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Bounded rank perturbations of a regular matrix pencil

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In this paper we study the possible Kronecker invariants of an arbitrary matrix pencil obtained by bounded rank perturbation of a regular matrix pencil. We solve this problem in the case of a perturbation of minimal rank.

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1. Introduction

Bounded rank perturbations of matrix pencils has been active field of study [2,5,6,10,13,16,20–24,28–30]. One of the main problems considering bounded rank perturbations of matrix pencils arise when the involved pencils are *regular*, see e.g. [1,3,4,7–9,18]. In particular, in [1] the main result is a solution to the following problem:

Theorem 1. *[1, Theorem 3] Let $B(\lambda)$, $C(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ be two regular matrix pencils. Let $\beta_1 \mid \cdots \mid \beta_{n+m}$ and $\gamma_1 \mid \cdots \mid \gamma_{n+m}$ be homogeneous invariant factors of $B(\lambda)$ and $C(\lambda)$, respectively. There exist matrix pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that*

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq m$$

if and only if

$$\beta_i \mid \gamma_{i+m} \quad \text{and} \quad \gamma_i \mid \beta_{i+m}, \quad i = 1, \ldots, n.$$

In Theorem 1 regularity of both of the involved pencils has been imposed. However, much more general, interesting and important problems can be stated, if we restrict this condition to only one of the involved pencils. This opens the question of the possible Kronecker invariants of an arbitrary pencil obtained by bounded rank perturbation of a regular pencil. And, at the same time, the question of the possible Kronecker invariants of a regular pencil obtained by bounded rank perturbation of a given (arbitrary) pencil. These are the problems that we address in this paper.

1.1. Problem formulation

Let $\mathbb{F}$ be an algebraically closed field. Two matrix pencils $D(\lambda) \in \mathbb{F}[\lambda]^{u\times v}$ and $D'(\lambda) \in \mathbb{F}[\lambda]^{u\times v}$ are *strictly equivalent* if and only if there exist invertible matrices $P \in \mathbb{F}^{u\times u}$ and $Q \in \mathbb{F}^{v\times v}$ such that

$$D'(\lambda) = PD(\lambda)Q.$$

The strict equivalence between two matrix pencils $D(\lambda)$ and $D'(\lambda)$ is denoted by $D(\lambda) \sim D'(\lambda)$.

A canonical form for the strict equivalence relation is usually called *the Kronecker canonical form*, and the corresponding invariants are called *the Kronecker invariants*. The set of Kronecker invariants of a matrix pencil consists of invariant factors, infinite elementary divisors, column minimal indices, and row minimal indices, for details see [17, Chapter XII], and also Section 2.1 below. In this paper, a special role is played by *regular* matrix pencils. The matrix pencil $D(\lambda)$ is called regular if it is square and $\det D(\lambda) \not\equiv 0$.

Let ρ be a nonnegative integer. In this paper we are considering bounded rank perturbations which transform a matrix pencil into a regular one. Hence, the involved matrix pencils have to be square. So, let $B(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ be a regular matrix pencil, that is $\operatorname{rank} B(\lambda) = n + m$, and let $C(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ be an arbitrary matrix pencil, with $\operatorname{rank} C(\lambda) = n$. We aim to find necessary and sufficient conditions for the existence of matrix pencils $B'(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ and $C'(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ such that $B'(\lambda) \sim B(\lambda)$, and $C'(\lambda) \sim C(\lambda)$, and such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq \rho.$$

Since $\operatorname{rank} C(\lambda) = n$ and $\operatorname{rank} B(\lambda) = n + m$, we have that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \geq \operatorname{rank} B'(\lambda) - \operatorname{rank} C'(\lambda) = m. \tag{1}$$

Hence, the minimal possible value of ρ is exactly m. In this paper we focus on the case when $\rho = m$, and give an explicit and constructive solution to the following problem:

Problem 1. Let $B(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ be a regular matrix pencil, and let $C(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ be a matrix pencil with $\operatorname{rank} C(\lambda) = n$. Find necessary and sufficient conditions for the existence of matrix pencils $B'(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ and $C'(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ such that $B'(\lambda) \sim B(\lambda)$, and $C'(\lambda) \sim C(\lambda)$, and such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) = m.$$

Note that the bounded perturbation problem with

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq \rho$$

has no solution for $\rho < m$.

Problem 1 is equivalent to the classical bounded perturbation problem of describing the possible Kronecker invariants of a regular pencil $C(\lambda) + X(\lambda)$, where the pencil $C(\lambda)$ is prescribed, and the pencil $X(\lambda)$ is such that the rank of $X(\lambda)$ is m. Indeed, clearly in Problem 1 one can fix one of the two pencils, say, $C'(\lambda) = C(\lambda)$, and then take $X(\lambda) := B'(\lambda) - C(\lambda)$.

At the same time, Problem 1 is equivalent to the classical bounded perturbation problem of describing the possible Kronecker invariants of a pencil $B(\lambda) + X(\lambda)$, where the pencil $B(\lambda)$ is prescribed and regular, and the pencil $X(\lambda)$ is such that the rank of $X(\lambda)$ is m, and $\operatorname{rank}(B(\lambda) + X(\lambda)) = \operatorname{rank} B(\lambda) - m$. Again, clearly in Problem 1 one can fix one of the two pencils, say, $B'(\lambda) = B(\lambda)$, and then take $X(\lambda) := C'(\lambda) - B(\lambda)$.

It is worth mentioning connection with the paper [6], where low rank perturbations of complex matrix pencils were studied in the generic case. It is shown that for perturbation with suitably low ranks, the property that the rank of the perturbed matrix pencil is the sum of the ranks of the unperturbed one and of the perturbation matrix pencil, is a generic property. More precisely, in [6, Theorem 3.1] it was proved that the set of the perturbation matrix pencils satisfying this property, is dense and open in the set of all pencils with the given rank. The topic of this paper deals precisely with such low rank perturbations: the rank of the perturbed pencil $B(\lambda)$ is equal to the sum of the ranks of the unperturbed matrix pencil $C(\lambda)$ and the perturbation pencil $X(\lambda)$.

In the paper [6] various necessary conditions on the Kronecker invariants of the involved pencils are obtained. In this paper, we give much stronger result in the case when $B(\lambda)$ is a regular pencil, and we obtain explicit necessary and sufficient condition for the existence of the desired perturbation in this case.

Although Problem 1 does not represent the most general bounded rank perturbation problem for a given regular pencil, it is an important step in that direction. The main feature is that we allow that $C(\lambda)$ has no kind of structural or any other restriction. This makes Problem 1 very general, and different comparing to the existing results in the literature, see e.g. [3,4,8,9,18].

One particularly interesting property of the obtained necessary and sufficient conditions is that the column and row minimal indices of $C(\lambda)$ appear in them only as a union. Therefore, one needs to know only the union of column and row minimal indices, and does not need to identify them separately.

Finally, the methods we are using come from Matrix Theory and Matrix Pencils Completion Problems, and they work perfectly for this problem.

2. Matrix pencils and partitions of integers

2.1. Kronecker canonical form and Kronecker invariants

The canonical form for the strict equivalence relation – *the Kronecker canonical form*, and the corresponding Kronecker invariants of the matrix pencils are well studied. The set of Kronecker invariants of a matrix pencil consists of invariant factors, infinite elementary divisors, column minimal indices, and row minimal indices. In this paper we consider invariant factors and infinite elementary divisors of a pencil unified as *homogeneous invariant factors*, for all details see e.g. [17, Chapter XII].

The number of Kronecker invariants of a pencil can be expressed in terms of the size and the rank of a matrix pencil as follows: a pencil $D(\lambda) \in \mathbb{F}[\lambda]^{u \times v}$ with $\tau = \operatorname{rank} D(\lambda)$, has τ homogeneous invariant factors, $u - \tau$ row minimal indices, and $v - \tau$ column minimal indices. Also, the sum of the column minimal indices, the row minimal indices and the degrees of the homogeneous invariant factors of $D(\lambda)$ equals its rank (τ). For all details about Kronecker canonical form, and Kronecker invariants, see e.g. [17, Chapter XII].

Throughout the paper all polynomials will be monic and homogeneous in two variables, λ and μ, over an algebraically closed field $\mathbb{F}$. A polynomial $p \in \mathbb{F}[\lambda, \mu]$ is called homogeneous of degree k, if $p(c\lambda, c\mu) = c^k p(\lambda, \mu)$, for every $c \in \mathbb{F}$. We denote the degree of a homogeneous polynomial p, by $d(p)$. A homogeneous polynomial $p \in \mathbb{F}[\lambda, \mu]$ is called monic if the coefficient of the monomial with the largest exponent of λ is equal to 1.

For any chain of homogeneous polynomials $\alpha_1 \mid \cdots \mid \alpha_n$ by convention we set

$$\alpha_i := 1, \quad \text{for} \quad i \leq 0; \quad \text{and} \quad \alpha_i := 0, \quad \text{for} \quad i \geq n+1. \tag{2}$$

For any polynomial p, we have $1 \mid p$ and $p \mid 0$.

As said before, in this paper *regular* matrix pencils are of particular interest. Their main feature is that the full set of Kronecker invariants of a regular matrix pencil consists only of its homogeneous invariant factors.

