  is 5-Poncelet inscribed in  $\Gamma$  if and only if:*

$$\det \begin{pmatrix} C_2(t) & C_3(t) \\ C_3(t) & C_4(t) \end{pmatrix} = 0,$$

which is equivalent to:

$$\alpha_5 + \beta_5 t + \gamma_5 t^2 + \delta_5 t^3 = 0,$$

with

$$\begin{aligned} \alpha_5 &= -64t_3^3 + 128ab \left( a + b + 1 - x_0^2 - y_0^2 \right) t_3 - 256a^2 b^2, \\ \beta_5 &= 192t_3^2 + 128t_3 \left( (a + b)(x_0^2 + y_0^2 - a - b - 1) - 2ab \right) + 128ab \left( 3a + 3b + x_0^2 + y_0^2 - 1 \right), \\ \gamma_5 &= 64 \left( 6a + 6b - 2x_0^2 - 2y_0^2 - 1 \right) t_3 - 128 \left( a^2 + 4ab + ax_0^2 + ay_0^2 - a + b^2 + bx_0^2 + by_0^2 - b \right), \\ \delta_5 &= -256t_3 + 64 \left( 2a + 2b + 2x_0^2 + 2y_0^2 - 1 \right), \end{aligned}$$

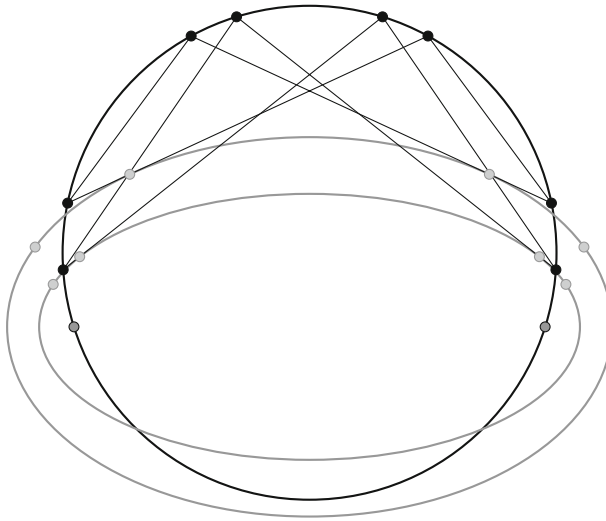
and  $t_3$  is given by (2.4).

### 3 Isoperiodic families of Poncelet polygons

In this section, we are searching for a circle  $\Gamma$  and a confocal pencil  $\mathcal{C}(t)$ , such that  $\mathcal{C}(t)$  is  $n$ -Poncelet inscribed in  $\Gamma$  for each  $t \notin \{a, b\}$ , with a fixed  $n$ . In such a case, we will say that  $(\Gamma, \mathcal{C}(t))$  is an isoperiodic family.

Notice that, according to Corollary 2.5, there are no isoperiodic families for  $n = 3$ .

We will focus our search to the cases  $n \in \{4, 6\}$ , because of the simplicity of the Cayley’s conditions for those values.



**Fig. 2** A circle and any conic with the foci on the circle are a 4-Poncelet pair

### 3.1 Quadrangles

In order to simplify the calculations, we will start with the analysis of the cases when the centre of the circle  $\Gamma$  belongs to one of the axes of the confocal family.

The first is case when the centre of  $\Gamma$  is on the  $y$ -axis, i.e.  $x_0 = 0$ .

The condition for quadrangles,  $C_3(t) = 0$ , according to Corollary 2.6, becomes:

$$(a - b + y_0^2 - 1) (2(a - b + y_0^2 + 1)t - (a^2 - b^2 + 2by_0^2 + 2b - y_0^4 + 2y_0^2 - 1)) = 0.$$

Having in mind that  $a > b$ , which implies that the second factor in the above expression is non-constant, we can conclude that  $C_3(t) = 0$  for all  $t$  is equivalent to  $a - b + y_0^2 = 1$ . It is easy to see that this is equivalent to the foci of the family  $\mathcal{C}(t)$  lying on the circle  $\Gamma$ , see Fig. 2.

**Remark 3.1** Notice that the Corollary 2.5 implies that there is a unique value of  $t$  such that the condition for triangles is satisfied. In the case when the foci of  $\mathcal{C}(t)$  lie on  $\Gamma$ , this is seemingly in contradiction with the conclusion that the condition for quadrangles is satisfied for any  $t$ .

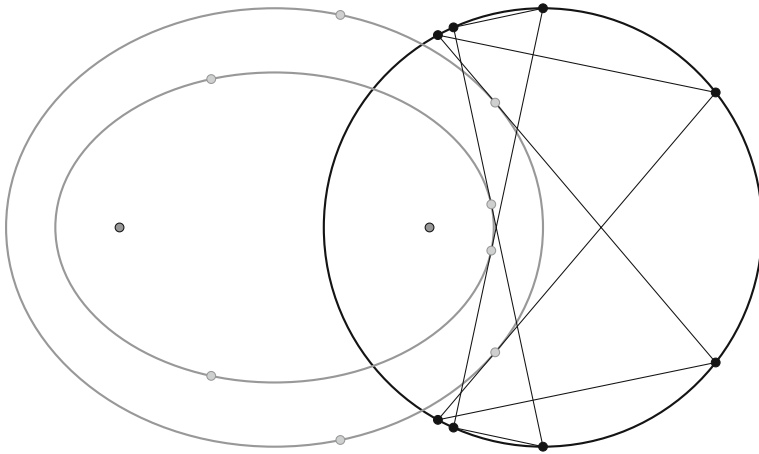
When we calculate  $t_3$  using the formula (2.4) from Corollary 2.5, with the condition  $a - b + y_0^2 = 1$ , we get  $t_3 = b$ , which corresponds to the singular conic  $\mathcal{C}(b)$  from the confocal family  $\mathcal{C}(t)$ .

Consider now the other symmetric case:  $y_0 = 0$ . The condition for quadrangles,  $C_3(t) = 0$ , is:

$$(a - b - x_0^2 + 1) (2(x_0^2 + 1 - a + b)t + (a^2 - b^2 - 2ax_0^2 - 2a + x_0^4 - 2x_0^2 + 1)) = 0, \tag{3.1}$$

so we see that this condition is satisfied for each  $t$  whenever either:

- $x_0^2 = a - b + 1$ ; or
- $-a + b + x_0^2 + 1 = a^2 - b^2 - 2ax_0^2 - 2a + x_0^4 - 2x_0^2 + 1 = 0$ .



**Fig. 3** Each conic from the confocal family is inscribed in quadrangles whose vertices belong to the circle. The foci are symmetric to each other with respect to the circle

Notice that the second possibility leads to  $y_0 = 0$  and  $a = b + 1$ , which a special case represented in Fig. 2, when the centre of the circles coincides with the centre of the confocal family.

On the other hand, we can see the geometric meaning of the first possibility by calculating the cross ratio of the foci with the endpoints of the diameter of  $\Gamma$  which lies on the  $x$ -axis:

$$\frac{\sqrt{a-b} - (x_0 - 1)}{\sqrt{a-b} - (x_0 + 1)} \cdot \frac{-\sqrt{a-b} - (x_0 + 1)}{-\sqrt{a-b} - (x_0 - 1)} = -1.$$

Thus, we get that the foci are symmetric to each other with respect to  $\Gamma$ , see Fig. 3.

**Remark 3.2** Similarly as in Remark 3.1, we should check the case when both Cayley’s conditions, for triangles and quadrangles, are satisfied:  $C_2(t) = C_3(t) = 0$ . The value of  $t_3$  from Corollary 2.5 is  $t_3 = a$  when the foci are symmetric with respect to the circle  $\Gamma$ .  $\mathcal{C}(a)$  is a degenerate conic coinciding with the  $y$ -axis, which is outside  $\Gamma$  when the foci are symmetric with respect to that circle.

