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NONLINEAR EVOLUTION OF WATER WAVES AND THEIR DISPERSION RELATION IN COASTAL WATERS

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Abstract. Preliminary results of the derivation of a new phase-resolving (deterministic) spatio-temporal nonlinear model of water wave evolution in non-deep waters with constant bathymetry are presented in this paper. The model is the first of its kind to include a nonlinear dispersion relation and cubic (four-wave) interactions. Simulations show the importance of nonlinear dispersion for bound wave components which, if not properly accounted for, result in inaccurate transfer of wave energy. We also investigate the relative importance of cubic nonlinearity, as compared to a quadratic (three-wave interactions) one, and show that it is non-negligible. The model provides the first step before the incorporation of these extensions into a phase-averaged (stochastic) formulation, which can then be used as a more accurate nonlinear source term for wave forecasting models.

1. Introduction

Waves are one of the most ubiquitous phenomena in nature, manifesting as acoustic, electromagnetic, etc. Surface water waves, in particular, are one of the most well-known, and have been capturing the minds of artists, poets, and scientists for centuries. They are commonly generated via wind forcing [**14**, **16**], which is why they are often called wind waves. These waves act as an interface between the oceans and the atmosphere making them instrumental in the exchange of various fluxes [**5**, **6**], and are one of the driving forces in the morphology of coastlines [**3**]. Nonlinear wave-wave interactions are (together with dissipation, shoaling, and wind forcing) one of the main processes governing the evolution of wave fields [**17**]. Modeling the nonlinear evolution of wave fields in open seas and oceans is a challenging task on its own [**11**]. This task is even more complicated in the rapidly varying coastal environments.

For nonlinear interactions to exchange a significant amount of energy, the interacting components need to close resonances in both wavenumber and frequency domains. One rare property (as opposed to many other types of waves) of water

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waves is their dispersive nature: waves of different frequencies travel at different speeds. This can be easily deduced from the linear wave dispersion relation:

$$\omega^2 = gk \tanh kh, \tag{1.1}$$

where ω is the angular frequency, g the gravitational acceleration, k is the wavenumber, and h is the bottom depth. This nonlinear relation makes it impossible for groups of three waves to resonate in non-shallow depths ($\mathbf{k_1} = \mathbf{k_2} + \mathbf{k_3}, \omega_1 = \pm\omega_2 \pm \omega_3$), hence four-wave interactions are a primary mean of energy transfer in deep waters. Three-wave interactions simply result in the formation of bound waves (forced oscillations) which follow a different dispersion relation. However, as waves enter the coastal areas, the changing bathymetry in intermediate waters can act as an additional component enabling a class III Bragg resonance [**12**]. In shallow waters ($\tanh kh \approx kh$), the nature of the dispersion relation changes to a linear one, enabling a direct resonance between the three-wave pairs.

The difference between free and bound waves is illustrated in figure 1, using an example of pendulum. We assume that the interacting waves close resonances in the frequency domain, but not necessarily in the wavenumber domain as defined by the dispersion relation. If a sum of wavenumbers of interacting waves does not fall on the resonance circle ($\mathbf{k_1} \neq \mathbf{k_2} + \mathbf{k_3}$), this results in the generation of a bound wave. The most famous example of this is the second-order Stokes wave, in which the bound wave modulates the shape of the wave that generated it. If however the interacting wavenumbers resonate (e.g. $\mathbf{k_1} = \mathbf{k_2} + \mathbf{k_3} + \mathbf{k_4}$), this results in transfer of energy (E) between the interacting waves. A similar analogy can be viewed via a simple pendulum. If one were to apply forcing at non-resonant frequency

