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Polynomial Entropy of Induced Maps of Circle and Interval Homeomorphisms

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Abstract
We compute the polynomial entropy of the induced maps on hyperspace for a homeomorphism f of an interval or a circle with finitely many non-wandering points. Also, we give a generalization for the case of an interval homeomorphism with an infinite non-wandering set.

Keywords Polynomial entropy · Hyperspaces · Interval homeomorphisms · Circle homeomorphisms · Induced maps

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1 Introduction

Every continuous map on a compact metric space X induces a continuous map (called the *induced map*) on the hyperspace 2^X of all closed subsets. If X is connected, and so a continuum, we consider the hyperspace $C(X)$ of subcontinua of X (which is also a continuum). One can also consider the hyperspace X^{*k} of all nonempty subsets with at most k points (for $k \in \mathbb{N}$). A natural question is what are the possible relations between the given (individual) dynamics on X and the induced one (collective dynamics) on the hyperspace. Various results in this direction were obtained in the last decades. Without attempting to give complete references, we mention just a few: Borsuk and

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Ulam [7], Bauer and Sigmund [5], Román-Flores [23], Banks [4], Acosta, Illanes and Méndez-Lango [1].

The topological entropy of the induced map was studied by Kwietniak and Oprocha in [14], Lampart and Raith [19], Hernández and Méndez [12], Arbieto and Bohorquez [2], and others.

In [19] the authors showed that, if f is an interval or a circle homeomorphism, the topological entropy of the induced map on the hyperspace of subcontinua is zero. This is also obtained as a corollary in [2], for Morse-Smale diffeomorphisms of a circle.

One of the measures of complexity of a system with zero topological entropy is the polynomial entropy. The very definitions of topological and polynomial entropies differ in the fact that the former measures the overall *exponential* complexity of the orbit structure, and the latter analyzes the growth rate at the *polynomial* scale. The polynomial entropy is a tool for distinguishing systems with zero topological entropy. For example, a rotation on the circle on the one hand, and a homeomorphism with both periodic and wandering points on the other hand, both have topological entropy equal to zero, but it is clear that the first system is simpler than the second one. Labrousse proved that the polynomial entropy distinguishes between these two systems (see Theorem 1 in [17]).

The topological and the polynomial entropy have some properties in common: they are both conjugacy invariants, they do not depend on a metric but only on the induced topology, they both have the finite union property and fulfill the product formula. However, there are some properties that differ: the power formula, the σ-union property, the variational principle (see [20, 21]). In addition, the topological entropy of a system is equal to the topological entropy of the same map, restricted to the non-wandering set, which is a closed and invariant subset; this does not hold for the polynomial entropy.

The notion of the polynomial entropy was first introduced by Marco in [20] and [21] in the context of Hamiltonian integrable systems. It was further investigated in different contexts by Labrousse [15–17], Labrousse and Marco [18], Bernard and Labrousse [6], Artigue, Carrasco–Olivera and Monteverde [3], Haseaux and Le Roux [11], Roth, Roth and Snoha [24], Correa and de Paula [8], etc.

As opposed to the topological entropy, which depends only on the dynamics restricted to the non-wandering set, the wandering set is visible to the polynomial entropy. In the case when non-wandering set is finite (for example, Morse-Smale systems), there is a technique for computing the polynomial entropy developed in [11] by Hauseux and Le Roux. Originally, they invented a simple coding procedure for homeomorphisms with only one non-wandering point, where the polynomial entropy is particularly well adapted, since the growth of the number of wandering orbits is always at least linear and at most polynomial. Hauseux and Le Roux also proved that the polynomial entropy localizes near a certain finite set (singular set), in order to compute the polynomial entropy of Brouwer homeomorphisms. This method was slightly generalized in [13], to the case of a map which is only continuous and the non-wandering set is finite. The coding procedure was also used in [8] for the computation of the polynomial entropy of Morse-Smale systems on surfaces.

In this paper we compute the polynomial entropy of the induced maps $C(f)$, f^{*k} and 2^f for a homeomorphism f of a circle or an interval with a finite non-wandering set. The polynomial entropy of such a homeomorphism of an interval is known to be 1 (this easily follows from Lemma 3.1 in [17] and the finite union property of the polynomial entropy), and of a circle is either 0 or 1 (see Theorem 1 in [17]). The hyperspace of subcontinua of a one-dimensional space is quite simple and can be identified with a two-dimensional object. Our computation uses the coding method and reduction to the singular sets.

Denote by $h_{pol}(\cdot)$ the polynomial entropy of a map and by $C(f)$, f^{*k} and 2^f the induced maps on $C(X)$, X^{*k} and 2^X, respectively. These are the statements of our results.

Theorem A *Let $f : [0, 1] \to [0, 1]$ be a homeomorphism with a finite non-wandering set. Then $h_{pol}(C(f)) = 2$, $h_{pol}(f^{*k}) = k$ and $h_{pol}(2^f) = \infty$.* □

Theorem B *Let $f : S^1 \to S^1$ be a homeomorphism with a finite non-wandering set. Then $h_{pol}(C(f)) = 2$, $h_{pol}(f^{*k}) = k$ and $h_{pol}(2^f) = \infty$.* □

The fact that $h_{pol}(2^f) = \infty$ in Theorem A and Theorem B follows from already known results. Namely, the topological entropy of 2^f is strictly positive in this case (see Theorem 20 in [14]); it is actually also infinite (see Theorem 3.1 in [9]). Here we derive it in a different way, as a consequence of the fact that $h_{pol}(f^{*k}) = k$.