We summarize the results of the above analysis as follows:

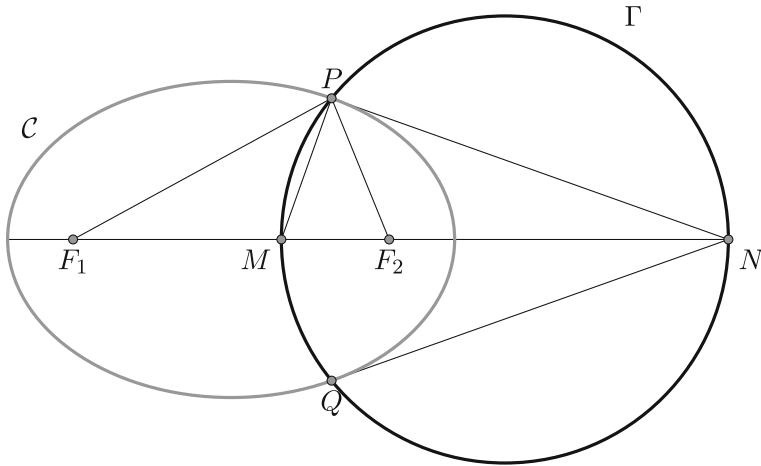
**Proposition 3.3** *If the focal points of the family  $\mathcal{C}(t)$  either belong to  $\Gamma$  or are symmetric with respect to  $\Gamma$ , then  $\Gamma$  and  $\mathcal{C}(t)$  are a 4-Poncelet pair for each  $t \notin \{a, b\}$ .*

**Proof** The focal points belong to  $\Gamma$  or they are symmetric with respect to  $\Gamma$  if and only if  $x_0 = 0$  and  $a - b + y_0^2 = 1$  or, respectively,  $y_0 = 0$  and  $a - b - x_0^2 + 1 = 0$ . In both of those cases, we get that  $\alpha_4 = \beta_4 = 0$  (see Corollary 2.6), thus the corresponding Cayley condition  $C_3(t) = 0$  is satisfied for each  $t$ , which implies that  $\Gamma$  and  $\mathcal{C}(t)$  are a 4-Poncelet pair for any  $t$  for which  $\mathcal{C}(t)$  is non-degenerate.  $\square$

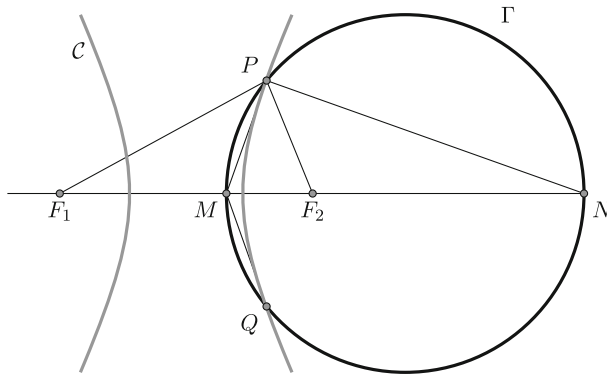
Now we will provide two alternative proofs.

*Second proof of Proposition 3.3, based on synthetic geometry.* First we consider the case when the foci  $F_1, F_2$  are symmetric with respect to  $\Gamma$ . In that case, the line  $F_1F_2$  intersects  $\Gamma$  at points  $M, N$ , such that  $MN$  is a diameter of  $\Gamma$ , and:

$$[F_1, F_2; M, N] = -1. \tag{3.2}$$



**Fig. 4** The path  $NPNQ$  is inscribed in the circle  $\Gamma$  and circumscribed about ellipse  $C$



**Fig. 5** The path  $MPMQ$  is inscribed in the circle  $\Gamma$  and circumscribed about hyperbola  $C$

Exactly one of the points  $M, N$  is between  $F_1$  and  $F_2$ , and we will assume that is  $M$ , as shown in Fig. 4

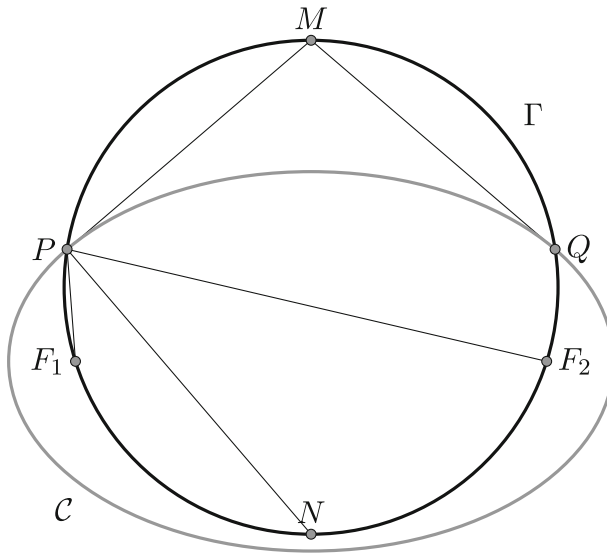
Let  $C$  be any conic with foci  $F_1, F_2$  and  $P$  be an intersection point of  $C$  and  $\Gamma$ . Since  $\angle MPN = 90^\circ$  and, from (3.2) we get  $[PF_1, PF_2; PM, PN] = -1$ , we get that the lines  $PM, PN$  are the bisectors of the angles determined by the lines  $PF_1, PF_2$ . Since  $PF_1, PF_2$  satisfy the billiard reflection law off  $C$ , we will have that one of the lines  $PM, PN$  is tangent to  $C$  and the other one is orthogonal to that conic at  $P$ . Denote by  $Q$  the point symmetric to  $P$  with respect to the line  $MN$ . Now it is easy to see that  $\{P, Q\} = C \cap \Gamma$ .

If  $C$  is ellipse, then  $N$  is outside that conic, and  $M$  within it, so  $PN$  and  $QN$  are tangent to  $C$ , thus  $NPNQ$  is a polygonal line inscribed in  $\Gamma$  and circumscribed about conic  $C$ , i.e. those two conics are a 4-Poncelet pair.

On the other hand, if  $C$  is a hyperbola,  $MPMQ$  is a polygonal line circumscribed about that conic and inscribed in  $\Gamma$ , so again those two conics are a 4-Poncelet pair, as shown in Fig. 5.

Second, we consider the case when both foci  $F_1, F_2$  belong to the circle  $\Gamma$ . Let  $M, N$  denote the mid-points of two arcs of  $\Gamma$  having the foci  $F_1, F_2$  as the end-points, see Fig. 6.





**Fig. 6** The path  $MPMQ$  is inscribed in the circle  $\Gamma$  and circumscribed about conic  $C$

Let  $C$  be any conic with foci  $F_1, F_2$ , such that  $\Gamma$  is not entirely contained within that conic.

Denote by  $P$  an intersection point of  $C$  and  $\Gamma$ . Then lines  $PM$  and  $PN$  and the bisectors of the two angles determined by the lines  $PF_1$  and  $PF_2$ , thus the focal property of the conics implies that one of  $PM, PN$  is tangent to  $C$  and the other orthogonal to that conic.

Assume  $PM$  is the tangent and let  $Q$  be the point symmetric to  $P$  with respect to  $MN$ . We have that  $MPMQ$  is inscribed in  $\Gamma$  and circumscribed about  $C$ , thus those two conics are a 4-Poncelet pair. □

**Corollary 3.4** *If the focal points of a conic  $C$  are symmetric with respect to  $\Gamma$ , then  $\Gamma$  and  $C$  have at most two intersection points.*

**Corollary 3.5** *Let  $F$  be a point on the circle  $\Gamma$ . Any circle  $C$  with the center at  $F$  is 4-Poncelet inscribed in  $\Gamma$ .*

**Proof** Let  $P, Q$  be the intersection points of the circles  $C$  and  $\Gamma$ , point  $O$  the centre of  $\Gamma$ , and  $N$  be the second intersection point of the line  $FO$  with  $\Gamma$ , see Fig. 7.