2.2. Partitions of integers

By a partition we mean a non-increasing sequence of integers. Also, for every set of non-increasing integers $a_1 \geq \cdots \geq a_s$, we can define the corresponding partition $\mathbf{a} = (a_1, \ldots, a_s)$. For any partition $\mathbf{a} = (a_1, \ldots, a_s)$ we shall assume that $a_i := +\infty$, for $i \leq 0$, and $a_i := -\infty$, for $i > s$.

For two partitions $\mathbf{a} = (a_1, \ldots, a_s)$ and $\mathbf{b} = (b_1, \ldots, b_t)$, by $\mathbf{a} \cup \mathbf{b}$ we denote the partition whose parts are $a_1, \ldots, a_s, b_1, \ldots, b_t$, i.e. the partition obtained by putting the sequence $a_1, \ldots, a_s, b_1, \ldots, b_t$ in non-increasing ordering. Also, we set $\sum_{i=u}^{v} a_i = 0$ whenever $u > v$.

Definition 1. [19] Consider partitions $\mathbf{a} = (a_1, \ldots, a_s)$ and $\mathbf{b} = (b_1, \ldots, b_s)$. If

$$\sum_{i=1}^{j} a_i \leq \sum_{i=1}^{j} b_i, \quad j = 1, \ldots, s-1, \tag{3}$$

$$\sum_{i=1}^{s} a_i = \sum_{i=1}^{s} b_i, \tag{4}$$

then we say that $\mathbf{a}$ is *majorized* by $\mathbf{b}$, and we write $\mathbf{a} \prec \mathbf{b}$.

The following simple combinatorial lemmas will be useful.

Lemma 1. *Let $\mathbf{a} = (a_1, \ldots, a_s)$ and $\mathbf{b} = (b_1, \ldots, b_s)$ be partitions such that*

$$\mathbf{a} \prec \mathbf{b}.$$

Let $v \in \{1, \ldots, s\}$. Let $\tilde{\mathbf{a}} = (\tilde{a}_1, \ldots, \tilde{a}_s)$ be the partition obtained as non-increasing ordering of $a_1, \ldots, a_v + 1, \ldots, a_s$, and let $\tilde{\mathbf{b}} = (\tilde{b}_1, \ldots, \tilde{b}_s)$ be the partition obtained as non-increasing ordering of $b_1, \ldots, b_v + 1, \ldots, b_s$. Then

$$\tilde{\mathbf{a}} \prec \tilde{\mathbf{b}}.$$

Proof. Let $l \in \{1, \dots, v\}$ be the minimal index such that $a_l = a_v$, and let $L \in \{1, \dots, v\}$ be the minimal index such that $b_L = b_v$. Then $\tilde{\mathbf{a}} = (a_1, \dots, a_{l-1}, a_l + 1, a_{l+1}, \dots, a_s)$, and $\tilde{\mathbf{b}} = (b_1, \dots, b_{L-1}, b_L + 1, b_{L+1}, \dots, b_s)$. Therefore, since $\mathbf{a} \prec \mathbf{b}$, from (3) and (4), we have that

$$\sum_{i=1}^{j} \tilde{a}_i \leq \sum_{i=1}^{j} \tilde{b}_i, \tag{5}$$

for all indices j such that $j \leq l-1$, as well as all indices j with $j \geq L$. Hence, if $L \leq l$, we have that (5) holds for all $j \in \{1, \dots, s\}$.

If $l < L$, let j be such that $l \leq j < L$. Then we must have $\sum_{i=1}^{j} a_i < \sum_{i=1}^{j} b_i$. Indeed, otherwise (by (3)) we would have $\sum_{i=1}^{j} a_i = \sum_{i=1}^{j} b_i$, which together with $\sum_{i=1}^{j-1} a_i \leq \sum_{i=1}^{j-1} b_i$, and $\sum_{i=1}^{L} a_i \leq \sum_{i=1}^{L} b_i$ (both following from (3)), implies $a_j \geq b_j$, and $\sum_{i=j+1}^{L} a_i \leq \sum_{i=j+1}^{L} b_i$, respectively. However, these two inequalities cannot hold simultaneously, since by assumption we have $a_j = a_{j+1} = \cdots = a_L$, and $b_j \geq b_{j+1} \geq \cdots \geq b_{L-1} > b_L$. Therefore $\sum_{i=1}^{j} a_i < \sum_{i=1}^{j} b_i$, and so $\sum_{i=1}^{j} \tilde{a}_i \leq \sum_{i=1}^{j} \tilde{b}_i$.

Altogether, we have shown that (5) holds for all $j \in \{1, \dots, s\}$, and since the total sums are clearly equal by (4), we have $\tilde{\mathbf{a}} \prec \tilde{\mathbf{b}}$, thus proving the lemma. □

Lemma 2. *Let* $\mathbf{a} = (a_1, \dots, a_s)$ *and* $\mathbf{b} = (b_1, \dots, b_s)$ *be partitions such that*

$$\mathbf{a} \prec \mathbf{b}.$$

Let $1 \leq u \leq s$ *be an integer, let* $\mathbf{a}'$ *be the partition obtained as non-increasing ordering of* $a_1, \dots, a_u, a_{u+1} + 1, \dots, a_s + 1$, *and let* $\mathbf{b}'$ *be the partition obtained as non-increasing ordering of* $b_1, \dots, b_u, b_{u+1} + 1, \dots, b_s + 1$. *Then*

$$\mathbf{a}' \prec \mathbf{b}'.$$

Proof. The proof follows by repeated application of Lemma 1. Indeed, let $\mathbf{a^{(j)}} = (a_1^{(j)}, \dots, a_s^{(j)})$, for $j = 0, \dots, s-u$, be partitions defined recursively by: $\mathbf{a^{(0)}} := \mathbf{a}$, while for $j = 1, \dots, s-u$, partition $\mathbf{a^{(j)}} = (a_1^{(j)}, \dots, a_s^{(j)})$ is obtained as non-increasing ordering of $a_1^{(j-1)}, \dots, a_{u+j}^{(j-1)} + 1, \dots, a_s^{(j-1)}$.

Analogously, let $\mathbf{b^{(j)}} = (b_1^{(j)}, \dots, b_s^{(j)})$, for $j = 0, \dots, s-u$, be partitions defined recursively by: $\mathbf{b^{(0)}} := \mathbf{b}$, while for $j = 1, \dots, s-u$, partition $\mathbf{b^{(j)}} = (b_1^{(j)}, \dots, b_s^{(j)})$ is obtained as non-increasing ordering of $b_1^{(j-1)}, \dots, b_{u+j}^{(j-1)} + 1, \dots, b_s^{(j-1)}$.

Now, since $\mathbf{a} \prec \mathbf{b}$, by repeated use of Lemma 1, we obtain that

$$\mathbf{a^{(j)}} \prec \mathbf{b^{(j)}}, \qquad \text{for every} \qquad j = 0, \dots, s-u. \tag{6}$$

Finally, since $\mathbf{a}' = \mathbf{a^{(s-u)}}$ and $\mathbf{b}' = \mathbf{b^{(s-u)}}$, majorization (6) for $j = s-u$ gives $\mathbf{a}' \prec \mathbf{b}'$, as desired. □

In this paper *minimal paths* between given polynomial chains play a central role. This combinatorial concept, related to polynomial chains satisfying interlacing properties has been widely studied, and has applications in different branches of mathematics. Minimal paths appear frequently in matrix and matrix pencil completion problems (see e.g. [15,25–27,30–32]). We cite here Lemmas 1 and 2 from [15], in the notation that we will use throughout the paper.

Lemma 3. *[15, Lemmas 1,2] Let* $\bar{\beta} : \bar{\beta}_1 \mid \cdots \mid \bar{\beta}_{n+2m}$ *and* $\gamma : \gamma_1 \mid \cdots \mid \gamma_n$ *be two polynomial chains. Let*

$$\pi_j = \prod_{i=1}^{n+j} \operatorname{lcm}(\bar{\beta}_i, \gamma_{i-j}), \quad j = 0, \dots, 2m.$$

Then

$$\pi_0 \mid \pi_1 \mid \cdots \mid \pi_{2m}.$$

Also, let

$$\sigma_j = \frac{\pi_j}{\pi_{j-1}}, \quad \text{for} \quad j = 1, \dots, 2m.$$

Then

$$\sigma_1 \mid \cdots \mid \sigma_{2m}.$$

If in addition the following divisibility is satisfied

$$\bar{\beta}_i \mid \gamma_i \mid \bar{\beta}_{i+2m}, \quad i = 1, \dots, n, \tag{7}$$

then we also have [15, Lemma 3]

$$\pi_0 = \prod_{i=1}^{n} \gamma_i, \quad \text{and} \quad \pi_{2m} = \prod_{i=1}^{n+2m} \bar{\beta}_i.$$

Therefore if (7) holds, we have

$$\sum_{i=1}^{j} d(\sigma_{2m-i+1}) = \sum_{i=1}^{n+2m} d(\bar{\beta}_i) - \sum_{i=1}^{n+2m-j} d(\operatorname{lcm}(\bar{\beta}_i, \gamma_{i-2m+j})), \quad j = 0, \dots, 2m. \tag{8}$$

3. Auxiliary results

In this section we give a notation that will be used throughout the paper. Also, we have collected all theorems that will be used in the proof of the main result. They work perfectly for a solution to Problem 1.