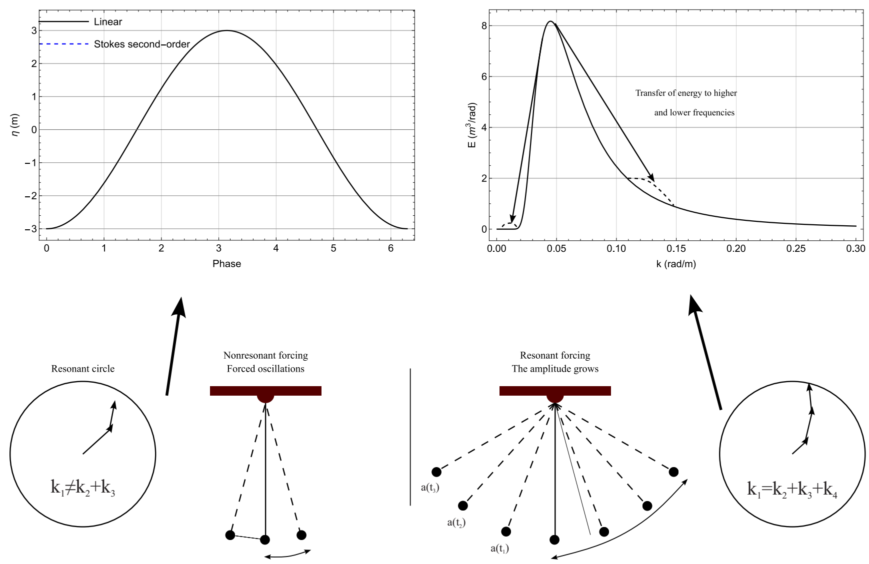

FIGURE 1. An illustration of the qualitative difference between bound and free waves.

to a pendulum, it would result in small forced oscillations. However, a forcing at the natural frequency of the pendulum would cause growth of the amplitude of oscillation with time. In the water wave context each spectral bin relates to a pendulum being forced by all other pendulums.

There are many approaches for modeling wave evolution in intermediate to shallow waters over mild slope. These range from Boussinesq models which solve the free surface elevation in the temporal domain [**13**, **15**], spectral deterministic and stochastic models which describe the evolution of individual frequency components [**1**, **9**, **10**, **18**], and even more complex numerical solvers of this free-surface flow (e.g., [**22**]). The spectral approaches focus on the nonlinear three-wave interactions, as they are the dominant source term in coastal areas, discarding the cubic interactions. This includes mild-slope models of the authors [**19**, **20**] which account for nonlinear interactions up to quadratic, although the potential importance of cubic terms was briefly discussed. One unique property of these two models is that they utilize a more accurate vertical velocity profile common in the Boussinesq models (see [**4**]).

The simple linear dispersion relation from equation (1.1) is often used in the models, which generally yields good agreement with the measurements. In this paper, we investigate the difference in the nonlinear energy transfer for purely free and bound waves. We also extend our previous works [**19**, **20**] to account for four-wave interactions. The preliminary results presented in this paper show that the contribution of cubic nonlinearity and the nonlinearity of the dispersion relation can be significant.

The paper is structured as follows. In section 2, the governing equations are defined and combined into a single equation. The nonlinear dispersion relation and wave evolution equation are derived in section 3. Examples of the effect of nonlinear dispersion and relative importance of cubic nonlinearity (compared to quadratic) are given in section 4. Conclusions and closing remarks are given in section 5.

2. Governing equations

In this section, we present the governing equations. The flow under consideration is assumed to be potential, with a still water level at $z = 0$. The flow is governed by the continuity (Laplace) equation, kinematic boundary condition at the surface and the flat bottom, and by a dynamic boundary condition at the surface:

$$\begin{aligned} \nabla_x^2\phi + \phi_{zz} &= 0, \quad -h < z < \eta, \\ \phi_z - \nabla_x\phi \cdot \nabla_x\eta - \eta_t &= 0, \quad z = \eta, \\ \phi_t + g\eta + ((\nabla_x\phi)^2 + \phi_z^2)/2 &= 0, \quad z = \eta, \\ \phi_z &= 0, \quad z = -h. \end{aligned} \tag{2.1}$$

Here, ϕ is the velocity potential, ∇_x and subscript z represents differentiation in the direction of wave propagation and in z. Initial steps of the derivation, for the most part, follow the works of [**4**] and [**20**], and are not shown here in detail. Three major differences are that we assume that the bottom is flat, that all waves are propagating in the same direction, and that we extend nonlinearity to ε^2, where $\varepsilon = ka$ is the wave slope. It is assumed that the wave field is discrete, with wave components defined as a function of frequency: $\omega_p = p\omega_1$, with ω_1 the lowest frequency under consideration.