In the proofs, we use the coding techniques adapted to a subset with only one non-wandering (hence fixed) point. If the non-wandering set is finite, then because of its invariance, it must be equal to the set of periodic points. Therefore, the non-wandering set of f^k, for some k, consists of finitely many fixed points. At this point, our proofs heavily rely on the finite union property of the polynomial entropy (that does not hold for an infinite union), see the third property on page 5. Our result is indeed false if the set of non-wandering points of a homeomorphism of the circle is infinite. For example, consider any rotation of the circle. The non-wandering set is the whole circle. Since the rotation is an isometry, so is every induced map, and the polynomial entropy is equal to zero. However, A can be easily generalized to the case of an infinite non-wandering set, that is not equal to the whole interval (see Remark 7).

2 Preliminaries

2.1 Hyperspaces and Induced Maps

For a compact metric space (X, d), the hyperspace 2^X is the set of all nonempty closed subsets of X. The topology on 2^X is induced by the Hausdorff metric

$$d_H(A, B) := \inf\{\varepsilon > 0 \mid A \subset U(B, \varepsilon),\ B \subset U(A, \varepsilon)\},$$

where

$$U(A, \varepsilon) := \{x \in X \mid d(x, A) < \varepsilon\}.$$

The space 2^X is compact and the topology induced by d_H is the Vietoris topology.

We will also consider two closed subspaces of 2^X, with the induced metric. The first one is X^{*k}, the space of all finite subsets of cardinality at most k, with the same topology. The set X^{*k} is called *k-fold symmetric product* of X.

If X is also connected (and so a continuum), then the set $C(X)$ of all connected and closed nonempty subsets of X is also a continuum. The set $C(X)$ is called the *hyperspace of subcontinua* of X.

If $f : X \to X$ is continuous, then it induces continuous maps

$$\begin{aligned} 2^f : 2^X &\to 2^X, & 2^f(A) &:= \{f(x) \mid x \in A\} \\ C(f) : C(X) &\to C(X), & C(f)(A) &:= \{f(x) \mid x \in A\} \\ f^{*k} : X^{*k} &\to X^{*k}, & f^{*k}(A) &:= \{f(x) \mid x \in A\}. \end{aligned}$$

If f is a homeomorphism, so are 2^X, $C(f)$ and f^{*k}.

Example 1 Let $f : [0, 1] \to [0, 1]$ be a homeomorphism.

The set $C([0, 1])$ consists of all compact and connected subsets, which are precisely segments $[a, b]$, for $0 \le a \le b \le 1$. Then $C(f)([a, b]) = [f(a), f(b)]$ or $C(f)([a, b]) = [f(b), f(a)]$ and, when $a = b$, $C(f)(\{a\}) = \{f(a)\}$.

The set X^{*k} consists of all subsets $\{x_1, \dots, x_l\}$, where $1 \le l \le k$ and $x_i \in [0, 1]$, for $i = 1, \dots l$. Then $f^{*k}(\{x_1, \dots, x_l\}) = \{f(x_1), \dots, f(x_l)\}$. Note that, if f is a homeomorphism, then the cardinality of the set $\{f(x_1), \dots, f(x_l)\}$ is l. If f is only continuous, then the cardinality of the set $\{f(x_1), \dots, f(x_l)\}$ is at most l.

2.2 Polynomial Entropy and Coding

Suppose that X is a compact metric space, and $f : X \to X$ is continuous. Denote by $d_n^f(x, y)$ the dynamic metric (induced by f and d):

$$d_n^f(x, y) = \max_{0 \le k \le n-1} d(f^k(x), f^k(y)).$$

Fix $Y \subseteq X$. For $\varepsilon > 0$, we say that a finite set $E \subset X$ is (n, ε)-*separated* if for every $x, y \in E$ it holds $d_n^f(x, y) \ge \varepsilon$. Let $S(n, \varepsilon; Y)$ denote the maximal cardinality of an (n, ε)-separated set E, contained in Y.

The *polynomial entropy* of the map f on the set Y is defined by

$$h_{pol}(f; Y) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log S(n, \varepsilon; Y)}{\log n}.$$

If $X = Y$ we abbreviate $h_{pol}(f) := h_{pol}(f; X)$. The polynomial entropy, as well as the topological entropy, can also be defined via coverings with sets of d_f^n-diameters less than ε, or via coverings by balls of d_n^f-radius less than ε, see p. 626 in [21]. We list some properties of the polynomial entropy that are important for our computations (for proofs see Propositions $1 - 4$ in [21]):

- $h_{pol}(f^k) = h_{pol}(f)$, for any $k \ge 1$

- if $Y \subset X$ is a closed, f-invariant set, then $h_{pol}(f; Y) = h_{pol}(f|_Y)$
- if $Y = \bigcup_{j=1}^{m} Y_j$ where Y_j are f-invariant, then $h_{pol}(f; Y) = \max\{h_{pol}(f; Y_j) \mid j = 1, \dots, m\}$
- If $f : X \to X$, $g : Y \to Y$ and $f \times g : X \times Y \to X \times Y$ is defined as $f \times g(x, y) := (f(x), g(y))$, then $h_{pol}(f \times g) = h_{pol}(f) + h_{pol}(g)$
- $h_{pol}(f)$ does not depend on a metric but only on the induced topology
- $h_{pol}(\cdot)$ is a *conjugacy invariant* (meaning if $f : X \to X$, $g : X' \to X'$, $\varphi : X \to X'$ is a homeomorphism of compact spaces and $g \circ \varphi = \varphi \circ f$, then $h_{pol}(f) = h_{pol}(g)$).
- If $f : X \to X$ and $g : X' \to X'$ are *semi-conjugated*, meaning that $\varphi : X \to X'$ is a continuous surjective map of compact spaces and $g \circ \varphi = \varphi \circ f$, then $h_{pol}(f) \geq h_{pol}(g)$.