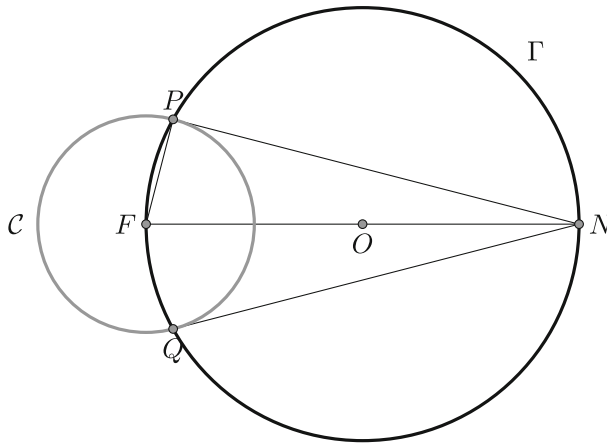
Then  $NP$  is orthogonal to  $FP$ , since  $FN$  is a diameter of  $\Gamma$  and  $P$  belongs to  $\Gamma$ . Thus,  $NP$  is tangent to  $C$  at  $P$ . Similarly,  $NQ$  is tangent to  $C$  at  $Q$ . Thus,  $NPNQ$  is a Poncelet quadrilateral inscribed in  $\Gamma$  and circumscribed about  $C$ . □

*Third proof of Proposition 3.3, based on the theory of invariants.* Denote by  $I_j, j \in \{1, 2, 3, 4\}$ , the coefficients of  $D_t(\lambda)$  from (2.1) with  $\lambda^{4-j}$ . The condition that there exists a quadrilateral inscribed in  $\Gamma$  and circumscribed about  $C(t)$  can be expressed in terms of  $I_j$ :

$$8I_1I_4^2 - 4I_2I_3I_4 + I_3^3 = 0. \tag{3.3}$$

(see for example [6]\*Example 4.50.<sup>1</sup>

<sup>1</sup> There is a misprint in that example. Here we corrected the term  $I_3^2$  to  $I_3^3$



**Fig. 7** The path  $NPNQ$  is inscribed in the circle  $\Gamma$  and circumscribed about circle  $C$

As said above, (i) the focal points belong to  $\Gamma$  if and only if  $x_0 = 0$  and  $a - b + y_0^2 = 1$  and (ii) they are symmetric with respect to  $\Gamma$  if and only if  $y_0 = 0$  and  $a - b - x_0^2 + 1 = 0$ .

In the case (i), the coefficients  $I_j$  take the form (up to a common factor):

$$\begin{aligned}
 I_1 &= -(a - t)(b - t); \\
 I_2 &= I_1 - t(b - a - 1) + (b - a)a - b; \\
 I_3 &= 2(t - a); \\
 I_4 &= -1.
 \end{aligned}
 \tag{3.4}$$

By substituting the relations (3.4) for  $I_j$  into (3.3) one gets the identity for all  $t$ . Case (ii) can be treated analogously, which completes the proof.  $\square$

Finally, we will show that there are no other examples when  $\Gamma$  and  $C(t)$  are a 4-Poncelet pair for each  $t$ .

**Theorem 3.6** *There are quadrangles inscribed in circle  $\Gamma$  and circumscribed about conic  $C(t)$  for each  $t \notin \{a, b\}$  if and only if the focal points of the family  $C(t)$  either belong to  $\Gamma$  or are symmetric with respect to  $\Gamma$ .*

**Proof** Suppose that  $C(t)$  is 4-Poncelet inscribed in  $\Gamma$  for each  $t \notin \{a, b\}$ . Thus the Cayley condition  $C_3(t) = 0$  is satisfied for infinitely many values of  $t$ . Since  $C_3(t)$  is a polynomial in  $t$ , we have that  $C_3(t) = 0$  everywhere.

By Corollary 2.6,  $C_3(t) = \alpha_4 + \beta_4 t$  is a linear expression in  $t$ , so  $\alpha_4 = \beta_4 = 0$ . From there:

$$\alpha_4 + (a + b - x_0^2 - y_0^2 + 3)\beta_4 = 0,$$

which gives:

$$1 - (a - b)^2 + (a - b - 1)x_0^2 + (b - a - 1)y_0^2 = 0.$$

From there, we get:

$$y_0^2 = \frac{(a - b - 1)(a - b - x_0^2 + 1)}{-a + b - 1},$$

which we substitute into  $\beta_4 = 0$ :

$$\frac{x_0^2(a - b)^2(a - b - x_0^2 + 1)}{(a - b + 1)^2} = 0.$$

The first possibility,  $x_0 = 0$ , will imply  $y_0^2 = 1 + b - a$ , which means that the foci belong to  $\Gamma$ . The second case,  $a - b - x_0^2 - 1 = 0$ , implies  $y_0 = 0$  and that the foci are symmetric with respect to  $\Gamma$ . Finally, the case  $a - b = 0$  implies that  $x_0^2 + y_0^2 = 1$ , i.e. the origin is on  $\Gamma$  and it is also the centre of the concentric family of circles  $\mathcal{C}(t)$ , which is a special case shown in Fig. 2, when  $\mathcal{C}(t)$  are concentric circles.

The second half of the equivalence is Proposition 3.3, thus the proof is concluded.  $\square$

### 3.2 Hexagons

The condition for existence of a hexagon inscribed in  $\Gamma$  and circumscribed about  $\mathcal{C}(t)$  is:

$$p_6(t) = \det \begin{pmatrix} C_3(t) & C_4(t) \\ C_4(t) & C_5(t) \end{pmatrix} = 0$$

which is equivalent to:

$$C_2(t)(\alpha_6 + \beta_6 t + \gamma_6 t^2 + \delta_6 t^3) = 0, \tag{3.5}$$

with

$$\begin{aligned} \delta_6 &= 64((a - b)^2 + 2(a - b)(y_0^2 - x_0^2) + (x_0^2 + y_0^2)^2), \\ \gamma_6 &= 128(a - b)(a - b - x_0^2 + y_0^2) - \frac{\delta_6^2}{256} - \frac{\delta_6}{4}(1 + 6a + 6b - 2x_0^2 - 2y_0^2). \end{aligned}$$

That condition is satisfied for each  $t$  if and only if  $\alpha_6 = \beta_6 = \gamma_6 = \delta_6 = 0$ . Now, we see that  $\gamma_6 = 0$  and  $\delta_6 = 0$  imply  $a = b$  or  $a - b - x_0^2 + y_0^2 = 0$ . Substituting  $a = b$  to the expression for  $\delta_6$ , we get  $x_0 = y_0 = 0$ , which means that  $\mathcal{C}(t)$  are concentric circles, and  $\Gamma$  shares the same centre with them. Obviously,  $\Gamma$  and  $\mathcal{C}(t)$  cannot be a 6-Poncelet pair for more than one value of  $t$  in this case.