The velocity potential is defined as an infinite series as:

$$\phi(x, z, t) = \sum_{n=0}^{\infty} z^n \phi_n(x, t).$$

By inserting this into the Laplace equation, a recursive relation can be obtained

$$\phi_{n+2} = -\nabla_x^2 \phi_n / ((n+1)(n+2)),$$

where ∇_x is horizontal gradient operator.

$$\phi(x, z, t) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}\nabla^{2n}}{(2n)!} \Phi + (-1)^n \frac{z^{2n+1}\nabla^{2n}}{(2n+1)!} W.$$

The velocity potential is expressed in terms of vertical velocity at the surface ($W = \phi_z\,(x, z = 0, t)$) and surface velocity potential ($\Phi = \phi\,(x, z = 0, t)$). This is simply an infinite Taylor series:

$$\phi(x, z, t) = \cos(z\nabla)\Phi + \frac{1}{\nabla} \sin(z\nabla) W. \tag{2.2}$$

With that in mind, the surface boundary conditions are expanded as a Taylor series around the mean water level ($z = 0$, pointing upward). The next step is to eliminate surface elevation and vertical velocity from the nonlinear terms, and to substitute them with velocity potential. By substituting surface elevation ($\eta = -\Phi_t/g + O(\varepsilon) + O(\varepsilon^2)$) and vertical velocity at the surface ($W = \eta_t + O(\varepsilon) + O(\varepsilon^2)$) for surface velocity potential (Φ) up to ε^2 in accuracy, the surface conditions are combined into:

$$-W - \frac{\Phi_{tt}}{g} = -\varepsilon Y + \varepsilon^2 G, \quad z = 0, \tag{2.3}$$

where Φ is the velocity potential at the surface, and Y and G terms account for quadratic and cubic nonlinearity. Note that the continuity equation (2.1) was used to eliminate double derivatives in z. For brevity, the definitions of Y and G are given in the appendix.

By Fourier expanding vertical velocity and velocity potential at the surface

$$\Phi(x, t) = \sum_{p=-N}^{N} \phi_p(x) e^{i\omega_p t}, \quad W(x, t) = \sum_{p=-N}^{N} w_p(x) e^{i\omega_p t}$$

together with some additional simplifications, equation (2.3) takes the form

$$\begin{aligned} w_p = \frac{\omega_p^2}{g}\phi_p &+ \varepsilon \sum_{s,l=-N}^{N} Y_{s,l}\,\delta(\omega_p + \omega_s + \omega_l)\phi_s\phi_l \\ &+ \varepsilon^2 \sum_{s,l,j=-N}^{N} G_{s,l,j}\,\delta(\omega_p + \omega_s + \omega_l + \omega_j)\phi_s\phi_l\phi_j, \end{aligned} \tag{2.4}$$

where w_p and ϕ_p are Fourier amplitudes of vertical velocity and velocity potential at $z = 0$, g is the acceleration due to gravity, and δ is the Kronecker delta. The Y and G terms are lengthy expressions in terms of wavenumbers and frequencies, and their formulation is given in the appendix. The subscripts denote the frequency, where w_p corresponds to the vertical velocity amplitude of a wave with frequency $\omega_p = p\omega_1$. Note that the ε terms merely indicate the order of the nonlinearity, and could be dropped from the equations.

The bottom boundary condition (after inserting eq. (2.2) into it) is defined as

$$\sin(h\nabla_x)\nabla\phi_p + \cos(h\nabla_x)w_p = 0.$$

By inserting the equation (2.4) into the bottom boundary condition and dividing by $\cos(h\nabla_x)$, the following governing equation is obtained:

$$\Big(\tan(h\nabla_x)\nabla_x + \frac{\omega_p^2}{g}\Big)\phi_p = -\varepsilon \sum_{s,l=-N}^{N} Y_{s,l}\delta(\omega_p + \omega_s + \omega_l)\phi_s\phi_l - \varepsilon^2 \sum_{s,l,j=-N}^{N} G_{s,l,j}\delta(\omega_p + \omega_s + \omega_l + \omega_j)\phi_s\phi_l\phi_j + O(\varepsilon^3), \tag{2.5}$$

The trigonometric functions are to be understood as infinite Taylor series in $h\nabla_x$, where the terms representing a change of the wavenumber ($\nabla_x k$) are discarded.