3.1. Notation

In this paper we are considering bounded rank perturbations which transform a matrix pencil into a regular one. Hence, the matrix pencils involved have to be square. The following notation will be used throughout the paper:

Let $B(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ be a regular matrix pencil having

$$\beta_1 \mid \cdots \mid \beta_{n+m} \quad - \quad \text{as homogeneous invariant factors.}$$

Let $C(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ be an arbitrary matrix pencil, $n = \operatorname{rank} C(\lambda)$, having

$$\begin{aligned} \gamma_1 \mid \cdots \mid \gamma_n \quad &- \quad \text{as homogeneous invariant factors,} \\ d_1 \geq \cdots \geq d_m \quad &- \quad \text{as column minimal indices,} \\ \bar{r}_1 \geq \cdots \geq \bar{r}_m \quad &- \quad \text{as row minimal indices.} \end{aligned}$$

Note that

$$\sum_{i=1}^{n} d(\gamma_i) + \sum_{i=1}^{m} d_i + \sum_{i=1}^{m} \bar{r}_i + m = \sum_{i=1}^{n+m} d(\beta_i) = n + m. \tag{9}$$

3.2. Auxiliary theorems

The first result we cite is the main result from [11], which studies a regular matrix pencil with a prescribed subpencil.

Theorem 2. *[11] Let $B(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ be the regular matrix pencil as given in Section 3.1. Let x be a nonnegative integer such that $x \leq m$, and let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+x)\times(n+m-x)}$ be a matrix pencil, with* $\operatorname{rank} A(\lambda) = n$*. Let $\alpha_1 \mid \cdots \mid \alpha_n$ be the homogeneous invariant factors of $A(\lambda)$, and let $c_1 \geq \cdots \geq c_{m-x}$ and $r_1 \geq \cdots \geq r_x$ be the column and row minimal indices of $A(\lambda)$, respectively. There exist matrix pencils $W(\lambda) \in \mathbb{F}[\lambda]^{(n+x)\times x}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{(m-x)\times(n+m-x)}$, and $Z(\lambda) \in \mathbb{F}[\lambda]^{(m-x)\times x}$ such that*

$$\left[\begin{array}{c|c} A(\lambda) & W(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array}\right] \sim B(\lambda)$$

if and only if

$$\beta_i \mid \alpha_i \mid \beta_{i+m}, \quad i = 1, \ldots, n, \tag{10}$$

$$\sum_{i=1}^{j}(c_i + 1) + \sum_{i=1}^{k}(r_i + 1) \leq \sum_{i=1}^{n+m} d(\beta_i) - \sum_{i=1}^{n+m-j-k} d(\operatorname{lcm}(\alpha_{i+j+k-m}, \beta_i)),$$
$$j = 0, \ldots, m - x, \quad k = 0, \ldots, x \tag{11}$$

In [12] we have studied the General Matrix Pencil Completion Problem in the case of *minimal* completions, i.e. when the size of the completion is the minimal possible. The following theorem is one of the main results in [12]. It deals with a completion of an arbitrary matrix pencil with a prescribed subpencil of the same rank, where the size of the completion is bounded by the difference in the number of column and row minimal indices of the involved pencils.

Theorem 3. *[12, Theorem 4.2] Let $C(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ be the matrix pencil as given in Section 3.1. Let x be a nonnegative integer, $x \leq m$, and let $A(\lambda) \in \mathbb{F}[\lambda]^{(n+x)\times(n+m-x)}$ be a matrix pencil, with $\operatorname{rank} A(\lambda) = n$. Let $\alpha_1 \mid \cdots \mid \alpha_n$ be the homogeneous invariant factors of $A(\lambda)$, and let $c_1 \geq \cdots \geq c_{m-x}$, and $r_1 \geq \cdots \geq r_x$ be the column and row minimal indices of $A(\lambda)$, respectively. Also, let r be the number of nonzero r_i's, c be the number of nonzero c_i's, d be the number of nonzero d_i's, and $\bar{r}$ be the number of nonzero $\bar{r}_i$'s.*

There exist matrix pencils $\bar{W}(\lambda) \in \mathbb{F}[\lambda]^{(n+x)\times x}$, $\bar{Y}(\lambda) \in \mathbb{F}[\lambda]^{(m-x)\times(n+m-x)}$, and $\bar{Z}(\lambda) \in \mathbb{F}[\lambda]^{(m-x)\times x}$ such that

$$\left[\begin{array}{c|c} A(\lambda) & \bar{W}(\lambda) \\ \hline \bar{Y}(\lambda) & \bar{Z}(\lambda) \end{array}\right] \sim C(\lambda)$$

if and only if

$$\gamma_i \mid \alpha_i, \quad i = 1, \dots, n, \qquad \text{and} \qquad \alpha_i \mid \gamma_{i+m}, \quad i = 1, \dots, n-m, \tag{12}$$

$$\bar{r} \geq r, \quad d \geq c, \tag{13}$$

$$r_i \geq \bar{r}_{i+m-x}, \quad i = 1, \dots, x, \quad c_i \geq d_{i+x}, \quad i = 1, \dots, m-x, \tag{14}$$

$$\sum_{i=1}^{h_{j'}} d_i - \sum_{i=1}^{h_{j'}-j'} c_i + \sum_{i=1}^{v_{k'}} \bar{r}_i - \sum_{i=1}^{v_{k'}-k'} r_i \leq \sum_{i=1}^{n} d(\alpha_i) - \sum_{i=1}^{n} d(\operatorname{lcm}(\alpha_{i-j'-k'}, \gamma_i)),$$
$$j' = 0, \dots, x, \quad k' = 0, \dots, m-x. \tag{15}$$

Here $h_{j'} = \min\{i | c_{i-j'+1} < d_i\}$, $j' = 1, \dots, x$, $h_0 = 0$, and $v_{k'} = \min\{i | r_{i-k'+1} < \bar{r}_i\}$, $k' = 1, \dots, m-x$, $v_0 = 0$.

Remark 1. We note that there is a typo in [12, Theorem 4.2]. The last summation in condition $(iv.2)$ of [12, Theorem 4.2] should start from $i = 1-j-k$, and not from $i = 1$, since there should be shift of indices in the summation in the formula (4.55) at the page 366 of [12]. So, the correct form of [12, Theorem 4.2] is the one given in the present paper (Theorem 3 – condition (15)).

Finally, we shall use the combinatorial result from [14, Theorem 1]. It tackles interlacing of polynomial chains. Here we cite it in the form suitable for the purpose of this paper.

Theorem 4. *Let $\gamma : \gamma_1 \mid \cdots \mid \gamma_n$ and $\bar\beta : \bar\beta_1 \mid \cdots \mid \bar\beta_{n+2m}$ be polynomial chains. Let $X_1, \dots, X_{m-1}$, $Y_1, \dots, Y_{m-1}$ and Z be nonnegative integers.*

There exists a polynomial chain $\bar\alpha : \bar\alpha_1 \mid \cdots \mid \bar\alpha_{n+m}$ satisfying

$$\bar\alpha_i \mid \gamma_i \mid \bar\alpha_{i+m}, \quad i = 1, \dots, n, \tag{16}$$

$$\bar\beta_i \mid \bar\alpha_i \mid \bar\beta_{i+m}, \quad i = 1, \dots, n+m, \tag{17}$$

$$\textstyle\sum_{i=1}^{n+m} d(\bar\alpha_i) = Z, \tag{18}$$

$$\textstyle\sum_{i=1}^{n+m-j} d(\mathrm{lcm}(\gamma_{i-m+j}, \bar\alpha_i)) \le X_j, \quad j = 1, \dots, m-1, \tag{19}$$

$$\textstyle\sum_{i=1}^{n+2m-k} d(\mathrm{lcm}(\bar\alpha_{i-m+k}, \bar\beta_i)) \le Y_k, \quad k = 1, \dots, m-1, \tag{20}$$

if and only if

$$\bar\beta_i \mid \gamma_i \mid \bar\beta_{i+2m}, \quad i = 1, \dots, n, \tag{21}$$

$$\sum_{i=1}^{n+2m-j-k} d(\mathrm{lcm}(\bar\beta_i, \gamma_{i-2m+j+k})) \le X_j + Y_k - Z, \quad j, k = 0, \dots, m. \tag{22}$$

Here $X_m = \sum_{i=1}^{n} d(\gamma_i)$, $X_0 = Z$, $Y_0 = \sum_{i=1}^{n+2m} d(\bar\beta_i)$ and $Y_m = Z$.

To end this section we give a purely combinatorial result that will be used in the sufficiency part of the proof of the main result of the paper.