A set $A \subset X$ is *wandering* if $f^n(A) \cap A = \emptyset$, for all $n \geq 1$. A point $p \in X$ is *wandering* if there exists a wandering neighbourhood $U \ni p$.

A point that is not wandering is said to be *non-wandering*. We denote the set of all non-wandering points by $NW(f)$. The set $NW(f)$ is closed and f-invariant. Also, we denote the set of all fixed points by $Fix(f)$.

We now give a brief description of the computation of the polynomial entropy for maps with a finite non-wandering set, by means of a coding and a local polynomial entropy. This construction was first done in [11] for homeomorphisms with only one non-wandering (hence fixed) point, and then modified in [13] for continuous maps with finitely many non-wandering points. Let Y be any f-invariant subset of X.

We first define a coding relative to a family of sets $\mathcal{F}$. Let

$$\mathcal{F} = \{Y_1, Y_2, \dots, Y_L\}$$

where $Y_j \subseteq X \backslash \mathrm{NW}(f)$ and

$$Y_\infty := Y \setminus \bigcup_{j=1}^{L} Y_j.$$

Let $\underline{x} = (x_0, \dots, x_{n-1})$ be a finite sequence of elements in Y. We say that a finite sequence $\underline{w} = (w_0, \dots, w_{n-1})$ of elements in $\mathcal{F} \cup \{Y_\infty\}$ is a *coding* of $\underline{x}$ relative to $\mathcal{F}$ if $x_j \in w_j$, for every $j = 0, \dots, n-1$. We will refer to $\underline{w}$ as a word and to w_j as a letter.

Let $\mathcal{A}_n(\mathcal{F}; Y)$ be the set of all codings of all orbits

$$(x, f(x), \dots, f^{n-1}(x))$$

of length n relative to $\mathcal{F}$, for all $x \in Y$. If $\sharp\mathcal{A}_n(\mathcal{F}; Y)$ denotes the cardinality of $\mathcal{A}_n(\mathcal{F}; Y)$, we define the *polynomial entropy of f, on the set Y, relative to the family $\mathcal{F}$* as the number:

$$h_{pol}(f, \mathcal{F}; Y) := \limsup_{n \to \infty} \frac{\log \sharp\mathcal{A}_n(\mathcal{F}; Y)}{\log n}.$$

We abbreviate $h_{pol}(\mathcal{F}; Y) := h_{pol}(f, \mathcal{F}; Y)$ whenever there is no risk of confusion.

Example 2 Let $f : [0, 1] \to [0, 1]$ be an increasing homeomorphism such that $f(x) < x$, for $x \in (0, 1)$ (hence $f(0) = 0$ and $f(1) = 1$). Let Y be the interval $[0, 1)$. We define $Y_1 = [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon] \subset [0, 1] \setminus NW(f)$, for $\varepsilon > 0$ small enough so that $f(Y_1) \cap Y_1 = \emptyset$. Let $\mathcal{F} = \{Y_1\}$. The elements of the set $\mathcal{A}_n(\mathcal{F}; Y)$ are exactly all the words of the form $(Y_\infty, \dots, Y_\infty, Y_1, Y_\infty, \dots, Y_\infty)$, as well as the word $(Y_\infty, Y_\infty, \dots, Y_\infty)$. Namely, for all $n \in \mathbb{N}$, we can choose $x \in (0, 1)$ so that $f^n(x) \in Y_1$. It follows that $\mathcal{A}_n(\mathcal{F}; Y)$ contains words of length n in which Y_1 takes any position i, $1 \leq i \leq n$. Also, the letter Y_1 can appear in a word at most once, because $f(Y_1) \cap Y_1 = \emptyset$. Now we have that $\sharp\mathcal{A}_n(\mathcal{F}; Y) = n + 1$ and $h_{pol}(\mathcal{F}; Y) = 1$. Note that $Y_1 \subset [0, 1] \setminus NW(f)$ could be any interval that satisfies $f(Y_1) \cap Y_1 = \emptyset$.

For $Z \subseteq X \setminus \mathrm{NW}(f)$, set

$$M(Z) := \sup_{x \in X} \sharp\{n \mid f^n(x) \in Z\}.$$

If $Z \subseteq X \setminus \mathrm{NW}(f)$ is compact, the number $M(Z)$ is finite, since Z can be covered by a finite number of open wandering sets, and every orbit can intersect a wandering set at most once (as in [11]).

We will use the following property of $h_{pol}(\mathcal{F}; Y)$ in order to localize our computation to a singular set.

Proposition 1 [Proposition 3.1 in [13]] *Let $\mathcal{F}$ and $\mathcal{F}'$ be two families of subsets of $X \setminus \mathrm{NW}(f)$ with $M(\cup\mathcal{F}) < +\infty$. Let $Y \subset X$ be an f-invariant subset with exactly one non-wandering point. If for every $Y'_j \in \mathcal{F}'$ there exists $Y_j \in \mathcal{F}$ such that $Y'_j \subseteq Y_j$, then $h_{pol}(\mathcal{F}'; Y) \leq h_{pol}(\mathcal{F}; Y)$.* □

Next we define the local polynomial entropy for a finite set

$$\mathcal{S} = \{x_1, x_2, \dots, x_l\} \subset X \setminus NW(f).$$

We choose a decreasing sequence of neighbourhoods $\{U_{j,n}\}_{n \in \mathbb{N}}$ of $x_j \in \mathcal{S}$ which form a basis of neighbourhoods of x_j. It follows from Proposition 1 that the sequence

$$\left\{h_{pol}\left(\{U_{1,n}, U_{2,n}, \dots, U_{l,n}\}; Y\right)\right\}_{n \geq 1}$$

is decreasing and converges, as well as that its limit does not depend on the choice of neighbourhoods. Define:

$$h_{pol}^{loc}(\mathcal{S}; Y) := \lim_{n \to \infty} h_{pol}\left(\{U_{1,n}, U_{2,n}, \dots, U_{l,n}\}; Y\right).$$

Finally, we relate the polynomial entropy to a singular set.