3. A discussion on nonlinear wave properties

3.1. Nonlinear dispersion. By expanding the velocity potential as

$$\Phi(x,t) = \sum_{p=-N}^{N} \hat{\phi}_p(x)e^{i\omega_p t} = \sum_{p=-N}^{N} b_p e^{i(\omega_p t - k_p x)}, \tag{3.1}$$

and inserting it into equation (2.5), the following relationship between wavenumber (k) and frequency is obtained:

$$(-gk_p \tanh k_p h + \omega_p^2)b_p = O(\varepsilon^2)$$

Note that the the b_p term (amplitude of velocity potential) is still allowed to vary in x, albeit at longer scales compared to the oscillatory behaviour.

The right-hand side (nonlinear part) of the equation (2.5) is often truncated, which results in the well-known linear dispersion relation (equation 1.1). By including the resonant components of a cubic order, an $O(\varepsilon^2)$ correction to the dispersion relation can be obtained from the Zakharov equation [**21**]. This correction has importance in deep waters which allows for large spatial and temporal evolution scales. Note that terms of $O(\varepsilon)$ are commonly disregarded.

The linear dispersion relation was shown to be a reasonable approximation in most cases. However, modeling generation of infragravity (IG) waves using the model of [**4**] can significantly overestimate the energy transfer, as shown in [**2**]. It is hypothesized that one of the reasons for this is that the IG wave regime often has little free wave energy [**7, 8**], in which case the dispersion relation is dominated by bound waves. The common approach practically approximates the spatial derivative in equation (2.5) using the linear solution of the wavenumber k_p for free waves. Clearly this approach will cause discrepancies when bound wave energy outweighs the free wave one. To accommodate this property in the IG regime, the spatial derivative is simply defined as a difference or a sum of forcing wavenumbers which represents the bound wavenumber:

$$k_p = k_l \pm k_s. \tag{3.2}$$

The simplest case is the famous Stokes wave solution, where a single wave (ω_s) generates a bound wave ($\omega_p = 2\omega_s, k_p = 2k_s$) which travels at a same speed as the original wave and modulates the shape of the free surface (sharper crests).

This solution is only accurate for a single pair of forcing waves. In a more general case, where bound and free energy are roughly proportional or when multiple wave frequencies are contributing to the nonlinear forcing, it is necessary to compute the evolution of dispersion relation in parallel to the wave evolution. However, the derivation and analysis of the full dispersion evolution equation goes beyond the scope of this paper.

3.2. Evolution model accounting for cubic nonlinearity. To obtain the model of wave evolution in space, an operator is applied to equation (2.5). It is defined as:

$$H = \frac{\nabla_x + ik'^{x}_{p,l}}{\tan(h\nabla_x)\nabla_x + \frac{\omega_p^2}{g}}.$$

By applying the H operator to the equation (2.5) with a fully expanded formulation of velocity potential from the equation (3.1), it is possible to eliminate the fast oscillatory terms. This results in an evolution equation describing a slow evolution of the amplitude.

One thing to keep in mind is that the k'_p term in the numerator is obtained from the linear dispersion relation for ω_p. In the case of the exact resonance, it is necessary to take the limit for the H operator:

$$H = \frac{ig}{2\omega_p C_{g,p}},$$

where $C_{g,p} = \partial\omega_p/\partial k_p$ is the group velocity (speed of propagation of energy).