Lemma 4. *Let m and x be nonnegative integers such that $m \ge x$. Let $d_1 \ge \cdots \ge d_m$ be nonnegative integers, and let $c_1 \ge \cdots \ge c_{m-x}$ be such that $c_i = d_{x+i}$, for $i = 1, \dots, m-x$. Let*

$$h_j := \min\{i \in \{1, \dots, m\} \mid c_{i-j+1} < d_i\}, \quad j = 1, \dots, x, \ \textit{and} \quad h_0 := 0. \tag{23}$$

Then

$$\sum_{i=1}^{h_j} d_i - \sum_{i=1}^{h_j-j} c_i = \sum_{i=1}^{j} d_i, \quad j = 0, \dots, x. \tag{24}$$

Proof. Let

$$t := \min\{i \in \{1, \dots, x+1\} \mid d_i = d_{x+1}\} - 1,$$
$$s := \max\{i \in \{1, \dots, m-x\} \mid c_i = c_1\}.$$

Then by (23) we have

$$h_j = \begin{cases} j, & j = 0, \dots, t, \\ j + s, & j = t+1, \dots, x. \end{cases}$$

Hence (24) for $j = 0, \dots, t$ is trivially satisfied. As for $j = t+1, \dots, x$, the left hand side of (24) becomes

$$\sum_{i=1}^{h_j} d_i - \sum_{i=1}^{h_j - j} c_i = \sum_{i=1}^{j+s} d_i - \sum_{i=1}^{s} c_i = \sum_{i=1}^{j} d_i + \sum_{i=j+1}^{j+s} d_i - \sum_{i=1}^{s} c_i. \tag{25}$$

By the definition of s we have $\sum_{i=1}^{s} c_i = sc_1$. Moreover, from the definitions of s and t, and since $c_i = d_{x+i}$, $i = 1, \dots, m-x$, we have that $d_i = c_1$, for $i = t+1, \dots, x+s$. Hence, for j such that $t+1 \leq j \leq x$, we have $\sum_{i=j+1}^{j+s} d_i = sc_1$, and so (25) becomes

$$\sum_{i=1}^{h_j} d_i - \sum_{i=1}^{h_j - j} c_i = \sum_{i=1}^{j} d_i, \quad j = t+1, \dots, x.$$

Altogether we have proved (24), as desired.

Remark 2. We note that Lemma 4 will be used when applying Theorem 3 in the case when the column and row minimal indices of the pencil $A(\lambda)$ are specific subsets of the column and the row minimal indices of the pencil $C(\lambda)$. In particular, by using the notation from Theorem 3, if $r_i = \bar{r}_{i+m-x}$, $i = 1, \dots, x$, and $c_i = d_{i+x}$, $i = 1, \dots, m-x$, conditions (13) and (14) are trivially satisfied, while by Lemma 4, the condition (15) simplifies to

$$\sum_{i=1}^{j'} d_i + \sum_{i=1}^{k'} \bar{r}_i \leq \sum_{i=1}^{n} d(\alpha_i) - \sum_{i=1}^{n} d(\operatorname{lcm}(\alpha_{i-j'-k'}, \gamma_i)), \ \ j' = 0, \dots, x, \ \ k' = 0, \dots, m-x.$$

4. Main result

By using the notation from Section 3.1, in the following theorem we give an explicit and constructive solution to Problem 1, which is the main result of the paper.

Let $B(\lambda)$ and $C(\lambda)$ be matrix pencils as given in Section 3.1. Let $\bar{\beta} : \bar{\beta}_1 \mid \cdots \mid \bar{\beta}_{n+2m}$ be the polynomial chain given by

$$\bar{\beta}_1 = \bar{\beta}_2 = \cdots = \bar{\beta}_m := 1 \tag{26}$$

$$\bar{\beta}_{i+m} := \beta_i, \quad i = 1, \dots, n+m. \tag{27}$$

Then for the polynomial chains $\bar{\beta} : \bar{\beta}_1 \mid \cdots \mid \bar{\beta}_{n+2m}$ and $\gamma : \gamma_1 | \cdots | \gamma_n$ we can define polynomial chain $\sigma : \sigma_1 | \cdots | \sigma_{2m}$ as in Lemma 3. In particular, we have

$$d(\sigma_{2m}) \geq \ldots \geq d(\sigma_1), \tag{28}$$

and so $(d(\sigma_{2m}), \ldots, d(\sigma_1))$ is a partition.

Now, by using the above notation, we can give our main result.

Theorem 5. *Let $B(\lambda)$ and $C(\lambda)$ be matrix pencils as in Section 3.1. There exist matrix pencils $B'(\lambda) \sim B(\lambda)$, and $C'(\lambda) \sim C(\lambda)$, such that*

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) = m,$$

if and only if

$$(i) \quad \gamma_i | \beta_{i+m}, \quad i = 1, \ldots, n, \qquad \beta_i | \gamma_{i+m}, \quad i = 1, \ldots, n-m,$$

$$(ii) \quad \mathbf{d} \cup \bar{\mathbf{r}} \prec (d(\sigma_{2m}), \ldots, d(\sigma_{m+1}), d(\sigma_m) - 1, \ldots, d(\sigma_1) - 1)$$

Proof. *Necessity of conditions:* Let us assume that there exist matrix pencils $B'(\lambda) \sim B(\lambda)$, and $C'(\lambda) \sim C(\lambda)$, such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) = m. \tag{29}$$

Since the rank of the matrix $B'(\lambda) - C'(\lambda)$ is equal to m, this means that the matrix pencil $B'(\lambda) - C'(\lambda)$ can be put by strict equivalence operations in the form:

$$m - x \left\{ \begin{bmatrix} \mathbf{0} & \overbrace{*}^{x} \\ * & * \end{bmatrix} \right.$$

for some $m \geq x \geq 0$. In other words, there exist invertible matrices $P, Q \in \mathbb{F}^{(n+m)\times(n+m)}$, such that

$$P(B'(\lambda) - C'(\lambda))Q = \left[\begin{array}{c|c} \mathbf{0} & * \\ \hline * & * \end{array}\right]$$

where $\mathbf{0} \in \mathbb{F}[\lambda]^{(n+x)\times(n+m-x)}$, is a matrix pencil whose all entries are equal to 0. The last means that the matrix pencils $PB'(\lambda)Q$ and $PC'(\lambda)Q$ are of the form:

$$PB'(\lambda)Q = \left[\begin{array}{c|c} A(\lambda) & W(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array}\right],$$

$$PC'(\lambda)Q = \left[\begin{array}{c|c} A(\lambda) & \bar{W}(\lambda) \\ \hline \bar{Y}(\lambda) & \bar{Z}(\lambda) \end{array}\right],$$

for some matrix pencils $A(\lambda) \in \mathbb{F}[\lambda]^{(n+x)\times(n+m-x)}$, $W(\lambda), \bar{W}(\lambda) \in \mathbb{F}[\lambda]^{(n+x)\times x}$, $Y(\lambda), \bar{Y}(\lambda) \in \mathbb{F}[\lambda]^{(m-x)\times(n+m-x)}$, and $Z(\lambda), \bar{Z}(\lambda) \in \mathbb{F}[\lambda]^{(m-x)\times x}$.

Finally, since $PB'(\lambda)Q \sim B'(\lambda) \sim B(\lambda)$, and $PC'(\lambda)Q \sim C'(\lambda) \sim C(\lambda)$, we have that

$$\left[\begin{array}{c|c} A(\lambda) & W(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array}\right] \sim B(\lambda), \tag{30}$$

and

$$\left[\begin{array}{c|c} A(\lambda) & \bar{W}(\lambda) \\ \hline \bar{Y}(\lambda) & \bar{Z}(\lambda) \end{array}\right] \sim C(\lambda). \tag{31}$$

From the form of matrix pencils (30) and (31), we have that:

$$\operatorname{rank} A(\lambda) \leq \operatorname{rank} B(\lambda) \leq \operatorname{rank} A(\lambda) + m,$$
$$\operatorname{rank} A(\lambda) \leq \operatorname{rank} C(\lambda) \leq \operatorname{rank} A(\lambda) + m.$$

Since $\operatorname{rank} B(\lambda) = n + m$, and $\operatorname{rank} C(\lambda) = n$, we have that rank of $A(\lambda)$ must be equal to n. Let us denote the Kronecker invariants of $A(\lambda)$ by

$\alpha_1 \mid \cdots \mid \alpha_n$	–	the homogeneous invariant factors,
$c_1 \geq \cdots \geq c_{m-x}$	–	the column minimal indices,
$r_1 \geq \cdots \geq r_x$	–	the row minimal indices.

Since $B(\lambda)$ is a regular pencil, we can apply Theorem 2 for the completion (30). Thus, we obtain conditions (10) and (11).

Moreover, for the completion (31) we can apply Theorem 3, and in such a way obtain conditions (12)-(15).

From (10) and (12) we obtain (i).

From (11) we have

$$\sum_{i=1}^{n+m-j-k} d(\operatorname{lcm}(\alpha_{i+j+k-m}, \beta_i)) \leq \sum_{i=1}^{n+m} d(\beta_i) - \sum_{i=1}^{j}(c_i + 1) - \sum_{i=1}^{k}(r_i + 1), \tag{32}$$

for all $j = 0, \ldots, m - x$, $k = 0, \ldots, x$. And from (15) we have

$$\sum_{i=1}^{n} d(\operatorname{lcm}(\alpha_{i-j'-k'}, \gamma_i) \leq \sum_{i=1}^{n} d(\alpha_i) - \sum_{i=1}^{h_{j'}} d_i + \sum_{i=1}^{h_{j'}-j'} c_i - \sum_{i=1}^{v_{k'}} \bar{r}_i + \sum_{i=1}^{v_{k'}-k'} r_i, \tag{33}$$

for all $j' = 0, \ldots, x$, $k' = 0, \ldots, m - x$.

Let us consider polynomial chain $\bar{\beta}_1 \mid \cdots \mid \bar{\beta}_{n+2m}$ as given in (26)-(27), and let us define a polynomial chain $\bar{\alpha}_1 \mid \cdots \mid \bar{\alpha}_{n+m}$ in the following way

$$\bar{\alpha}_1 = \bar{\alpha}_2 = \cdots = \bar{\alpha}_m := 1$$
$$\bar{\alpha}_{i+m} := \alpha_i, \quad i = 1, \ldots, n.$$

Then from (10) and (12) we have (16) and (17).