We say that the subsets $U_1, \dots, U_L$ of $X \setminus \mathrm{NW}(f)$ are *mutually singular* if for every $M > 0$, there exist x and positive integers $n_1, \dots, n_L$ such that

$$f^{n_j}(x) \in U_j, \quad |n_i - n_j| > M, \quad \text{for every } i \neq j.$$

The points $x_1, \ldots, x_L \in X \backslash \mathrm{NW}(f)$ are mutually singular if every family of respective neighbourhoods $U_1, \ldots, U_L$, $U_j \ni x_j$, satisfying $U_j \subset X \backslash \mathrm{NW}(f)$, is mutually singular. We say that a finite set is *singular* if it consists of mutually singular points. Also note that a singleton (which consists of one wandering point) is always singular.

Proposition 2 [Propositions 3.2 and 3.3 in [13]] *Let $Y \subseteq X$ be an f-invariant set containing exactly one non-wandering point. Then it holds:*

(a) $h_{pol}(f; Y) = \sup\{h_{pol}(\{K\}; Y) \mid K \subseteq X \setminus \mathrm{NW}(f),\ K \textit{ compact}\}$
(b) $h_{pol}(f; Y) = \sup\{h_{pol}(\{K \cap Y\}; Y) \mid K \subseteq X \setminus \mathrm{NW}(f),\ K \textit{ compact}\}$
(c) $h_{pol}(f; Y) = \sup\left\{h_{pol}^{loc}(\mathcal{S}; Y) \mid \mathcal{S} \subset \overline{Y},\ \mathcal{S} \textit{ singular}\right\}$. □

The following corollary is of importance for our result, so we will prove it.

Corollary 3 [Corollary 3.3.1 in [13]] *Let $Y \subseteq X$ be an f-invariant set containing exactly one non-wandering point. The polynomial entropy $h_{pol}(f; Y)$ is bounded from above by the maximal cardinality of a singular set contained in $\overline{Y}$.*

Proof The local polynomial entropy of a finite set $\{x_1, \ldots, x_l\} \subset X \setminus \mathrm{NW}(f)$ is bounded from above by its cardinality. Indeed, one can choose wandering neighbourhoods $Y_j \ni x_j$, such that every letter Y_j appears in the coding of any orbit $(x, f(x), \ldots, f^{n-1}(x))$ at most once. Therefore there are at most $n(n-1)\cdots(n-l+1) \le n^l$ possible codings. We obtain

$$h_{pol}^{loc}(\{x_1, \ldots, x_l\}; Y) \le h_{pol}(\{Y_1, \ldots, Y_l\}; Y) = \limsup_{n \to \infty} \frac{\log \sharp \mathcal{A}_n(\{Y_1, \ldots, Y_l\}; Y)}{\log n} \le l.$$

By taking the supremum over all sets of mutually singular points and using the statement (c) in Proposition 2, we conclude that $h_{pol}(f; Y) \le l$. □

In particular, if the maximal cardinality of a singular set is two, we have a more precise statement.

Proposition 4 *Let $Y \subseteq X$ be an f-invariant set containing exactly one non-wandering point. Suppose that the maximal cardinality of a singular set contained in $\overline{Y}$ equals 2. If for every singular set $\mathcal{S} = \{x_1, x_2\} \subset Y$ and every open $Y_j \ni x_j$, $Y_j \subset X \backslash NW(f)$, there exists a positive integer L such that for all $k \ge L$ it holds $f^k(Y_1) \cap Y_2 \neq \emptyset$, then $h_{pol}(f; Y) = 2$.*

Proof It follows from Corollary 3 that $h_{pol}(f; Y) \le 2$. To prove the other inequality, note that the assumed condition on Y_1 and Y_2 implies that for all positive integer m and for all $k \ge L$, there exists x with $f^m(x) \in Y_1$, $f^{m+k}(x) \in Y_2$. Therefore, for every $k \ge L$ there exists a coding of a form

$$(\underbrace{Y_\infty, \ldots, Y_\infty}_{m}, Y_1, \underbrace{Y_\infty, \ldots, Y_\infty}_{k}, Y_2, Y_\infty, \ldots, Y_\infty)$$

where m is any number and $k \ge L$. We get

$$\sharp \mathcal{A}_n(\{Y_1, Y_2\}; Y) \ge \sum_{m=0}^{n-L-1} (n - L - m) \sim \frac{1}{2} n^2,$$

so $h_{pol}(\{Y_1, Y_2\}; Y) \geq 2$. Since this holds for all Y_j with $x_j \in Y_j \subset X \backslash NW(f)$, $j = 1, 2$, the same is true for $h_{pol}(\{x_1, x_2\}; Y)$. $\square$

We will use the following notations:

$$\mathcal{O}(x) := \{f^n(x) \mid n \in \mathbb{Z}\}$$

for the *orbit* of a point x and

$$W^s(p) := \{x \in X \mid f^n(x) \to p, n \to \infty\}$$

for the *stable set* of a (fixed) point p.

3 Proofs

In order to use the methods described in Sect. 2.2 we need to establish that the sets $NW(C(f))$ and $NW\left(f^{*k}\right)$ are finite. Since $h_{pol}(f^m) = h_{pol}(f)$, we can assume that $NW(f) = Fix(f)$.