In both cases, the spatial evolution equation for the amplitude of velocity potential is:

$$\frac{\partial b_p}{\partial x} = -\sum_{s,l=-N}^{N} W_{s,l}\delta(\omega_p + \omega_s + \omega_l) b_s b_l e^{i\int k_p - ks - k_l dx}$$
$$- \sum_{s,l,j=-N}^{N} Q_{s,l,j}\delta(\omega_p + \omega_s + \omega_l + \omega_j) b_s b_l b_j e^{i\int k_p - ks - k_l - k_j dx}.$$

The surface amplitude of velocity potential was further expanded using $b_p e^{ik_p x} = \phi_p$, and the W and Q terms are defined as $W_{s,l} = H_{s,l}F_{s,l}$ and $Q_{s,l,j} = H_{s,l,j}G_{s,l,j}$ respectively. The approximation of the spatial derivatives in these nonlinear terms should use equation (3.2) when appropriate.

From here it is straightforward to incorporate either free or bound wave formulation of the dispersion relation into the model. An example of the effect of the nonlinear dispersion relation on the magnitude of the W-terms and a comparison of magnitudes of quadratic and cubic nonlinear terms are given in the following section.

4. Examples

4.1. Comparison of interaction terms for free and bound waves. In this section, we investigate the differences in the nonlinear interaction term ($W_{s,l}$) for purely bound and free waves. The overall energy transfer from three-wave interactions is proportional to it, so it is imperative to accurately account for it. For simplicity, we use a simple bichromatic wave field (two waves with periods of 1 and 2/3 seconds), to evaluate differences due to different dispersion relation for superharmonic (transfer of energy to higher frequency) and subharmonic (vice versa) interactions. It is assumed that the wave at a difference frequency (period of 2 seconds) is a bound wave.

While the magnitude of the interaction term for the transfer of energy from free waves is identical to that from previous works (see figure 2), the magnitude of energy transfer from a bound lower frequency harmonic to the waves that force it is far stronger. An underestimation of back-transfer from low to high frequencies is a potential explanation for the discrepancy observed in [**2**], however further validation is needed.

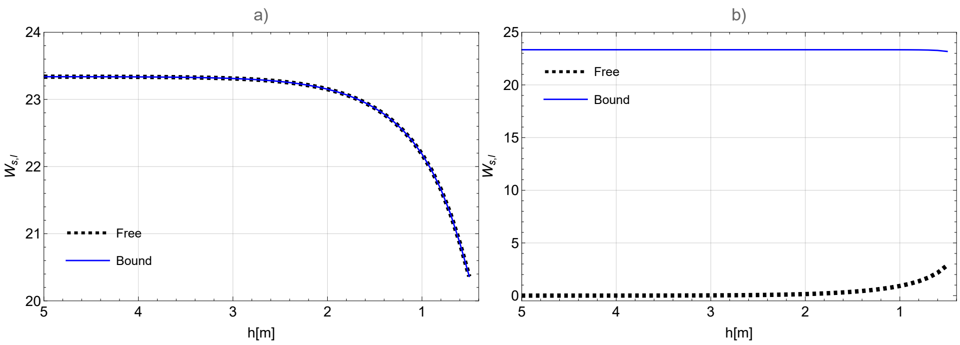

FIGURE 2. Nonlinear interaction terms for free and bound waves as a function of bottom depth. a) The magnitude of the nonlinear term governing subharmonic interaction of two free waves ($W_{3,-2}$). b) The magnitude of the term governing superharmonic interaction of a bound and free wave ($W_{1,2}$) is given in subfigure b. The results shown with the black line are computed under the assumption that all waves follow the linear dispersion relation, and for the results shown with the blue line, it is assumed that wave components 2 and 3 (k_2 and k_3) follow linear dispersion relation and wave component 1 ($k_1 = k_3 - k_2$) follows the bound wave dispersion relation.

We note that the wave regime is often complex, and can not be described as a purely free or purely bound wave. For the IG regime, for example, free and bound wave energies are comparable. Here, as a first step, we limit our analysis to simple test cases presented here to show that one cannot accurately model the combination of both bound and free wave energy by assuming free wave coefficients.