Now, condition (32) becomes

$$\sum_{i=1}^{n+2m-s} d(\operatorname{lcm}(\bar{\alpha}_{i-m+s}, \bar{\beta}_i)) \leq \sum_{i=1}^{n+2m} d(\bar{\beta}_i) - \max_{\substack{j=0,\ldots,m-x \\ k=0,\ldots,x \\ j+k=s}} \left[\sum_{i=1}^{j} (c_i + 1) + \sum_{i=1}^{k} (r_i + 1) \right], \tag{34}$$

for $s = 0, \ldots, m$. Also, condition (33) becomes

$$\sum_{i=1}^{n+m-t} d(\operatorname{lcm}(\bar{\alpha}_i, \gamma_{i-m+t})) \leq Z - \max_{\substack{j'=0,\ldots,x \\ k'=0,\ldots,m-x \\ j'+k'=t}} \left[\sum_{i=1}^{h_{j'}} d_i - \sum_{i=1}^{h_{j'}-j'} c_i + \sum_{i=1}^{v_{k'}} \bar{r}_i - \sum_{i=1}^{v_{k'}-k'} r_i \right] \tag{35}$$

for $t = 0, \ldots, m$. Here (18) holds. Let us denote by

$$X_t := Z - \max_{\substack{j'=0,\ldots,x, \\ k'=0,\ldots,m-x, \\ j'+k'=t}} \left[\sum_{i=1}^{h_{j'}} d_i - \sum_{i=1}^{h_{j'}-j'} c_i + \sum_{i=1}^{v_{k'}} \bar{r}_i - \sum_{i=1}^{v_{k'}-k'} r_i \right], \quad t = 0, \ldots, m.$$

Then (35) becomes

$$\sum_{i=1}^{n+m-t} d(\operatorname{lcm}(\bar{\alpha}_i, \gamma_{i-m+t})) \leq X_t, \quad t = 0, \ldots, m. \tag{36}$$

We note that $X_0 = Z$, and $X_m = \sum_{i=1}^{n} d(\gamma_i)$ (because of (14), (16) and (36)). Also, let us denote by

$$Y_s := \sum_{i=1}^{n+2m} d(\bar{\beta}_i) - \max_{\substack{j=0,\ldots,m-x, \\ k=0,\ldots,x, \\ j+k=s}} \left[\sum_{i=1}^{j} (c_i + 1) + \sum_{i=1}^{k} (r_i + 1) \right], \quad s = 0, \ldots, m.$$

Then (34) becomes

$$\sum_{i=1}^{n+2m-s} d(\mathrm{lcm}(\bar{\beta}_i, \bar{\alpha}_{i-m+s})) \leq Y_s, \quad s = 0, \dots, m. \tag{37}$$

We note that $Y_0 = \sum_{i=1}^{n+2m} d(\bar{\beta}_i)$, and $Y_m = Z$, by definition. Now, from (16), (17), (18), (36) and (37), by applying Theorem 4 we get (21) and (22).

By getting back to the initial notation, i.e. by the definitions of X_t, Y_s and Z, the inequalities (22) become

$$\sum_{i=1}^{j}(c_i+1) + \sum_{i=1}^{k}(r_i+1) + \sum_{i=1}^{h_{j'}} d_i - \sum_{i=1}^{h_{j'}-j'} c_i + \sum_{i=1}^{v_{k'}} \bar{r}_i - \sum_{i=1}^{v_{k'}-k'} r_i \leq \tag{38}$$

$$\leq \sum_{i=1}^{n+2m} d(\bar{\beta}_i) - \sum_{i=1}^{n+2m-j'-k'-j-k} d(\mathrm{lcm}(\bar{\beta}_i, \gamma_{i-2m+j'+k'+j+k}))$$

for all $j, k' = 0, \dots, m-x$, and all $j', k = 0, \dots, x$.

By (8) we have

$$\sum_{i=1}^{n+2m} d(\bar{\beta}_i) - \sum_{i=1}^{n+2m-j'-k'-j-k} d(\mathrm{lcm}(\bar{\beta}_i, \gamma_{i-2m+j'+k'+j+k})) = \sum_{i=1}^{j+k+j'+k'} d(\sigma_{2m-i+1}),$$

for all $j, k' = 0, \dots, m-x$, and all $j', k = 0, \dots, x$.

Let $s, t \in \{0, \dots, m\}$. Let $\omega \in \{0, \dots, x\}$ and $\nu \in \{0, \dots, m-x\}$ be such that

$$h_{\omega+1} > s \geq h_\omega, \quad \text{and} \quad v_{\nu+1} > t \geq v_\nu.$$

Recall that $h_u = \min\{i | c_{i-u+1} < d_i\}$, for $u = 1, \dots, x$, and $v_u = \min\{i | r_{i-u+1} < \bar{r}_i\}$, for $u = 1, \dots, m-x$, and by convention we have $h_0 = 0$, $v_0 = 0$. Let $h_{x+1} = m+1$, and $v_{m-x+1} = m+1$. From definition we have

$$0 = h_0 < h_1 < \cdots < h_x < h_{x+1} = m+1, \tag{39}$$

$$0 = v_0 < v_1 < \cdots < v_{m-x} < v_{m-x+1} = m+1, \tag{40}$$

and therefore, in particular, ω and ν are well defined. Also (39) and (40) imply that $h_\omega \geq \omega$ and $v_\nu \geq \nu$.

Since $s < h_{\omega+1}$ we have that $c_{i-w} \geq d_i$, for $i \leq s$. This holds from the definition of $h_{\omega+1}$ for $\omega < x$, and from (14) for $\omega = x$. Therefore

$$\sum_{i=h_w-w+1}^{s-w} c_i \geq \sum_{i=h_w+1}^{s} d_i,$$

which gives

$$\sum_{i=1}^{s} d_i - \sum_{i=1}^{s-\omega} c_i \leq \sum_{i=1}^{h_\omega} d_i - \sum_{i=1}^{h_\omega-\omega} c_i. \tag{41}$$

Completely analogously we get

$$\sum_{i=1}^{t} \bar{r}_i - \sum_{i=1}^{t-\nu} r_i \leq \sum_{i=1}^{v_\nu} \bar{r}_i - \sum_{i=1}^{v_\nu-\nu} r_i. \tag{42}$$

Now, from (41) and (42) we have

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} \bar{r}_i \leq \sum_{i=1}^{h_\omega} d_i - \sum_{i=1}^{h_\omega-\omega} c_i + \sum_{i=1}^{s-\omega} c_i + \sum_{i=1}^{v_\nu} \bar{r}_i - \sum_{i=1}^{v_\nu-\nu} r_i + \sum_{i=1}^{t-\nu} r_i.$$

Hence,

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} \bar{r}_i + (s+t-\omega-\nu) \leq$$

$$\leq \sum_{i=1}^{h_\omega} d_i - \sum_{i=1}^{h_\omega-\omega} c_i + \sum_{i=1}^{s-\omega} (c_i+1) + \sum_{i=1}^{v_\nu} \bar{r}_i - \sum_{i=1}^{v_\nu-\nu} r_i + \sum_{i=1}^{t-\nu} (r_i+1).$$

The last inequality together with (38) for $j' = \omega$, $k' = \nu$, $j = s - \omega$ and $k = t - \nu$, gives

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} \bar{r}_i + s + t - \omega - \nu \leq \sum_{i=1}^{s+t} d(\sigma_{2m-i+1}), \quad s,t = 0,\dots,m. \tag{43}$$

Since $s \geq \omega$, and $t \geq \nu$, we get $s + t \geq \omega + \nu$. Also, $x \geq \omega$, and $m - x \geq \nu$, and thus $m \geq \omega + \nu$. Hence

$$\min(s+t, m) \geq \omega + \nu,$$

and so

$$s + t - \omega - \nu \geq \max(0, s + t - m).$$

So, from (43), for $s + t \in \{0, \dots, m\}$, we get

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} \bar{r}_i \leq \sum_{i=1}^{s+t} d(\sigma_{2m-i+1}) \tag{44}$$

while for $s+t \in \{m+1, \ldots, 2m\}$, we get

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} \bar{r}_i + s + t - m \leq \sum_{i=1}^{s+t} d(\sigma_{2m-i+1}), \quad s,t = 0, \ldots, m. \tag{45}$$

Finally, (44) and (45), together with (9) and (i), give

$$\mathbf{d} \cup \bar{\mathbf{r}} \prec (d(\sigma_{2m}), \ldots, d(\sigma_{m+1}), d(\sigma_m) - 1, \ldots, d(\sigma_1) - 1)$$

i.e. we get condition (ii), as desired.

Sufficiency of conditions: Let us assume that conditions (i) and (ii) are satisfied. Before proceeding, we need to introduce some additional notation: let us consider the partition $\mathbf{d} \cup \bar{\mathbf{r}}$, which is of the length $2m$. Take the m smallest elements from it. Among those m elements, some belong to the partition $\mathbf{d}$, and some belong to the partition $\bar{\mathbf{r}}$. Denote the number of those elements belonging to $\bar{\mathbf{r}}$ by z.