Proposition 5 *Let $f : [0, 1] \to [0, 1]$ or $f : S^1 \to S^1$ be a homeomorphism such that the set $NW(f) = Fix(f)$ is finite. Then $NW(C(f)) = Fix(C(f))$ and $NW\left(f^{*k}\right) = Fix\left(f^{*k}\right)$ are also finite for every $k \geq 1$.*

Proof Let us prove the proposition for $C(f)$, $f : I \to I$. The other case can be proved analogously. The set $Fix(C(f))$ is finite as it consists of all intervals $[a, b] \subseteq I$, where $a, b \in Fix(f)$. Note that the condition $NW(f) = Fix(f)$ implies that f is increasing. Suppose that there exists $[x, y] \in NW(C(f)) \setminus Fix(C(f))$ and that $x \notin Fix(f)$. Since $NW(f) = Fix(f)$, there exist $a, b \in Fix(f)$ (possibly equal) with

$$f^n(x) \to a, \quad f^n(y) \to b, \quad n \to \infty.$$

Let $\varepsilon < d(x, a)/2$. For $n \geq n_0$, where n_0 is big enough, it holds $d(f^n(x), a) < \varepsilon$, $d(f^n(y), b) < \varepsilon$ so we have

$$\begin{aligned} d_H(C(f)^n([x, y]), [x, y]) &\geq d_H([a, b], [x, y]) - d_H\left(C(f)^n([x, y]), [a, b]\right) \\ &\geq d(a, x) - \varepsilon > \varepsilon. \end{aligned}$$

$\square$

3.1 Proof of Theorem A

Let $I := [0, 1]$. We can assume that f is increasing, since if not, f^2 is. We will first compute the polynomial entropy of $C(f)$. We can identify the space $C(I)$ with the set

$$\{(x, y) \in [0, 1]^2 \mid 0 \leq x \leq y \leq 1\},$$

which is the upper triangle A in the square $[0,1] \times [0,1]$. The homeomorphism $\varphi : C(I) \to A$ is given by $\varphi : [x, y] \to (x, y)$. The map $C(f)$ is conjugated to $f \times f|_A$ via φ (where $f \times f : [0,1] \times [0,1] \to [0,1] \times [0,1]$, $(f \times f)(x, y) = (f(x), f(y))$).

Denote the lower triangle in $[0,1] \times [0,1]$ by B. Since both A and B are closed and $f \times f$-invariant, we have

$$h_{pol}(f \times f) = \max\{h_{pol}(f \times f|_A), h_{pol}(f \times f|_B)\}.$$

On the other hand

$$h_{pol}(f \times f) = 2h_{pol}(f).$$

Since $f \times f|_A$ and $f \times f|_B$ are topologically equivalent systems (the map $(x, y) \mapsto (y, x)$ realizes a conjugacy), we have

$$h_{pol}(C(f)) = 2h_{pol}(f) \tag{1}$$

It is not hard to see that $h_{pol}(C(f)) = 2$, because $h_{pol}(f) = 1$. Let $0 = p_0 < p_1 < \ldots < p_k = 1$ denote the fixed points. We apply again $h_{pol}(f) = \max_j h_{pol}\left(f|_{[p_{j-1},p_j]}\right)$.

As in 2, $h_{pol}(f|_{[p_{j-1},p_j]}) \geq 1$ (this also follows from the fact that $f|_{[p_{j-1},p_j]}$ possesses a wandering point, see Proposition 2.1 in [17] for a more general statement). The set $Y = (p_{j-1}, p_j]$ is $f|_{[p_{j-1},p_j]}$-invariant and contains exactly one non-wandering point. So, by Corollary 3 and the fact that f has only one singular point in $(p_{j-1}, p_j]$ we obtain

$$h_{pol}\left(f|_{[p_{j-1},p_j]}\right) = \max\left\{h_{pol}\left(f|_{[p_{j-1},p_j]}; (p_{j-1}, p_j]\right), h_{pol}\left(f|_{\{p_{j-1}\}}\right)\right\} \leq 1.$$

(For a different, more explicit proof of $h_{pol}(f|_{[p_{j-1},p_j]}) = 1$ see [17].)

Let us prove that $h_{pol}\left(f^{*k}\right) = k$. Since $f^{\times k} := f \times \ldots \times f$ and f^{*k} are semi-conjugated via

$$\pi : I^k \to I^{*k}, \quad \pi : (x_1, \ldots, x_k) \mapsto \{x_1, \ldots, x_k\},$$

we have

$$h_{pol}\left(f^{*k}\right) \leq h_{pol}\left(f^{\times k}\right) = k. \tag{2}$$

We want to prove the other inequality. For a permutation σ of $\{1, \ldots, k\}$, define

$$A_\sigma := \{(x_1, \ldots, x_k) \in I^k \mid x_{\sigma(1)} \leq \ldots \leq x_{\sigma(k)}\}.$$

As before, we see that $I^k = \bigcup_\sigma A_\sigma$, A_σ's are $f^{\times k}$-invariant, so

$$h_{pol}\left(f^{\times k}\right) = \max_\sigma \left\{h_{pol}\left(f^{\times k}|_{A_\sigma}\right)\right\}.$$

Since all $f^{\times k}|_{A_\sigma}$ are mutually conjugated we have

$$h_{pol}\left(f^{\times k}|_{A_\sigma}\right) = h_{pol}\left(f^{\times k}|_{A_\rho}\right)$$

for every two permutations σ and ρ and therefore

$$h_{pol}\left(f^{\times k}|_{A_\sigma}\right) = \max_{\sigma}\left\{h_{pol}\left(f^{\times k}|_{A_\sigma}\right)\right\} = h_{pol}\left(f^{\times k}\right) = k$$

for every σ.