If  $a - b = x_0^2 - y_0^2$ , the condition  $\delta_6 = 0$  gives  $x_0 = 0$  or  $y_0 = 0$ . Since  $a > b$ , we can have only  $y_0 = 0$  and the centre of  $\Gamma$  is at one of the foci of the family  $\mathcal{C}(t)$ . Now, calculating the coefficients  $\alpha_6, \beta_6$  and substituting  $y_0 = 0, x_0^2 = a - b$ , we get:

$$\begin{aligned} \alpha_6 &= -(4a - 4b - 1)(16a^2 - 16ab + 8a - 4b - 3), \\ \beta_6 &= 4(4a - 4b - 1)(4a - 4b + 1). \end{aligned}$$

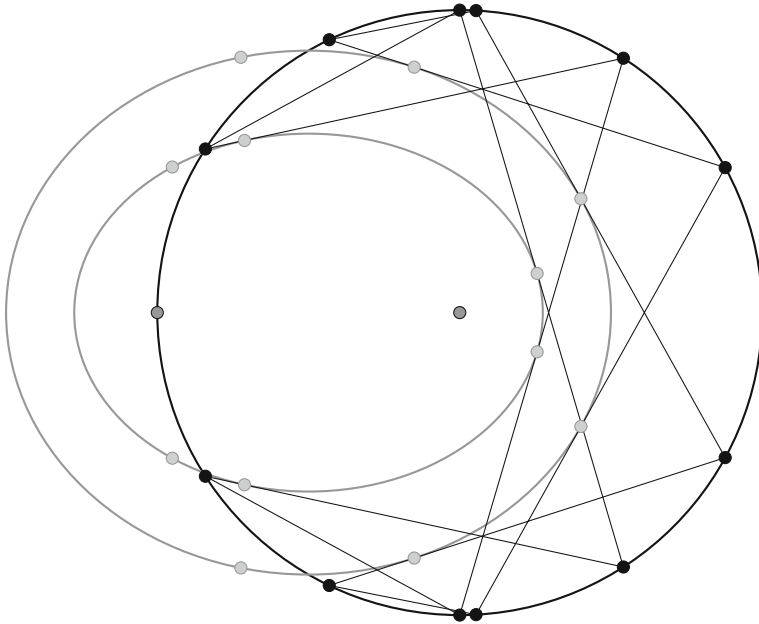
Since  $a > b$ , the condition  $\alpha_6 = \beta_6 = 0$  implies  $4a - 4b = 1$ . Thus, the distance between the foci equals  $2\sqrt{a - b} = 1$ , so we can conclude:

**Theorem 3.7** *Each non-degenerate conic  $\mathcal{C}(t)$  is 6-Poncelet inscribed in  $\Gamma$  if and only if one of the focal points belongs to  $\Gamma$  and the other one is at the centre of  $\Gamma$ , see Fig. 8.*

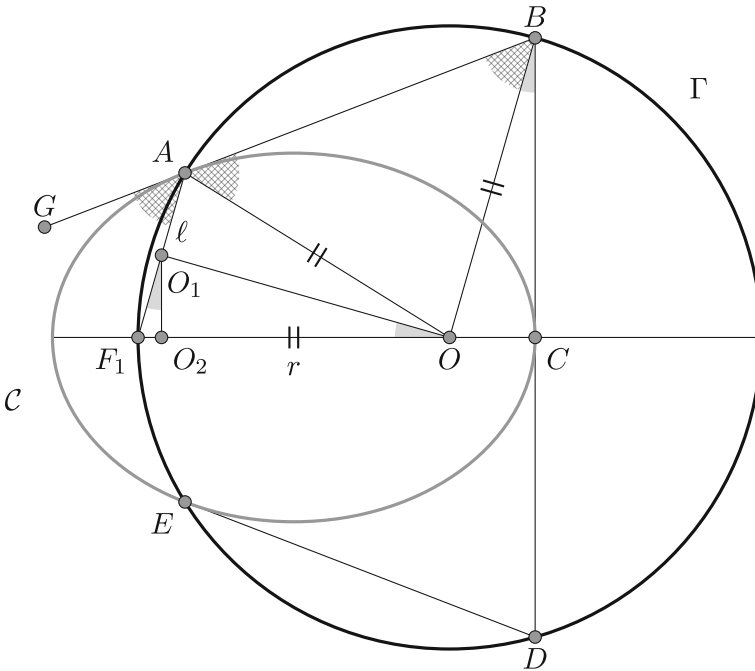
*Alternative geometric proof of one direction of Theorem 3.7.*

Denote by  $O$  and  $r$  the center and the radius of  $\Gamma$ . Let  $F_1$  be a point of  $\Gamma$ ;  $C$  an ellipse with foci  $F_1$  and  $O$ ;  $A$  an intersection point of  $\Gamma$  and  $C$ ;  $B$  the point of  $\Gamma$  such that  $AB$  is tangent to  $C$ ;  $E, D$  the points symmetric to  $A, B$  respectively with respect to  $OF_1$ , as shown in Fig. 9.

Our goal is to show that closed polygonal line  $ABDEDBA$  is circumscribed about  $C$ .



**Fig. 8** Each conic from the confocal family is inscribed in hexagons whose vertices belong to the circle



**Fig. 9** The path  $ABDEDBA$  is a polygonal line inscribed in  $\Gamma$  and circumscribed about  $C$

Denote by  $C$  the intersection of  $BD$  and the line  $F_1O$ . We want to show that  $C$  and  $A$  belong to the same ellipse with foci  $F_1$  and  $O$ . Denote  $F_1A =: \ell$ . We observe that  $F_1A + OA = \ell + r$ . Similarly,  $F_1C + OC = r + 2OC$ . Thus, we need to show that  $2OC = \ell$ .

Denote by  $G$  any point on the line  $AB$ , which is at the opposite to  $B$  side of  $A$ . Observe that  $\angle OAB = \angle F_1AG$  and  $\angle OAB = \angle OBA$ . The first fact follows from the focal property of ellipses and the second from  $\triangle OAB$  being isosceles, since  $O$  is the center of the circle  $\Gamma$ . From there we get  $F_1A \parallel OB$ .

Let  $OO_1$  be the altitude of the isosceles triangle  $OF_1A$ . Construct the normal line from  $O_1$  to  $F_1O$ , with  $O_2$  denoting the footing of the normal at  $F_1O$ . We have that  $\angle F_1O_1O_2 = \angle OBC$  as angles with parallel sides. Similarly,  $\angle F_1O_1O_2 = \angle F_1OO_1$  as angles with orthogonal sides. Thus,  $\angle F_1OO_1 = \angle OBC$ .

We showed that the triangles  $\triangle F_1OO_1$  and  $\triangle OBC$  are congruent. Thus,  $F_1O_1 = OC$  and we finally get:

$$OC = \frac{1}{2}F_1A.$$

Similar proof is valid when the conic  $C$  is a hyperbola. □

**Remark 3.8** In the case when  $C$  is ellipse, we assume in the proof above that  $\Gamma$  is not entirely contained within the ellipse  $C$ . Thus, the proof implies that the circle and the ellipse have at most two intersection points.

### 4 Do isoperiodic families exist for $n \neq 4, 6$ ?

Any Poncelet  $n$ -polygon can be also considered as a closed  $mn$ -polygon, for any  $m \in \mathbb{N}$ , where its edges and vertices are repeated  $m$  times each. This is our motivation to refine the notion of  $n$ -Poncelet pairs from Definition 2.2.

**Definition 4.1** We will say that conics  $\Gamma$  and  $C$  are a *strict  $n$ -Poncelet pair* or that  $C$  is *strictly  $n$ -inscribed in  $\Gamma$*  if the following conditions are satisfied:

- there is a polygon with  $n$  sides inscribed in  $\Gamma$  and circumscribed about  $C$ ;
- for any  $m$  that divides  $n$  and distinct from  $n$ , there is no polygonal line with  $m$  sides inscribed in  $\Gamma$  and circumscribed about  $C$ .

Notice that Cayley’s conditions also reflect the existence of strict and non-strict Poncelet pairs of conics, since, for non-prime numbers, those conditions have factors corresponding to the divisors of the numbers, see [3]. For example, in the calculations of this paper, we can see that the condition for hexagons (3.5) has a factor corresponding to triangles, i.e.  $p_6(t) = p_3(t)q_6(t)$ , for some polynomial  $q_6(t)$ .

More generally, for  $m|n$ , we have that  $p_m(t) = 0$  implies  $p_n(t) = 0$ . We will denote by  $q_n(t)$  the largest factor of  $p_n(t)$  which is co-prime with all polynomials  $p_m(t)$ , where  $m|n$  and  $m < n$ .