Next, we shall divide the partition $\mathbf{d} = (d_1, \ldots, d_m)$ into two. Let us define the partition

$$\mathbf{d}'' = (d''_1, \ldots, d''_z), \quad \text{where} \quad d''_i := d_i, \quad i = 1, \ldots, z.$$

And let us define a partition

$$\mathbf{d}' = (d'_1, \ldots, d'_{m-z}), \quad \text{where} \quad d'_i := d_{i+z}, \quad i = 1, \ldots, m-z. \tag{46}$$

Thus

$$\mathbf{d} = \mathbf{d}'' \cup \mathbf{d}'.$$

Analogously, let us divide the partition $\bar{\mathbf{r}} = (\bar{r}_1, \ldots, \bar{r}_m)$ into two. Let us define a partition

$$\bar{\mathbf{r}}'' = (\bar{r}''_1, \ldots, \bar{r}''_{m-z}), \quad \text{where} \quad \bar{r}''_i = \bar{r}_i, \quad i = 1, \ldots, m-z.$$

And let us define a partition

$$\bar{\mathbf{r}}' = (\bar{r}'_1, \ldots, \bar{r}'_z), \quad \text{where} \quad \bar{r}'_i = \bar{r}_{i+m-z}, \quad i = 1, \ldots, z. \tag{47}$$

Thus

$$\bar{\mathbf{r}} = \bar{\mathbf{r}}'' \cup \bar{\mathbf{r}}'.$$

Moreover, let us denote by $\mathbf{e}' = (e'_1, \ldots, e'_m)$ the partition

$$(\bar{r}'_1 + 1, \ldots, \bar{r}'_z + 1) \cup (d'_1 + 1, \ldots, d'_{m-z} + 1),$$

and let us denote by $\mathbf{e}'' = (e''_1, \ldots, e''_m)$ the partition $\bar{\mathbf{r}}'' \cup \mathbf{d}''$. Finally, let $\mathbf{e} = \mathbf{e}'' \cup \mathbf{e}'$.

Then from condition (ii) and by Lemma 2 we have

$$\mathbf{e} \prec (d(\sigma_{2m}), \ldots, d(\sigma_1)). \tag{48}$$

Our aim is to prove that there exist homogeneous polynomials $\alpha_1 | \cdots | \alpha_n$ such that the following two sets of conditions are valid:
condition (10) together with

$$\sum_{i=1}^{j} e'_i \le \sum_{i=1}^{n+m} d(\beta_i) - \sum_{i=1}^{n+m-j} d(\operatorname{lcm}(\alpha_{i-m+j}, \beta_i)), \quad j = 1, \ldots, m-1, \tag{49}$$

$$\sum_{i=1}^{m} e'_i = \sum_{i=1}^{n+m} d(\beta_i) - \sum_{i=1}^{n} d(\alpha_i), \tag{50}$$

and
condition (12) together with

$$\sum_{i=1}^{k} e''_i \le \sum_{i=1}^{n} d(\alpha_i) - \sum_{i=1}^{n} d(\operatorname{lcm}(\alpha_{i-k}, \gamma_i)), \quad k = 1, \ldots, m-1, \tag{51}$$

$$\sum_{i=1}^{m} e''_i = \sum_{i=1}^{n} d(\alpha_i) - \sum_{i=1}^{n} d(\gamma_i). \tag{52}$$

To that end, first note that by condition (i) we have (21). Also, by (8), the majorization (48) gives

$$\sum_{i=1}^{j} e''_i + \sum_{i=1}^{k} e'_i \le \sum_{i=1}^{n+2m} d(\bar{\beta}_i) - \sum_{i=1}^{n+2m-j-k} d(\operatorname{lcm}(\bar{\beta}_i, \gamma_{i-2m+j+k})), \ \ j, k = 0, \ldots, m. \tag{53}$$

Let us denote by

$$Z = n - \left(\sum_{i=1}^{z} \bar{r}'_i + \sum_{i=1}^{m-z} d'_i \right),$$

$$X_j = Z - \sum_{i=1}^{j} e''_i, \quad j = 1, \ldots, m-1,$$

and

$$Y_k = \sum_{i=1}^{n+2m} d(\bar{\beta}_i) - \sum_{i=1}^{k} e'_i, \quad k = 1, \dots, m-1.$$

Also, let $X_0 = Z$, $X_m = \sum_{i=1}^{n} d(\gamma_i)$, $Y_0 = \sum_{i=1}^{n+2m} d(\bar{\beta}_i)$, and $Y_m = Z$.

Then (53) becomes (22).

Such defined integers $X_1, \dots, X_{m-1}$, $Y_1, \dots, Y_{m-1}$ and Z are nonnegative. Hence, since conditions (21) and (22) hold, by Theorem 4 there exist homogeneous polynomials $\bar{\alpha}_1 \mid \cdots \mid \bar{\alpha}_{n+m}$ such that

$$\bar{\alpha}_i \mid \gamma_i \mid \bar{\alpha}_{i+m}, \quad i = 1, \dots, n, \tag{54}$$

and conditions (16)-(20) hold.

Now, by the definition of X_j, (19) gives

$$\sum_{i=1}^{j} e''_i \le Z - \sum_{i=1}^{n+m-j} d(\mathrm{lcm}(\bar{\alpha}_i, \gamma_{i-m+j})), \quad j = 1, \dots, m-1, \tag{55}$$

and by the definition of Y_k, (20) gives

$$\sum_{i=1}^{k} e'_i \le \sum_{i=1}^{n+2m} d(\bar{\beta}_i) - \sum_{i=1}^{n+2m-k} d(\mathrm{lcm}(\bar{\alpha}_{i-m+k}, \bar{\beta}_i)), \quad k = 1, \dots, m-1. \tag{56}$$

Also, (18) implies that

$$\sum_{i=1}^{n+m} d(\bar{\alpha}_i) = n - (\sum_{i=1}^{z} \bar{r}'_i + \sum_{i=1}^{m-z} d'_i). \tag{57}$$

Hence $\sum_{i=1}^{n+m} d(\bar{\alpha}_i) \le n$, and thus

$$\bar{\alpha}_1 = \cdots = \bar{\alpha}_m = 1.$$

Let

$$\alpha_i := \bar{\alpha}_{i+m}, \quad i = 1, \dots, n.$$

Now, we directly obtain that (56) equals (49), and (55) equals (51). Also, (54) and (16), imply (10) and (12). Finally, (18) gives (50) and (52), as desired.

Furthermore, from the definition of the homogeneous polynomials $\alpha_1 \mid \cdots \mid \alpha_n$, and (57), we have $\sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{z} \bar{r}'_i + \sum_{i=1}^{m-z} d'_i = n$. Therefore, there exists a matrix pencil $A(\lambda) \in \mathbb{F}[\lambda]^{(n+z)\times(n+m-z)}$ having $\alpha_1 \mid \cdots \mid \alpha_n$ as the homogeneous invariant factors, $\bar{\mathbf{r}}'$ as the row minimal indices, and $\mathbf{d}'$ as the column minimal indices.

By applying Theorem 2, from conditions (10), (49) and (50), we obtain the existence of matrix pencils $W(\lambda) \in \mathbb{F}[\lambda]^{(n+z)\times z}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{(m-z)\times(n+m-z)}$, and $Z(\lambda) \in \mathbb{F}[\lambda]^{(m-z)\times z}$ such that

$$\left[\begin{array}{c|c} A(\lambda) & W(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array}\right] \sim B(\lambda).$$

Since the column and row minimal indices of $A(\lambda)$ are subsets of column and row minimal indices of $C(\lambda)$, respectively, as given in (46) and (47), by Lemma 4, and since conditions (12), (51) and (52) are satisfied, we can apply Theorem 3 (see also Remark 2). Hence, there exist pencils $\bar{W}(\lambda) \in \mathbb{F}[\lambda]^{(n+z)\times z}$, $\bar{Y}(\lambda) \in \mathbb{F}[\lambda]^{(m-z)\times(n+m-z)}$, and $\bar{Z}(\lambda) \in \mathbb{F}[\lambda]^{(m-z)\times z}$ such that

$$\left[\begin{array}{c|c} A(\lambda) & \bar{W}(\lambda) \\ \hline \bar{Y}(\lambda) & \bar{Z}(\lambda) \end{array}\right] \sim C(\lambda).$$

Let

$$B'(\lambda) = \left[\begin{array}{c|c} A(\lambda) & W(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array}\right], \quad \text{and} \quad C'(\lambda) = \left[\begin{array}{c|c} A(\lambda) & \bar{W}(\lambda) \\ \hline \bar{Y}(\lambda) & \bar{Z}(\lambda) \end{array}\right].$$

Since

$$\operatorname{rank}\left(\left[\begin{array}{c|c} A(\lambda) & W(\lambda) \\ \hline Y(\lambda) & Z(\lambda) \end{array}\right] - \left[\begin{array}{c|c} A(\lambda) & \bar{W}(\lambda) \\ \hline \bar{Y}(\lambda) & \bar{Z}(\lambda) \end{array}\right]\right) = \operatorname{rank} \begin{array}{c} \\ m-z\{ \end{array} \begin{bmatrix} \mathbf{0} & \overbrace{*}^{z} \\ * & * \end{bmatrix} \leq m,$$

we have

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq m, \tag{58}$$

and because of (1), we must have an equality in (58), as desired.