4.2. Magnitudes of quadratic and cubic nonlinearity. Equation (2.4) is used to evaluate the relative magnitudes of quadratic and cubic nonlinear terms ($Y_{s,l}\phi_s\phi_l$ and $G_{s,l,j}\phi_s\phi_l\phi_j$), after expanding amplitudes of velocity potential in wavenumber domain ($\phi_p = b_p e^{-ik_p x}$). A simple flat spectrum is used to define the wave field, where the lowest frequency considered is $\omega_1 = 2\pi/40$, and all of the energy is concentrated in wave components 6 to 10 with $b_6 = \cdots = b_{10} = 1\text{m}^2$. Results are computed as a function of depth, from 50 m to 1.5 m (see figure 3).

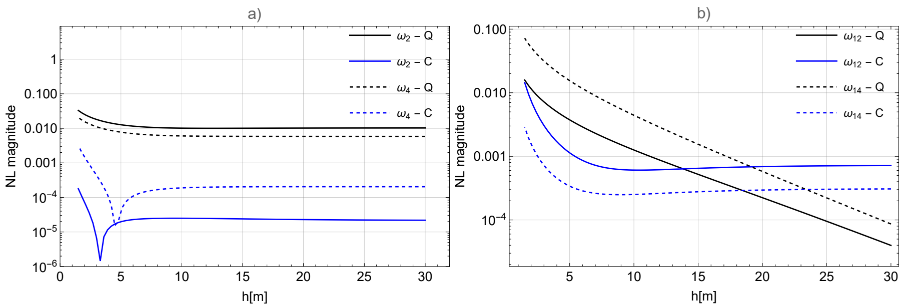

FIGURE 3. Magnitude of the quadratic and cubic nonlinear terms. a) Subharmonic interactions for wave components with frequencies ω_2 and ω_4. b) Superharmonic interactions for wave components with frequencies ω_{12} and ω_{14}. Note that $\omega_p = p\omega_1$, and that the Q and C in legend indicate quadratic and cubic interactions respectively.

It should be stressed that quadratic and cubic interactions transfer most of the energy to different frequencies. Quadratic interactions transfer energy to sum and difference frequency of the two interacting components, while the cubic interactions transfer energy to side-bands. To this end, we have showcased the results for two amplitudes closer in frequency to the initial spectral components (ω_4 and ω_{12}), and two further away from it (ω_2 and ω_{14}).

For the generation of low frequency components, the quadratic terms are dominant throughout the range of depths. For superharmonic interactions, the quadratic terms tend to zero in deep waters, but grow quickly in more shallow waters, overtaking the cubic terms by an order of magnitude. Still, the overall contribution of the cubic terms is not negligible, at least for certain frequencies.

5. Discussion and conclusions

We have presented preliminary findings in this paper, which lay down the foundation for further development of a spectral shoaling model up to cubic nonlinearity.

A new formulation of the quasi-Boussinesq model accounting for nonlinearity up to the cubic order has been derived. The quadratic terms have shown to be dominant as expected in a simple test case. Nevertheless, the contribution of the cubic terms has been found to be non-negligible for superharmonic interactions. This stresses the importance of including the cubic terms in the nonlinear models, as no matter how accurate the quadratic models are they can not account for cubic effects. The main reason behind not considering cubic terms is computational efficiency, however with an ever increasing availability of computational resources, inclusion of cubic terms into models is becoming increasingly feasible.

We have investigated the effects of nonlinear dispersion relation on the transfer of energy and shown a significant discrepancy in the magnitude of the interaction terms for free and bound waves. The results showcasing the effect of the dispersion relation on the nonlinear generation of lower frequency waves are especially interesting, as they significantly improve the accuracy of the model. Modeling of IG waves (which are generated through subharmonic interaction) is becoming an increasingly important task with the launch of satellite altimeters which are able to measure their wavenumber range. Current operational global models for IG waves are largely empirical, and do not give a full description of many parameters (e.g. directionality). The end goal of this line of research is to implement a nonlinear triad source term into WAVEWATCH III (currently being tested) and to develop a dedicated model of IG wave evolution.