Define:

$$A(k) := \{(x_1, \ldots, x_k) \in I^k \mid x_1 \leq \ldots \leq x_k\}$$
$$\hat{A}(k, I) := \left\{\{x_1, \ldots, x_k\} \in I^{*k} \mid x_1 < \ldots < x_k\right\}.$$

Whenever there is no risk of confusion we will abbreviate $\hat{A}(k) = \hat{A}(k, I)$. If $k = 2$ we are done, since $\pi|_{A(2)} : A(2) \to I^{*2}$ is a homeomorphism of compact sets, so $h_{pol}\left(f^{*2}\right) = h_{pol}\left(f^{\times 2}|_{A(2)}\right) = 2$.

For $k > 2$ we need the following auxiliary fact.

Lemma 6 *Suppose that $Fix(f) = \{0, 1\}$. Then $h_{pol}\left(f^{*k}; \hat{A}(k)\right) = k$.*

The rest of the proof follows easily from Lemma 6. Indeed, if $Fix(f) = \{0, 1\}$ then we have:

$$h_{pol}\left(f^{*k}\right) \geq h_{pol}\left(f^{*k}; \hat{A}(k))\right) = k \overset{(2)}{\Rightarrow} h_{pol}\left(f^{*k}\right) = k.$$

If $Fix(f) = \{0, p_1, \ldots, p_{k-1}, 1\}$ with $0 < p_1 < \ldots < p_{k-1} < 1$, define

$$T := \left\{\{x_1, \ldots, x_j\} \mid j \in \{1, \ldots, k\},\ x_i \in [0, p_1]\right\} \subset I^{*k}$$

and apply Lemma 6 to $f|_{[0,p_1]}$. Since T is f^{*k}-invariant, we conclude

$$\begin{aligned} h_{pol}\left(f^{*k}\right) \geq h_{pol}\left(f^{*k}|_T\right) &= h_{pol}\left(\left(f|_{[0,p_1]}\right)^{*k}\right) \\ &\geq h_{pol}\left(\left(f|_{[0,p_1]}\right)^{*k}; \hat{A}(k, [0, p_1])\right) = k. \end{aligned}$$

From here and (2) we obtain $h_{pol}\left(f^{*k}\right) = k$. □

Proof of Lemma 6 Consider the following covering of $A(k)$:

- $\tilde{A}(k) := \{(x_1, \ldots, x_k) \in I^k \mid x_1 < \ldots < x_k\}$
- $A_j(k) := \{(x_1, \ldots, x_k) \in I^k \mid x_1 \leq \ldots \leq x_j = x_{j+1} \leq \ldots \leq x_k\}$, for $j = 1, \ldots, k-1$.

It is obvious that $\tilde{A}(k)$ and $A_j(k)$ are $f^{\times k}$-invariant. Therefore

$$h_{pol}\left(f^{\times k}\right) = h_{pol}\left(f^{\times k}|_{A(k)}\right) = \max\left\{h_{pol}\left(f^{\times k}; \tilde{A}(k)\right), h_{pol}\left(f^{\times k}|_{A_j(k)}\right)\right\}.$$

Notice that $f^{\times k}|_{A_j(k)}$ is conjugated to $f^{\times(k-1)}|_{A(k-1)}$, so

$$h_{pol}\left(f^{\times k}|_{A_j(k)}\right) = h_{pol}\left(f^{\times(k-1)}|_{A(k-1)}\right) = k - 1.$$

Since $h_{pol}\left(f^{\times k}\right) = k$ and $h_{pol}\left(f^{\times k}|_{A_j(k)}\right) = k - 1$, we conclude that $h_{pol}\left(f^{\times k}; \tilde{A}(k)\right) = k$.

Notice that

$$\pi|_{\tilde{A}(k)} : \tilde{A}(k) \to \hat{A}(k)$$

is a homeomorphism that establishes a conjugacy between $f^{\times k}|_{\tilde{A}(k)}$ and $f^{*k}|_{\hat{A}(k)}$. Although polynomial entropy is a conjugacy invariant only when the domain is compact (while the sets $\hat{A}(k)$ and $\tilde{A}(k)$ are not), we can indirectly prove that

$$h_{pol}\left(f^{*k}; \hat{A}(k)\right) = h_{pol}\left(f^{\times k}; \tilde{A}(k)\right) = k.$$

We wish to apply the coding method from Proposition 2. Note that for $k > 2$ the sets $\tilde{A}(k)$ and $\hat{A}(k)$ do not contain any non-wandering points. We can add the point $(0, \ldots, 0)$ to the set $\tilde{A}(k)$ and $\{0\}$ to $\hat{A}(k)$, keeping the same notations: this will not change the entropy and π will still be a homeomorphism between the two. In this way we achieve that the assumptions from Proposition 2 are fulfilled.

Note that π induces a bijection between the sets

$$\left\{K \cap \tilde{A}(k) \mid K \subset I^k \setminus NW\left(f^{\times k}\right),\ K \text{ compact}\right\}$$

and

$$\left\{L \cap \hat{A}(k) \mid L \subset I^{*k} \setminus NW\left(f^{*k}\right),\ L \text{ compact}\right\}.$$

If $\underline{w} = (w_0, \ldots, w_{n-1})$ is a coding of an orbit $\left(x, f^{\times k}(x) \ldots, (f^{\times k})^{n-1}(x)\right)$ in $\tilde{A}(k)$ (consisting of letters $K \cap \tilde{A}(k)$ and $Y_\infty := \tilde{A}(k) \setminus K$), then $(\pi(w_0), \ldots, \pi(w_{n-1}))$ is the coding of an orbit $\left(\pi(x), f^{*k}(\pi(x)), \ldots, (f^{*k})^{n-1}(\pi(x))\right)$ in $\hat{A}(k)$ (consisting of letters $\pi(K \cap \tilde{A}(k))$ and $Y_\infty := \hat{A}(k) \setminus \pi(K \cap \tilde{A}(k))$), and vice versa. Therefore, for a fixed compact K, the sets $\mathcal{A}_n\left(\{K \cap \tilde{A}(k)\}; \tilde{A}(k)\right)$ and $\mathcal{A}_n\left(\{\pi(K \cap \tilde{A}(k))\}; \hat{A}(k)\right)$ have the same cardinality. Applying (b) from Proposition 2 we finish the proof of Lemma 6.