Corollary 6. *Let $C(\lambda)$ be the matrix pencil given in Section 3.1. Let Ψ be a homogeneous polynomial of degree $n+m$. There exists a regular matrix pencil $B'(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ having Ψ as the characteristic polynomial (product of its homogeneous invariant factors), and such that*

$$\operatorname{rank}(B'(\lambda) - C(\lambda)) = m,$$

if and only if

$$\prod_{i=1}^{n} \gamma_i \mid \Psi. \tag{59}$$

Proof. The necessity of the condition (59) follows trivially by applying Theorem 5.

As for the sufficiency, let

$$\Omega = \frac{\Psi}{\prod_{i=1}^{n} \gamma_i}$$

Let us define polynomials

$$\begin{aligned} \beta_{n+m} &:= \Omega\gamma_n, \\ \beta_{i+m} &:= \gamma_i, \qquad i = 1, \dots, n-1, \\ \beta_i &:= 1, \qquad i = 1, \dots, m. \end{aligned}$$

Let $B(\lambda)$ be a regular matrix pencil having $\beta_1 \mid \cdots \mid \beta_{n+m}$ as the homogeneous invariant factors. By definition of the polynomial chain $\sigma_1 \mid \cdots \mid \sigma_{2m}$, we have $\sigma_{2m} = \Omega$, while $\sigma_i = 1$, for all $i = 1, \dots, 2m-1$. Since

$$d(\Omega) = n + m - \sum_{i=1}^{n} d(\gamma_i) = m + \sum_{i=1}^{m} d_i + \sum_{i=1}^{m} \bar{r}_i,$$

we have that conditions (i) and (ii) from Theorem 5 are satisfied. Hence, by applying Theorem 5 we obtain the existence of a regular pencil $B'(\lambda) \sim B(\lambda)$ such that (59) is valid.

5. Conclusion

For a regular matrix pencil $B(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ and an arbitrary matrix pencil $C(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$, with $\operatorname{rank} C(\lambda) = n$, we give explicit necessary and sufficient conditions for the existence of matrix pencils $B'(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ and $C'(\lambda) \in \mathbb{F}[\lambda]^{(n+m)\times(n+m)}$ such that $B'(\lambda) \sim B(\lambda)$, and $C'(\lambda) \sim C(\lambda)$, and such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) = m.$$

This result is a substantial step towards general solution of bounded rank perturbation problem for a regular pencil. It is a novel approach, and uses recent combinatorial results on partitions of integers [14]. The main novelty and difference to the existing results is that we allow $C(\lambda)$ to be arbitrary, i.e. we do not impose any kind of restriction to $C(\lambda)$. This makes the main result of the paper very general, and different from the existing results in the literature. Moreover, we in fact solve the minimal bounded rank perturbation problem for a regular matrix pencil, since $\operatorname{rank}(B'(\lambda) - C'(\lambda)) \geq m$ always holds.

Finally, by using novel approach via combinatorics of partitions of integers and results from Matrix Theory and Matrix Pencils Completion Problems, we have obtained relatively simple conditions. This is an advantage for the expected future application of the main result. One feature of the obtained conditions is that the column and row minimal indices appear in them only as a union, so one needs to know only the union of column and row minimal indices, and does not need to identify them separately. This means that for this problem we can consider a lighter set of Kronecker invariants, by joining column and row minimal indices together.

6. Examples

Example 1. Let $n = 10$, $m = 2$. Let $B(\lambda) \in \mathbb{F}[\lambda]^{12\times 12}$ be a regular matrix pencil having $\beta_1 = \cdots = \beta_9 = 1$, $\beta_{10} = \lambda^3$, $\beta_{11} = \lambda^4$, and $\beta_{12} = \lambda^4\mu$ as homogeneous invariant factors. More precisely, it has $1 \mid 1 \cdots \mid 1 \mid \lambda^3 \mid \lambda^4 \mid \lambda^4$ as invariant factors, and one infinite elementary divisor of degree 1.

Let $C(\lambda) \in \mathbb{F}[\lambda]^{12\times 12}$ be a matrix pencil of rank 10, having $\gamma_1 = \cdots = \gamma_9 = 1$, and $\gamma_{10} = \lambda$, as homogeneous invariant factors, $\mathbf{d} = (4, 1)$ as column minimal indices and $\bar{\mathbf{r}} = (3, 1)$ as row minimal indices. We note that $C(\lambda)$ has no infinite elementary divisors, and its homogeneous invariant factors are equal to its finite invariant factors.

Our goal is to determine whether there exists a perturbation matrix pencil $X(\lambda)$ with $\operatorname{rank} X(\lambda) = 2$, such that the matrix pencil $C(\lambda)+X(\lambda)$ is regular and strictly equivalent to $B(\lambda)$, and if such pencil exists, to construct $X(\lambda)$ explicitly.

We shall first check whether the conditions from Theorem 5 are satisfied. Indeed we have:

$$\beta_i \mid \gamma_{i+2}, \quad i = 1, \ldots, 8, \text{ and } \gamma_i \mid \beta_{i+2}, \quad i = 1, \ldots, 10,$$

which is condition (i) from Theorem 5. Furthermore, let $\bar{\beta}_1 = \bar{\beta}_2 = 1$, and let $\bar{\beta}_{i+2} = \beta_i$, $i = 1, \ldots, 12$. We can now compute polynomial chain $\sigma_1|\sigma_2|\sigma_3|\sigma_4$. By definition we have:

$$\pi_4 = \bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{\beta}_4 \operatorname{lcm}(\gamma_1, \bar{\beta}_5) \operatorname{lcm}(\gamma_2, \bar{\beta}_6) \cdots \operatorname{lcm}(\gamma_{10}, \bar{\beta}_{14}) = \lambda^3\lambda^4 \operatorname{lcm}(\lambda, \lambda^4\mu) = \lambda^{11}\mu,$$
$$\pi_3 = \bar{\beta}_1\bar{\beta}_2\bar{\beta}_3 \operatorname{lcm}(\gamma_1, \bar{\beta}_4) \operatorname{lcm}(\gamma_2, \bar{\beta}_5) \cdots \operatorname{lcm}(\gamma_{10}, \bar{\beta}_{13}) = \lambda^3 \operatorname{lcm}(\lambda, \lambda^4) = \lambda^7,$$
$$\pi_2 = \bar{\beta}_1\bar{\beta}_2 \operatorname{lcm}(\gamma_1, \bar{\beta}_3) \operatorname{lcm}(\gamma_2, \bar{\beta}_4) \cdots \operatorname{lcm}(\gamma_{10}, \bar{\beta}_{12}) = \operatorname{lcm}(\lambda, \lambda^3) = \lambda^3,$$
$$\pi_1 = \bar{\beta}_1 \operatorname{lcm}(\gamma_1, \bar{\beta}_2) \operatorname{lcm}(\gamma_2, \bar{\beta}_3) \cdots \operatorname{lcm}(\gamma_{10}, \bar{\beta}_{11}) = \operatorname{lcm}(\lambda, 1) = \lambda,$$
$$\pi_0 = \operatorname{lcm}(\gamma_1, \bar{\beta}_1) \operatorname{lcm}(\gamma_2, \bar{\beta}_2) \cdots \operatorname{lcm}(\gamma_{10}, \bar{\beta}_{10}) = \operatorname{lcm}(\lambda, 1) = \lambda$$

and so

$$\sigma_4 = \pi_4/\pi_3 = \lambda^4\mu$$
$$\sigma_3 = \pi_3/\pi_2 = \lambda^4$$
$$\sigma_2 = \pi_2/\pi_1 = \lambda^2$$
$$\sigma_1 = \pi_1/\pi_0 = 1$$

Hence,

$$(d(\sigma_4), d(\sigma_3), d(\sigma_2) - 1, d(\sigma_1) - 1) = (5, 4, 1, -1).$$

On the other hand, we have

$$\mathbf{d} \cup \bar{\mathbf{r}} = (4, 3, 1, 1), \tag{60}$$

and since

$$(4, 3, 1, 1) \prec (5, 4, 1, -1),$$

we have that the condition (ii) from Theorem 5 is also satisfied. Therefore, there exists $X(\lambda)$ with the desired properties. Now, we shall define explicitly one such perturbation pencil $X(\lambda)$, as it is done in the proof of Theorem 5.