**Proposition 4.2** For  $t \notin \{a, b\}$ , the conics  $\Gamma$  and  $C(t)$  form a strict  $n$ -Poncelet pair if and only if  $q_n(t) = 0$ .

**Lemma 4.2.1** Suppose that  $\Gamma$  and  $C(t)$  form an  $n$ -Poncelet pair for infinitely many values of  $t$ . If  $m$  is co-prime with  $n$ , then all values of  $t$  that satisfy the Cayley’s condition for  $m$  belong to the set  $\{a, b\}$ .

**Proof** Notice first that, in the condition  $p_n(t) = 0$  for the  $n$ -Poncelet pair,  $p_n(t)$  is a polynomial in  $t$ , where the coefficients are polynomials in  $a, b, x_0^2, y_0^2$ . If the polynomial  $p_n(t)$  has infinitely many zeroes, then it must be identically equal to zero. This means that  $\Gamma$  and  $\mathcal{C}(t)$  form an  $n$ -Poncelet pair in the complexified plane for each  $t \in \mathbb{C} \setminus \{a, b\}$ .

From there, if  $m$  and  $n$  are co-prime, the polynomial  $p_m(t)$  cannot have any zeroes in  $\mathbb{C}$ , other than  $a$  and  $b$ . □

**Proposition 4.3** *Suppose that  $\Gamma$  and  $\mathcal{C}(t)$  form a strict  $n$ -Poncelet pair for infinitely many values of  $t$ . If  $m$  is not a multiple of  $n$ , then all values of  $t$  that satisfy the Cayley’s condition  $p_m(t) = 0$  must be in the set  $\{a, b\}$ .*

**Proof** Similarly as in the proof of Lemma 4.2.1, we can show that the polynomial  $q_n(t)$  is identically equal to zero. This implies that  $\Gamma$  and  $\mathcal{C}(t)$  form a strict  $n$ -Poncelet pair in the complexified plane for each  $t \in \mathbb{C} \setminus \{a, b\}$ .

If for some  $t \in \mathbb{C} \setminus \{a, b\}$ , there exists  $m \neq n$  such that  $p_m(t) = 0$ , then either  $m|n$  or  $n|m$ . The former is impossible due to the definition of a strict  $n$ -Poncelet pair and the latter by the assumption of the Proposition, which concludes the proof. □

**Lemma 4.3.1** *If  $t_3 = a$ , then  $\Gamma$  and  $\mathcal{C}(t)$  form a 4-Poncelet pair for each  $t$ .*

**Proof** We have that  $t_3 = a$  is equivalent to:

$$(x_0^2 + y_0^2 - a + b - 1)^2 + 4(a - b)y_0^2 = 0.$$

Since  $a \geq b$ , that implies  $(a - b)y_0 = 0$  and  $x_0^2 + y_0^2 - a + b - 1 = 0$ . If  $a = b$ , we get  $x_0^2 + y_0^2 = 1$ , i.e.  $\mathcal{C}(t)$  is a family of concentric circles with the centre on  $\Gamma$ . If  $y_0 = 0$ , we have  $x_0^2 = a - b + 1$ .

According to the discussion from Sect. 3.1, each of these conclusions implies that  $\Gamma$  and  $\mathcal{C}(t)$  form a 4-Poncelet pair for each  $t$ . □

**Theorem 4.4** *If  $\Gamma$  and  $\mathcal{C}(t)$  form  $n$ -Poncelet pairs for infinitely many values of  $t$ , then  $n \in \{4, 6\}$ .*

**Proof** Suppose that  $n$  is a number not equal to 4 or 6, such that  $\Gamma$  and  $\mathcal{C}(t)$  form  $n$ -Poncelet pairs for infinitely many values of  $t$ . Based on Corollary 2.5, there is a unique value  $t = t_3$ , which if given by (2.4), such that the Cayley condition for triangles inscribed in  $\Gamma$  and circumscribed about  $\mathcal{C}(t)$  is satisfied. Thus  $n \neq 3$ .

Proposition 4.3 and Lemma 4.3.1 then imply  $t_3 = b$ .

According to Corollary 2.6, there is a quadrangle inscribed in  $\Gamma$  and circumscribed about  $\mathcal{C}(t)$  if and only if  $\alpha_4 + \beta_4 t = 0$ . After substituting  $t_3 = b$ , this is equivalent to:

$$(b - t) (1 - a + b - x_0^2 - y_0^2) = 0.$$

Since  $n \neq 4$ , we have:

$$1 - a + b - x_0^2 - y_0^2 \neq 0. \tag{4.1}$$

Thus, the equation  $\alpha_4 + \beta_4 t = 0$  has a unique solution  $t = b$ .

Let us consider now Poncelet pentagons. Since the coefficients  $C_2(t)$  and  $C_3(t)$  both take values 0 for  $t = b$ , that means that  $t = b$  is also a solution of the cubic equation from Corollary 2.7, describing Poncelet pentagon, i.e.

$$\alpha_5 + \beta_5 b + \gamma_5 b^2 + \delta_5 b^3 = 0.$$

Substituting  $t_3 = b$  in the expressions from Corollary 2.7 for  $\alpha_5, \beta_5, \gamma_5,$  and  $\delta_5,$  we get:

$$\alpha_5 + \beta_5 t + \gamma_5 t^2 + \delta_5 t^3 = -64(b - t)^2 \times ((t - a)(2a - 2b + 2x_0^2 + 2y_0^2 - 1) + a - b). \tag{4.2}$$

Thus  $n = 5$  only if:

$$2a - 2b + 2x_0^2 + 2y_0^2 - 1 = a - b = 0,$$

which implies  $2x_0^2 + 2y_0^2 = 1.$  Under those conditions, we calculate that  $t_3 = b - \frac{1}{16},$  which is in contradiction with the assumption  $t_3 = b.$  We conclude that  $n \neq 5.$

The equality (4.2) shows that the polynomial  $\alpha_5 + \beta_5 t + \gamma_5 t^2 + \delta_5 t^3$  has a double root at  $t = b.$  Considering the third root  $t_5$  of that polynomial, if it exists, we have the following:

- either  $t_5 \in \{a, b\};$  or
- the polynomial  $\alpha_5 + \beta_5 t + \gamma_5 t^2 + \delta_5 t^3$  does not have the third root, i.e.  $\delta_5 = 0.$

We split those possibilities into three subcases.

*Case 1: The third root of  $\alpha_5 + \beta_5 t + \gamma_5 t^2 + \delta_5 t^3$  equals  $a.$*  From (4.2), we see that then  $a = b.$  Substituting that into  $t_3 = b,$  we get  $x_0^2 + y_0^2 = 1,$  which implies  $n = 4.$

*Case 2: The third root of  $\alpha_5 + \beta_5 t + \gamma_5 t^2 + \delta_5 t^3$  equals  $b.$*  Relation (4.2) then implies  $a - b + x_0^2 + y_0^2 - 1 = 0,$  which is in contradiction with (4.1).

*Case 3: The polynomial  $\alpha_5 + \beta_5 t + \gamma_5 t^2 + \delta_5 t^3$  does not have the third root, i.e.  $\delta_5 = 0.$*  That means:  $2a - 2b + 2x_0^2 + 2y_0^2 - 1 = 0.$  Combining that with  $t_3 = b,$  we get  $16(a - b)x_0^2 = 1$  and  $16(a - b)y_0^2 = -(4a - 4b - 1)^2.$  Since  $a \geq b,$  those two equations give  $4a - 4b - 1 = 0,$   $y_0 = 0,$  and  $x_0 = \pm \frac{1}{2}.$  According to the discussion from Sect. 3.2, we have  $n = 6.$

This concludes the proof. □

## 5 Poncelet pairs of conics and Blaschke ellipses

### 5.1 Properties of the Blaschke transformations

In this section, we are going to review basic properties of Blaschke transformations in a way we will need it later. For a more comprehensive overview, see e.g. [5] and references therein.