Finally, to prove that $h_{pol}\left(2^f\right) = \infty$, we notice that X^{*k} is a closed and 2^f-invariant subset of 2^X and moreover $f^{*k} = 2^f|_{X^{*k}}$ so

$$h_{pol}\left(2^f\right) \geq h_{pol}\left(f^{*k}\right) = k$$

for every $k \in \mathbb{N}$. □

Remark 7 A can be generalized to the case of a homeomorphism f of an interval with an infinite, but not equal to the whole interval, non-wandering set. We thank the anonymous referee for raising this issue. Let us briefly explain this generalization.

1. The polynomial entropy of f is still equal to one. It is clear, as before, that $h_{pol}(f) \geq 1$. For the other inequality see Corollary 3.5 in [10].
2. The result concerning the polynomial entropy of $C(f)$ follows directly from (1).
3. Since $h_{pol}\left(f^{*k}\right) \leq h_{pol}\left(f^{\times k}\right)$, we have $h_{pol}\left(f^{*k}\right) \leq k$. The other inequality follows from Lemma 6 applied to $f|_{[p_1,q_1]}$. Indeed, as before, set $T := \left\{\{x_1, \ldots, x_j\} \mid j \in \{1, \ldots, k\},\ x_i \in [p_1, q_1]\right\} \subset I^{*k}$, T is f^{*k}-invariant. We have

$$\begin{aligned} h_{pol}\left(f^{*k}\right) &\geq h_{pol}\left(f^{*k}|_T\right) = h_{pol}\left(\left(f|_{[p_1,q_1]}\right)^{*k}\right) \\ &\geq h_{pol}\left(\left(f|_{[p_1,q_1]}\right)^{*k}; \hat{A}(k, [p_1, q_1])\right) = k. \end{aligned}$$

3.2 Proof of Theorem B

Let us first compute the polynomial entropy of $C(f)$. For that reason, we will distinguish between the following possibilities:

(1) the set $Fix(f)$ consists of only one point
(2) the set $Fix(f)$ has at least three different points
(3) the set $Fix(f)$ has exactly two points.

Case (1). Since f has only one fixed point, f preserves the orientation of the circle. If $Fix(f) = \{a\}$, the continuum map $C(f)$ has only two non-wandering points - $\{a\}$ and S^1. We will divide the set $C(S^1)$ into two closed invariant subsets:

- P is the set of all $[x, y]$ such that the point x is between points a and y counter-clockwise, including degenerate cases when the two or all three points are equal (meaning that $[x, a]$, $\{x\}$, $[a, y]$ and S^1 are in P); notice that $[x, y]$ does not contain a as an interior point, for $x \neq y$. See the circle on the left in Fig. 1.
- Q is the set of all $[x, y]$ such that the point y is between points a and x counter-clockwise, including degenerate cases when the two or all three points are equal (meaning that $[x, a]$, $\{a\}$, $[a, y]$ and S^1 are in Q); notice that $[x, y]$ contains a. See the circle in the middle in Fig. 1.

In this way we know that, for $[x, y], [z, w] \in P$ or $[x, y], [z, w] \in Q$, the following is true:

$$d(x, z) < r \quad \text{and} \quad d(y, w) < r \;\Rightarrow\; d_H([x, y], [z, w]) < r.$$

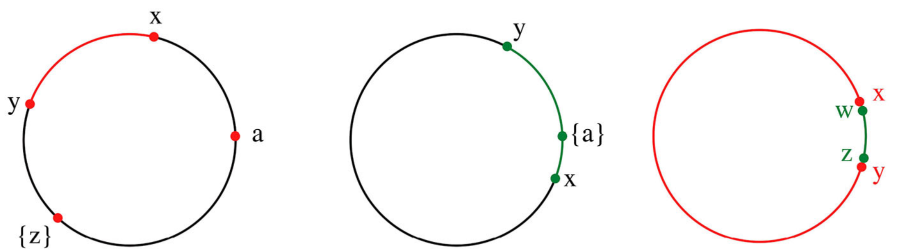

Fig. 1 Elements of set P (on the left) and Q (center); arcs $[x, y]$ and $[z, w]$ (right)

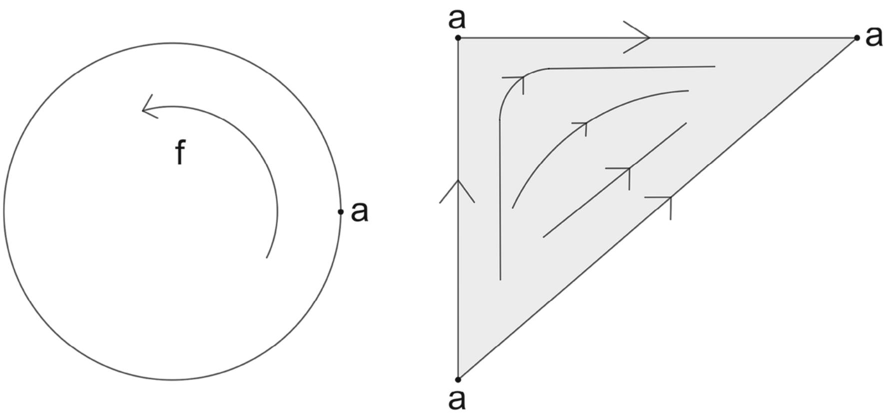

Fig. 2 Direction of f (on the left) and dynamics of $C(f)|_P$ (on the right)

(In general, this does not have to hold, since the endpoints of the arcs $[x, y] \in P$ and $[z, w] \in Q$ may be close, but not the corresponding intervals, see the circle on the right in Fig. 1.)