First we split the union of $\mathbf{d}$ and $\bar{\mathbf{r}}$ from (60) into two halves – larger and the smaller ones: $(4, 3 \mid 1, 1)$. The two smaller ones are one column minimal index, $d_2 = 1$, and one row minimal index $\bar{r}_2 = 1$. Thus, by using the notation from the sufficiency part of the proof of the main result, we have $z = 1$. Therefore we are looking for completions by $z = 1$ column and $m - z = 1$ row. That is, we are looking for matrix pencils $A(\lambda) \in \mathbb{F}[\lambda]^{11\times 11}$, $W(\lambda), \bar{W}(\lambda) \in \mathbb{F}[\lambda]^{11\times 1}$, $Y(\lambda), \bar{Y}(\lambda) \in \mathbb{F}[\lambda]^{1\times 11}$, and $Z(\lambda), \bar{Z}(\lambda) \in \mathbb{F}[\lambda]^{1\times 1}$, such that

$$B(\lambda) \sim \begin{bmatrix} A(\lambda) & W(\lambda) \\ Y(\lambda) & Z(\lambda) \end{bmatrix}, \text{ and } C(\lambda) \sim \begin{bmatrix} A(\lambda) & \bar{W}(\lambda) \\ \bar{Y}(\lambda) & \bar{Z}(\lambda) \end{bmatrix}. \tag{61}$$

To that end, we will choose the matrix pencil $A(\lambda)$ such that $\operatorname{rank} A(\lambda) = n = 10$. So it will have only one column and one row minimal index, c_1, and r_1, respectively. We choose these minimal indices to be the two smaller ones from the union $\mathbf{d} \cup \bar{\mathbf{r}}$ (60), and therefore we put

$$c_1 := d_2 = 1,$$
$$r_1 := \bar{r}_2 = 1.$$

Finally, for the homogeneous invariant factors of $A(\lambda)$ we will take polynomials $\alpha_1 \mid \cdots \mid \alpha_{10}$ satisfying

$$\beta_i \mid \alpha_i \mid \beta_{i+2}, \qquad i = 1, \ldots, 10, \tag{62}$$

$$\gamma_i \mid \alpha_i, \quad i = 1, \ldots, 10, \quad \text{and} \quad \alpha_i \mid \gamma_{i+2}, \quad i = 1, \ldots, 8, \tag{63}$$

$$\sum_{i=1}^{10} d(\alpha_i) = 10 - c_1 - r_1 = 8, \tag{64}$$

$$4 \leq \sum_{i=1}^{10} d(\alpha_i) - \sum_{i=1}^{10} d(\operatorname{lcm}(\gamma_i, \alpha_{i-1})), \tag{65}$$

$$1 + 1 \leq \sum_{i=1}^{12} d(\beta_i) - \sum_{i=1}^{11} d(\operatorname{lcm}(\beta_i, \alpha_{i-1})). \tag{66}$$

As shown in Theorem 4, the conditions (i) and (ii) and Theorem 4 ensure the existence of such polynomials α_i. We will take

$$\alpha_1 = \ldots = \alpha_8 := 1, \quad \alpha_9 = \alpha_{10} := \lambda^4.$$

Furthermore, the conditions (62)–(66), by Theorems 2 and 3, imply the existence of completions (61).

Explicitly, the completions (61) in our case can be obtained as follows. Let $A(\lambda)$ be in its Kronecker canonical form (all empty entries are considered to be zeros):

$$A(\lambda) = \left[\begin{array}{cccc|cccc|cc|c}
\lambda & 1 & & & & & & & & & \\
& \lambda & 1 & & & & & & & & \\
& & \lambda & 1 & & & & & & & \\
& & & \lambda & & & & & & & \\
\hline
& & & & \lambda & 1 & & & & & \\
& & & & & \lambda & 1 & & & & \\
& & & & & & \lambda & 1 & & & \\
& & & & & & & \lambda & & & \\
\hline
& & & & & & & & \lambda & 1 & \\
\hline
& & & & & & & & & & \lambda \\
& & & & & & & & & & 1
\end{array}\right]$$

Now, by calculating the Kronecker invariants of the pencil:

$$\left[\begin{array}{cccc|cccc|cc|c|c}
\lambda & 1 & & & & & & & & & & \\
& \lambda & 1 & & & & & & & & & \\
& & \lambda & 1 & & & & & & & & \\
& & & \lambda & & & & & & & & \\
\hline
& & & & \lambda & 1 & & & & & & 1 \\
& & & & & \lambda & 1 & & & & & \\
& & & & & & \lambda & 1 & & & & \\
& & & & & & & \lambda & & & & \\
\hline
& & & & & & & & \lambda & 1 & & \\
\hline
& & & & & & & & & & \lambda & \\
& & & & & & & & & & 1 & \lambda \\
\hline
& & & & & 1 & & & \lambda & & &
\end{array}\right] \tag{67}$$

we obtain that it is regular, having exactly $1 \mid 1 \mid \cdots \mid 1 \mid \lambda^3 \mid \lambda^4 \mid \lambda^4\mu$ as the homogeneous invariant factors (i.e. it has $1 \mid 1 \cdots \mid 1 \mid \lambda^3 \mid \lambda^4 \mid \lambda^4$ as invariant factors, and one infinite elementary divisor of degree 1), and no other Kronecker invariants. Hence it is strictly equivalent to $B(\lambda)$, and has the desired form (61) for $W(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}^T$, $Y(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \lambda & 0 & 0 \end{bmatrix}$, and $Z(\lambda) = [0]$.

Also, by calculating the Kronecker invariants of the pencil:

$$\left[\begin{array}{cccc|cccc|cc|c|c}
\lambda & 1 & & & & & & & & & & \\
& \lambda & 1 & & & & & & & & & \\
& & \lambda & 1 & & & & & & & & \\
& & & \lambda & & & & & & & & 1 \\
\hline
& & & & \lambda & 1 & & & & & & \\
& & & & & \lambda & 1 & & & & & \\
& & & & & & \lambda & 1 & & & & \\
& & & & & & & \lambda & & & & \\
\hline
& & & & & & & & \lambda & 1 & & \\
\hline
& & & & & & & & & & \lambda & \\
& & & & & & & & & & 1 & \\
\hline
& & & & \lambda & & & & & & &
\end{array}\right] \tag{68}$$

we obtain that its rank is 10, having exactly $1 \mid 1 \mid \cdots \mid 1 \mid \lambda$ as the (homogeneous) invariant factors, $\mathbf{d} = (4, 1)$ as column minimal indices and $\bar{\mathbf{r}} = (3, 1)$ as row minimal indices, and no infinite elementary divisors. Hence it is strictly equivalent to $C(\lambda)$, and has the desired form (61) for $\bar{W}(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, $\bar{Y}(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and $\bar{Z}(\lambda) = [0]$.

Now, the existence of completions (67) and (68) gives the desired perturbation matrix pencil $X(\lambda)$. Indeed, since (68) is strictly equivalent to $C(\lambda)$, it means that there exist invertible matrices $P, Q \in \mathbb{F}^{12\times 12}$ such that the matrix (68) is equal to $PC(\lambda)Q$. Then for the matrix $X(\lambda)$ we can take:

$$X(\lambda) = P^{-1} \left[\begin{array}{c|c} \mathbf{0} & W(\lambda) - \bar{W}(\lambda) \\ \hline Y(\lambda) - \bar{Y}(\lambda) & Z(\lambda) - \bar{Z}(\lambda) \end{array}\right] Q^{-1}.$$

Example 2. Let us use data from the previous Example 1, and let us assume that $C(\lambda)$ has the following form:

$$C(\lambda)=\left[\begin{array}{ccccc|ccc|cc|c|c}
\lambda & 1 & & & & & & & & & & \\
 & \lambda & 1 & & & & & & & & & \\
 & & \lambda & 1 & & & & & & & & \\
 & & & \lambda & 1 & & & & & & & \\
\hline
 & & & & & \lambda & 1 & & & & & \\
 & & & & & & \lambda & 1 & & & & \\
 & & & & & & & \lambda & & & & \\
 & & & & & 1 & & & & & & \\
\hline
 & & & & & & & & \lambda & 1 & & \\
\hline
 & & & & & & & & & & \lambda & \\
 & & & & & & & & & & 1 & \\
\hline
 & & & & & & & & & & & \lambda
\end{array}\right] \tag{69}$$

Then the invertible matrices P^{-1} and Q^{-1} which transform (68) into $C(\lambda)$ are

$$P^{-1}=\left[\begin{array}{cccc|cccc|cccc}
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&-1\\
\hline
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right] \quad \text{and} \quad Q^{-1}=\left[\begin{array}{cccc|cccc|cccc}
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0
\end{array}\right]$$

Hence

$$X(\lambda)=P^{-1}\left[\begin{array}{ccccccccccc|c}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&\lambda\\
\hline
0&0&0&0&-\lambda&1&0&0&\lambda&0&0&0
\end{array}\right]Q^{-1}=\left[\begin{array}{cccc|cccc|cc|c|c}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&-1&0&0&-\lambda&0&0&\lambda\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\lambda&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&1&0&0&\lambda&0&0&-\lambda
\end{array}\right].$$

It is straightforward to see that $\operatorname{rank} X(\lambda) = 2$, and

$$C(\lambda)+X(\lambda)=\left[\begin{array}{cccc|cccc|cc|c|c}
\lambda & 1 & & & & & & & & & & \\
 & \lambda & 1 & & & & & & & & & \\
 & & \lambda & 1 & & & & & & & & \\
 & & & \lambda & & & & & & & & \\
\hline
 & & & & & \lambda & 1 & & & & & \\
 & & & & & & \lambda & 1 & & & & \\
 & & & & & & & \lambda & & & & \\
 & & & & 1 & & & & -\lambda & & & \lambda \\
\hline
 & & & & & & & & \lambda & 1 & & \\
\hline
 & & & & & & & & & & \lambda & \\
 & & & & \lambda & & & & & & 1 & \\
\hline
 & & & & & 1 & & & \lambda & & &
\end{array}\right].$$

The pencil $C(\lambda)+X(\lambda)$ is regular, having $1 \mid \cdots \mid 1 \mid \lambda^3 \mid \lambda^4 \mid \lambda^4$ as invariant factors, and a unique infinite elementary divisor of degree 1. Hence its homogeneous invariant factors are $1 \mid \cdots \mid 1 \mid \lambda^3 \mid \lambda^4 \mid \lambda^4\mu$, and so it is strictly equivalent to $B(\lambda)$, as desired.

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Declaration of competing interest

No competing interest.

Data availability

No data was used for the research described in the article.

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