Denote a Möbius transformation for a given  $p \in \mathbb{C}$  as follows:

$$B_p(z) = \frac{p - z}{1 - z\bar{p}}. \tag{5.1}$$

**Lemma 5.0.1** *Let  $z, w, v \in \mathbb{C}, z \neq \bar{w}, z\bar{v} \neq 1, v\bar{w} \neq 1.$  Then*

$$\frac{1 - B_w(z)B_w(\bar{v})}{1 - z\bar{v}} = \frac{1 - |w|^2}{(1 - z\bar{w})(1 - \bar{v}w)}.$$

**Proof** It follows by straightforward calculation:

$$1 - B_w(z)B_w(\bar{v}) = \frac{(1 - z\bar{w})(1 - \bar{v}w) - (z - w)(\bar{v} - \bar{w})}{(1 - z\bar{w})(1 - \bar{v}w)} = \frac{(1 - z\bar{w})(1 - |\bar{w}|^2)}{(1 - z\bar{w})(1 - \bar{v}w)}.$$

□

**Corollary 5.1** *For  $v = z$  it holds:*

$$\frac{1 - |B_w(z)|^2}{1 - |z|^2} = \frac{1 - |w|^2}{|1 - z\bar{w}|^2}. \tag{5.2}$$

**Corollary 5.2** *The Möbius transformations (5.1) have the following properties:*

- If  $|z| < 1$  and  $|w| < 1$  then  $|B_w(z)| < 1$ .
- If  $|z| < 1$  and  $|w| \geq 1$  then  $|B_w(z)| \geq 1$ .
- If  $|z| > 1$  and  $|w| < 1$  then  $|B_w(z)| > 1$ .
- If  $|z| > 1$  and  $|w| > 1$  then  $|B_w(z)| < 1$ .

**Lemma 5.2.1** *For given  $p, q$ , such that  $|p| < 1, |q| < 1$ , there exists a unique  $c$  such that  $B_c(p) = q$ . That value of  $c$  satisfies:  $|c| < 1$ .*

**Proof** The condition  $B_c(p) = q$  leads to a system of two linear equations on two real variables  $c_1 = \operatorname{Re} c, c_2 = \operatorname{Im} c$ . The determinant of that system is

$$\Delta = 1 - |p|^2|q|^2.$$

Thus, from  $|p| < 1, |q| < 1$  it follows that  $\Delta \neq 0$ , and the system has a unique solution,  $c$ . From Corollary 5.2 it follows that  $|c| < 1$ .  $\square$

Let us denote  $\mathbb{T} = \{z \mid |z| = 1\}$  the unit circle and  $\mathbb{D} = \{z \mid |z| < 1\}$  the open unit disk bounded by  $\mathbb{T}$ .

The transformations defined by (5.1) are called the Blaschke transformations. For  $|p| < 1$ ,  $B_p$  is an automorphism of  $\mathbb{D}$ , see [5]. The composition of  $n$  Blaschke transformations defines a Blaschke product  $B$  of degree  $n$ . Following [5], one associates to  $B$  a Blaschke curve as the envelope of all polygons with vertices at the solutions of  $B(z) = \lambda$ , for each  $\lambda \in \mathbb{T}$ . Let us recall two statements from [5].

**Proposition 5.3** (Theorem 2.9, [5]) *Let  $B$  be a Blaschke product of degree 3 with zeros  $0, p, q$ . For  $\lambda \in \mathbb{T}$ , let  $z_1, z_2, z_3$  denote the three distinct solutions to  $B(z) = \lambda$ . Then the lines joining  $z_j$  and  $z_k$ , ( $j \neq k$ ) are tangent to the ellipse given by*

$$|w - p| + |w - q| = |1 - \bar{p}q|.$$

**Proposition 5.4** (Corollary 13.9, [5]) *Let  $B$  be a Blaschke product of degree 4. Then the associated Blaschke curve is an ellipse if and only if  $B$  is decomposable.*

## 5.2 Poncelet triangles and Blaschke ellipses

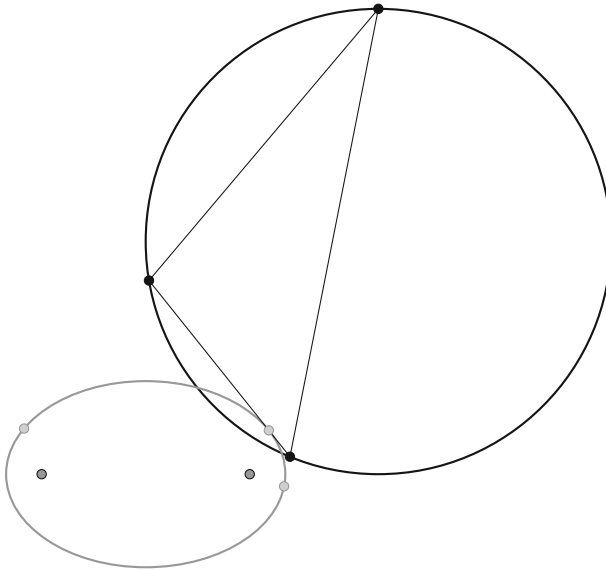
Now, we are going to study triangles inscribed in the unit circle and circumscribed about a conic from a given confocal family. Unlike [5] we, in principle, allow the possibility for the inscribed conic to be a hyperbola.

**Theorem 5.5** *Let  $p, q$  be two given points in  $\mathbb{D}$ . Then there exists a unique conic  $\mathcal{C}$  with the foci  $p, q$ , which is 3-Poncelet inscribed in  $\mathbb{T}$ . Moreover,  $\mathcal{C}$  is an ellipse. That ellipse is the Blaschke ellipse with the major axis of length  $|1 - \bar{p}q|$ .*

**Proof** We can construct the Blaschke 3-ellipse with the focal points  $p, q$ . This ellipse satisfies all the requirements of the theorem. Such a conic is unique, according to Corollary 2.5.  $\square$

**Remark 5.6** Previously existing proofs of the uniqueness, like one from [5, pp. 43–44], assume that the conics are completely situated within  $\mathbb{D}$ . They are based on a beautiful and intuitive argument, which however, does not seem to be applicable nor easily extendable in a more general situation of our interest, when conics are not necessarily completely situated within  $\mathbb{D}$ . Moreover, in principle, a triangle which is inscribed in  $\mathbb{T}$  and whose sides belong to tangents to an ellipse which is not completely contained in  $\mathbb{D}$  could exist, see Fig. 10. Our nontrivial conclusion is that in such a situation, at least one focal point lies outside  $\mathbb{D}$ .





**Fig. 10** A triangle inscribed in the circle

**Remark 5.7** Note that Theorem 5.5 states not only that  $\mathcal{C}$  is unique among the ellipses with focal points  $p, q$ , but that it is also unique among all conics with those focal points. In other words, we proved that there is no hyperbola with given properties.

**Theorem 5.8** Let  $ABC$  be a triangle inscribed in  $\mathbb{T}$  with the property that lines  $AB, AC, BC$  touch an ellipse in the points  $C_1, B_1, A_1$  respectively with the following orders of the points:  $B - A_1 - C, C_1 - B - A$ , and  $B_1 - C - A$ . Then at least one focal point of  $\mathcal{C}$  is outside  $\mathbb{D}$ .