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Appendix

The full definition of the surface governing equation is:

$$\begin{aligned}
-W-\frac{\Phi_{tt}}{g} &= -\varepsilon\Big(\frac{2\Phi_{tt}\Phi_{ttt}+\Phi_t\Phi_{tttt}}{g^3}+\frac{2(\nabla_x\Phi_t)(\nabla_x\Phi)+\Phi_t(\nabla_x^2\Phi)}{g}\Big)\\
&+\varepsilon^2\Big(\frac{-2\Phi_{ttt}(\nabla_x\Phi)(\nabla_x\Phi_t)-4\Phi_t(\nabla_x\Phi)(\nabla_x\Phi_{ttt})+2\Phi_{tt}^2(\nabla_x^2\Phi)^2+3\Phi_t^2(\nabla_x^2\Phi_{tt})}{2g^3}\\
&\qquad+\frac{-3g^2(\nabla_x\Phi)^2(\nabla_x^2\Phi)-\Phi_{tt}(4(\nabla_x\Phi)(\nabla_x\Phi_{tt})-6\Phi_t(\nabla_x^2\Phi_t))}{2g^3}\Big).
\end{aligned}$$

The nonlinear interaction terms are defined as:

$$Y_{s,l}=\frac{i\omega_s^2\omega_l^2\omega_p}{2g^3}-\frac{i\omega_s\omega_l\omega_p^3}{2g^3}-\frac{i\omega_p\nabla_s\nabla_l}{g}-\frac{i\omega_l\nabla_s^2}{2g}-\frac{i\omega_s\nabla_l^2}{2g},$$

$$\begin{aligned}
2g^3G_{s,l,j} &= \omega_j(\omega_l(3\omega_s^2+\omega_l\omega_j+6\omega_s(\omega_l+\omega_j))\nabla_s^2\\
&\quad-2(2(\omega_s^3+\omega_l^3)+2(\omega_s^2+\omega_l^2)\omega_j)\nabla_s\nabla_l)\\
&\quad-2\omega_j(\omega_s+\omega_l)\omega_j^2\nabla_s\nabla_l-(\omega_s\omega_l(\omega_s\omega_l+6(\omega_s+\omega_l)\omega_j+3\omega_j^2)
\end{aligned}$$

$$
\begin{aligned}
&- 3g^2\nabla_s\nabla_l)\nabla_j^2 + \omega_j\omega_s(\omega_s(6\omega_l + \omega_j) + 3\omega_l(\omega_l + 2\omega_j))\nabla_l^2 \\
&- 2\omega_l(2\omega_s^3 + 2\omega_s^2\omega_l + \omega_s\omega_l^2 + \omega_j(\omega_l^2 + 2\omega_l\omega_j + 2\omega_j^2))\nabla_s\nabla_j \\
&- (2\omega_s^3(\omega_l + \omega_k) + 4\omega_s^2(\omega_l^2 + \omega_j^2) + 4\omega_s(\omega_l^3 + \omega_j^3) \\
&+ 3g^2\nabla_s^2)\nabla_l\nabla_j - 3g^2\nabla_s\nabla_l^2\nabla_j
\end{aligned}
$$

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НЕЛИНЕАРНА ЕВОЛУЦИЈА ВОДЕНИХ ТАЛАСА И ЊИХОВА РЕЛАЦИЈА ДИСПЕРЗИЈЕ У ПРИОБАЉУ

Резиме. Прелиминарни резултати деривације новог детерминистичког и стохастичког модела еволуције таласа у запреминама воде које нису дубоке су представљени у овом раду. Ово је први модел овог типа у коме су укључени ефекти нелинеарне дисперзије и кубних (четири таласа) интеракција. Помоћу симулација је демонстрирана важност нелинеарне дисперзије на форсиране таласе, што ако се не узме у обзир резултира у погрешној процени трансфера енергије. Такође смо истражили релативну важност кубне нелинеарности, у поређењу са квадратном (интеракције три таласа), и показали да нису занемарљиве. Овај модел је први корак ка инкорporисању ових ефеката у стохастичку формулацију модела, која би могла да се користи за прецизнију прогнозу таласа.

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