**Remark 5.9** Observe in Fig. 10 what happens with rotation numbers under the assumptions of Theorem 5.8. Let us denote by  $\rho_A, \rho_B, \rho_C$  the arcs of the ellipse  $\mathcal{C}$  seen from the corresponding vertex of the triangle  $ABC$ . Then, obviously:

$$\rho_A = \rho_B + \rho_C.$$

### 5.3 Poncelet quadrangles and Blaschke ellipses

**Remark 5.10** We want to observe first that not all 4-Poncelet ellipses are Blaschke ellipses.

However, the following reformulation and generalization of a statement from [5], p. 175, holds:

**Theorem 5.11** Suppose  $p, q \in \mathbb{D}$ . Among all conics with the foci  $p, q$  there is a unique one which is 4-Poncelet inscribed in  $\mathbb{T}$ . That conic is an ellipse and it is the Blaschke ellipse.

**Proof** Using Lemma 5.2.1, we see that for given  $p, q \in \mathbb{D}$ , there is a unique value  $c$  such that  $B_c(p) = q$ . Moreover,  $c \in \mathbb{D}$ . Using now the construction from [5] one gets the Blaschke product with zeros  $0, c, p, q$ . Since  $B_c(p) = q, B_c(0) = c$ , this product is decomposable and thus, generates a Blaschke ellipse which is 4-Poncelet inscribed in  $\mathbb{T}$ . Using the Cayley-type condition derived above in Corollary 2.6 we see that there is at most one such 4-Poncelet inscribed conic with the foci  $p, q$ . Thus, we obtained the statement.  $\square$

The last Theorem has an interesting consequence.

**Corollary 5.12** *A hyperbola with the foci in  $\mathbb{D}$  cannot be a 4-Poncelet inscribed in  $\mathbb{T}$ .*

Corollary 5.12 may seem quite counter-intuitive. Thus, it may be instructive to compare it with the well-studied situation of billiards within ellipses. For every even natural number  $2k$  greater than two, there exist ellipses and trajectories of billiards within each of these ellipses with the period  $2k$  which are circumscribed about a hyperbola confocal with the boundary ellipse, see e.g. [6, 10]. Narrowing down our focus to four-periodic trajectories of elliptical billiards, one can notice that a hyperbola confocal to the boundary ellipse can be inscribed in a four-periodic trajectory if and only if the square of the major semi-axis of the boundary ellipse is more than two times larger than the square of its minor semi-axis. See the paragraph after formula (4.4), p. 18. [10].

## 6 View towards Painlevé VI equations

### 6.1 Picard solutions and Okamoto transformation

The Painlevé VI equations form one of the six families of remarkable second order ordinary differential equations introduced by Paul Painlevé and his school at the beginning of the XX century. The Painlevé VI equations are the four-parameter family

$$\begin{aligned} \frac{d^2y}{dx^2} = & \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) \left( \frac{dy}{dx} \right)^2 - \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) \frac{dy}{dx} \\ & + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left( \alpha + \beta \frac{x}{y^2} + \gamma \frac{x-1}{(y-1)^2} + \delta \frac{x(x-1)}{(y-x)^2} \right) \end{aligned} \quad (6.1)$$

with parameters  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ , see for example [12, 16, 17, 22] and references therein. We are here interested in two particular equations with parameter values

$$\alpha = \frac{1}{8}, \quad \beta = -\frac{1}{8}, \quad \gamma = \frac{1}{8}, \quad \delta = \frac{3}{8}, \quad (6.2)$$

and

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = \frac{1}{2}. \quad (6.3)$$

We will use the transformed Weierstrass  $\wp$ -function with periods  $2w_1$  and  $2w_2$  satisfying the equation

$$(\hat{\wp}'(z))^2 = \hat{\wp}(z)(\hat{\wp}(z) - 1)(\hat{\wp}(z) - x). \quad (6.4)$$

Apparently, more than a decade prior to Painlevé, Picard found the explicit solution of the above mentioned Painlevé VI equation with parameters (6.3), see [23]. Namely, the function

$$y_0(x) = \hat{\wp}(2c_1w_1(x) + 2c_2w_2(x)) \quad (6.5)$$

is the Picard solution [23] of the Painlevé VI equation (6.1) with constants  $\alpha = \beta = \gamma = 0$ ,  $\delta = 1/2$ , see for example (3.6) in [22].

The transformation

$$y(x) = y_0 + \frac{y_0(y_0 - 1)(y_0 - x)}{x(x - 1)y_0' - y_0(y_0 - 1)}, \quad (6.6)$$

follows from [22] and was written in this form in [12]. It relates the Picard solution  $y_0(x)$  (6.5) and the general solution  $y(x)$  of the Painlevé VI equation with constants (6.2).

In order to apply the Okamoto transformation, one needs an expression for  $y'_0$ . Such an expression is obtained in [12] as follows. Consider the elliptic curve  $\mathcal{E}$  parameterized by  $(u, v) = (\hat{\phi}, \hat{\phi}')$ , that is the curve of the equation

$$v^2 = u(u - 1)(u - x) \tag{6.7}$$

and its Abel map

$$A(p) = \int_{p_\infty}^p \frac{du}{v}, \quad p \in \mathcal{L},$$

into the Jacobian  $J(\mathcal{E}) = \mathbb{C}/\{n(2w_1) + m(2w_2) \mid n, m \in \mathbb{Z}\}$ , where  $p_\infty$  is the point at infinity on the compactified curve  $\mathcal{E}$ . Let us now take an arbitrary point  $2w_1c_1 + 2w_2c_2$  in the Jacobian, with some constants  $c_1$  and  $c_2$  and consider its Jacobi inversion, that is the point  $q_0$  on the curve such that

$$A(q_0) = \int_{p_\infty}^{q_0} \frac{du}{v} = 2w_1c_1 + 2w_2c_2.$$

From these data, denoting by  $q^*$  the elliptic involution of a point  $q$  of the curve, we build the following differential of the third kind:

$$\Omega_1 := \Omega_{q_0, q_0^*} - 4\pi ic_2\omega, \tag{6.8}$$

with  $\Omega_{q_0, q_0^*}$  being the normalized differential of the third kind with poles at  $q_0$  and  $q_0^*$  of residues 1 and  $-1$ , respectively, and  $\omega$  the normalized holomorphic differential  $w = \frac{1}{2w_1} \frac{du}{v}$ . Differential  $\Omega_1$  has two zeros paired by the elliptic involution, we denote by  $y$  their projection on the  $u$ -sphere. This projection, as a function of  $x$ , satisfies a Painlevé-VI equation, as stated in the following theorem.

**Theorem 6.1** ([12], Theorem 1) *If  $x$  varies, while  $c_1, c_2$  stay fixed, then  $y(x)$ , the position of zero of  $\Omega_1$ , is the Okamoto transformation (6.6) of  $y_0$  and satisfies Painlevé VI with constants (6.2)*

Moreover, the derivative of  $y'_0$  needed to apply the Okamoto transformation (6.6) can also be expressed in terms of  $\Omega_1$  :

$$\frac{dy_0}{dx} = -\frac{1}{4}\Omega_1(p_x) \frac{\omega(p_x)}{\omega(q_0)}. \tag{6.9}$$

Note that the  $a$ -period of  $\Omega_1$  is  $-4\pi ic_2$ , while the  $b$ -period is  $4\pi ic_1$ . Thus, the periods are preserved under the deformation described in Theorem 6.1.

## 6.2 From isoperiodic families to explicit solutions to a Painlevé VI equation

### 6.2.1 Quadrangles

In order to simplify calculations, we will consider the “super-symmetric” case, when both  $x_0$  and  $y_0$  are zero, meaning that the circle  $\Gamma = \Gamma_0$  is concentric with the confocal conics  $\mathcal{C}(t)$ . Then,  $a = b + 1$  and

$$\Gamma_0 : x^2 + y^2 = 1,$$

and

$$C(t) : \frac{x^2}{a-t} + \frac{y^2}{a-t-1} = 1.$$

The pencil of cubics  $\mathcal{E}_t$  given by (2.3) under the assumptions we just made becomes:

$$\mathcal{E}_t : \mu^2 = -(\lambda + 1)(\lambda(a - t) + 1)(\lambda(a - t - 1) + 1).$$

This pencil has a section of points  $P_0(t)$  of order 4 corresponding to  $\lambda = 0$  each. Each of the cubics  $\mathcal{E}_t$  has three finite ramification points:

$$\lambda_1(t) = \frac{1}{t-a}, \quad \lambda_2(t) = \frac{1}{t-a+1}, \quad \lambda_3(t) = -1.$$

Let us consider the following Möbius transformations of the basis:

$$u = \phi_t(\lambda) = \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1}.$$

Then  $\phi_t(\infty) = \infty$ ,  $\phi_t(\lambda_1) = 0$ ,  $\phi_t(\lambda_2) = 1$  and denote  $x := \phi_t(\lambda_3)$ . The Möbius transformations  $\phi_t$  induce morphisms from the elliptic curves  $\mathcal{E}_t$  onto the elliptic curve (6.7). Since  $P_0(t)$  corresponding to  $\lambda = 0$  are points of a fixed finite order (that order equals 4), we get the following:

**Proposition 6.2** *Let*

$$y_0(t) := \phi_t(0). \tag{6.10}$$

*Then  $y_0(t) = -(a - t - 1)$  is a Picard solution to the Painlevé VI  $(0, 0, 0, 1/2)$  as a function of  $x = (a - t - 1)^2$ .*

**Proof** By a direct computation we get

$$x = \phi_t(\lambda_3) = (a - t - 1)^2, \quad y_0(t) = \phi_t(0) = -(a - t - 1). \tag{6.11}$$

Since the points  $P_0(t)$  which correspond to  $\lambda = 0$  have all order 4, we see that for them  $c_1, c_2$  do not depend on  $t$ , thus also do not depend on  $x$ , so, by (6.5), they provide the Picard solution to the Painlevé VI  $(0, 0, 0, 1/2)$ .  $\square$

Before we apply the Okamoto transformation (6.6), let us verify one technical statement.

**Lemma 6.2.1** *For  $y_0$  from (6.10) the identity holds:*

$$\frac{(y_0 - 1)(y_0 - x)}{x(x - 1)y'_0 - y_0(y_0 - 1)} = -2. \tag{6.12}$$

**Proof** Observe that

$$\frac{dy_0}{dx} = \frac{1}{2y_0}. \tag{6.13}$$

The rest is a straightforward verification.  $\square$

**Theorem 6.3** *Let*

$$y(t) := -y_0(t). \tag{6.14}$$

*Then  $y(t) = a - t - 1$  is a solution to the Painlevé VI  $(1/8, -1/8, 1/8, 3/8)$  as a function of  $x = (a - t - 1)^2$ .*

**Proof** By applying the Okamoto transformation (6.6) and using Lemma 6.2.1 we get:  $y = y_0 - 2y_0 = -y_0$ . □

**Remark 6.4** Both solutions  $y = y(x)$  to Painlevé VI  $(1/8, -1/8, 1/8, 3/8)$  and  $y_0 = y_0(x)$  to Painlevé VI  $(0, 0, 0, 1/2)$  can be seen as instances of the solution parametrized as  $y(s) = s, x(s) = s^2$  where  $s = a - t - 1$  in one case and  $s = -(a - t - 1)$  in the other case. So, in some sense, here the Okamoto transformation maps the “same” solutions to each other, although the Okamoto transformation is not identity itself. It is an involution in this case. Let us also observe that the same solution was obtained by Hitchin in [16] in his consideration of points of order four on elliptic curves. Hitchin also observed in the same paper that  $y(s) = s, x(s) = s^2$  provides solutions to all Painlevé VI with

$$\alpha + \beta = 0, \quad \gamma + \delta = \frac{1}{2},$$

in (6.1). However, Hitchin did not consider the Okamoto transformation nor iso-periodic confocal pencils.

**Remark 6.5** We did not need to use (6.9) here since our iso-rotational confocal family provided an explicit form of the one-parameter family of cubics  $\mathcal{E}_t = \mathcal{E}_t(\mu, \lambda)$  with the section of marked points of order four lying above  $\lambda = 0$ . This section provides an explicit form of  $y_0$  (see (6.11)), which can be differentiated directly, as in (6.13).

### 6.2.2 Hexagons

Here  $x_0 = 1/2$  and  $y_0 = 0$  and  $b = a - 1/4$ . Thus,

$$\Gamma_0 : \left(x - \frac{1}{2}\right)^2 + y^2 = 1,$$

and

$$\mathcal{C}(t) : \frac{x^2}{a - t} + \frac{y^2}{a - t - \frac{1}{4}} = 1.$$

The pencil of cubics (2.3) is then given by:

$$\mathcal{E}_t : \mu^2 = -\frac{1}{16}((4s - 1)\lambda + 4)(4s\lambda^2 + (4s + 3)\lambda + 4)$$

where  $s = a - t$ .

Each of the cubics  $\mathcal{E}_t$  has three finite ramification points:

$$\lambda_1(t) = -\frac{4}{4s - 1}, \quad \lambda_{2,3}(t) = -\frac{1}{2} - \frac{3}{8s} \pm \frac{\sqrt{16s^2 - 40s + 9}}{8s}.$$

Let us consider Möbius transformations on the basis

$$\phi_t(\lambda) = \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1},$$

and denote

$$x(s) := \phi_t(\lambda_3) = \frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_1}, \quad y_0(s) := \phi_t(0) = \frac{\lambda_1}{\lambda_1 - \lambda_2}.$$

We calculate

$$x(s) = \frac{3 + 24s - 16s^2 - (4s - 1)\sqrt{16s^2 - 40s + 9}}{3 + 24s - 16s^2 + (4s - 1)\sqrt{16s^2 - 40s + 9}},$$

$$y_0(s) = \frac{32s}{3 + 24s - 16s^2 + (4s - 1)\sqrt{16s^2 - 40s + 9}},$$

and

$$\frac{dy_0}{dx} = \frac{dy_0}{ds} : \frac{dx}{ds} = -\frac{64s^3 - 80s^2 - 20s + 9 - (3 + 16s^2)\sqrt{16s^2 - 40s + 9}}{24(4s - 1)}$$

Here  $y_0$  is a Picard solution to the Painlevé VI  $(0, 0, 0, 1/2)$  as a function of  $x$ . We apply the Okamoto transformation (6.6) and calculate:

$$y(x) = y_0 + \frac{y_0(y_0 - 1)(y_0 - x)}{x(x - 1)\frac{dy_0}{dx} - y_0(y_0 - 1)}$$

$$= \frac{12 - 16s}{3 + 24s - 16s^2 + (4s - 1)\sqrt{16s^2 - 40s + 9}}, \tag{6.15}$$

$$x(s) = \frac{3 + 24s - 16s^2 - (4s - 1)\sqrt{16s^2 - 40s + 9}}{3 + 24s - 16s^2 + (4s - 1)\sqrt{16s^2 - 40s + 9}}.$$

**Theorem 6.6** *The function  $y(s)$ , as a function of  $x(s)$ , given by the last two formulas (6.15), is a solution to Painlevé VI  $(1/8, -1/8, 1/8, 3/8)$  equation, which corresponds to a 6-Poncelet inscribed family.*

**Remark 6.7** Hitchin [16] obtained the following parameterization of a solution to Painlevé VI  $(1/8, -1/8, 1/8, 3/8)$  equation which corresponds to 6-Poncelet polygons:

$$x(S) = \frac{S^3(S + 2)}{2S + 1}, \tag{6.16}$$

$$y(S) = \frac{S(1 + S + S^2)}{2S + 1}.$$

**Lemma 6.7.1** *The solutions (6.16) are obtained from the solutions (6.15) by the reparametrization*

$$s = \frac{2S^2 + 5S + 2}{4S}.$$

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