Suppose that f moves the points in S^1 in the positive direction, as in Fig. 2. The other case is treated in the same way.

We will first consider the map $C(f)|_P$. Its dynamics is depicted in Fig. 2. We have the following possibilities:

- $x \neq a$ and $y \neq a \Rightarrow C(f)^n([x, y]) \overset{n\to\pm\infty}{\longrightarrow} \{a\}$
- $x \neq a$ and $y = a \Rightarrow C(f)^n([x, y]) \overset{n\to\infty}{\longrightarrow} \{a\}$ and $C(f)^n([x, y]) \overset{n\to-\infty}{\longrightarrow} S^1$
- $[x, y] = \{x\} \Rightarrow C(f)^n([x, y]) \overset{n\to\pm\infty}{\longrightarrow} \{a\}$
- $x = a$ and $y \neq a \Rightarrow C(f)^n([x, y]) \overset{n\to\infty}{\longrightarrow} S^1$, $C(f)^n([x, y]) \overset{n\to-\infty}{\longrightarrow} \{a\}$
- $[x, y] = S^1 \Rightarrow C(f)^n([x, y]) \overset{n\to\pm\infty}{\longrightarrow} S^1$.

We can divide P into the sets $\{S^1\}$ and $Y := P\backslash\{S^1\}$ and compute

$$h_{pol}(C(f)|_P) = \max\left\{h_{pol}(C(f)|_P; Y), h_{pol}\left(C(f)|_P; \{S^1\}\right)\right\} = h_{pol}(C(f)|_P; Y)$$

since $h_{pol}(C(f)|_P; \{S^1\}) = h_{pol}(C(f)|_{\{S^1\}}) = 0$.

We claim that the arcs $[a, p]$, for $p \neq a$ and $[q, a]$ for $q \neq a$ are two mutually singular points x_1 and x_2 satisfying the conditions stated in Proposition 4.

Let us first prove that $[a, p]$ and $[q, a]$ are mutually singular. Fix an $\varepsilon > 0$, $\varepsilon < \min\{d(a, p), d(a, q)\}$ and $M > 0$. Choose $y \in B(p, \varepsilon) \subset S^1$ arbitrarily. Since $f^n(y) \to a$, when $n \to \infty$, there exists a non-negative integer $n_1 > M$ such that for $n \geq n_1$ it holds $d(f^n(y), a) < \varepsilon$. Let $\alpha \in B(q, \varepsilon) \subset S^1$ be any point. Since $f^{-n}(\alpha) \to a$, when $n \to \infty$, there exists $n_2 \geq n_1$ with $d(f^{-n_2}(\alpha), a) < \varepsilon$. We can increase n_2 if necessary to obtain that the point $f^{-n_2}(\alpha)$ is between a and y, and $f^{-n_2}(\alpha) \neq y$. Choose $x := f^{-n_2}(\alpha)$. Set $I := [x, y]$. We claim that the orbit of I intersects the ε-balls around $[a, p]$ and $[q, a]$ in the times with difference greater than M. Indeed, we have:

$$d_H(I, [a, p]) \leq \max\{d(x, a), d(y, p)\} < \varepsilon,$$

so $I \in B([a, p], \varepsilon)$ and

$$d(f^{n_2}(I), [q, a]) \leq \max\{d(f^{n_2}(x), q), d(f^{n_2}(y), a)\} = \max\{d(\alpha, q), d(f^{n_2}(y), a)\} < \varepsilon.$$

Therefore, $f^{n_2}(I) \in B([q, a], \varepsilon)$.

The next step is to show that any two arcs except the ones of the form $[a, p]$ and $[q, a]$ cannot be mutually singular. Suppose that $[p_1, q_1]$ and $[p_2, q_2]$ are two different arcs such that $p_j \neq a$ and $q_j \neq a$. We distinguish between the following two possibilities

(a) If $[p_1, q_1]$ and $[p_2, q_2]$ are not in the same orbit of $C(f)$, it is enough to show that there exist neighbourhoods $U \ni [p_1, q_1]$ and $V \ni [p_2, q_2]$ such that

$$C(f)^n(U) \cap V = \emptyset, \quad \text{for all } n \in \mathbb{Z}.$$

Let ε be a positive real number with the following properties:

- $\varepsilon < d_H([p_2, q_2], \mathcal{O}([p_1, q_1]))$, where $d_H([p_2, q_2], \mathcal{O}([p_1, q_1]))$ is the distance from $[p_2, q_2]$ to the orbit $\mathcal{O}([p_1, q_1])$, which is strictly positive, as $C(f)^n([p_1, q_1])$ converges to $\{a\}$, when $n \to \pm\infty$
- the balls (in $C(S^1)$) of radius ε around $[p_2, q_2]$ and $\{a\}$ are disjoint.

Since $C(f)^n([p_1, q_1])$ converges to $\{a\}$, when $n \to \pm\infty$, and the same holds for any $[p, q]$ close enough to $[p_1, q_1]$, we can find $\delta > 0$ and n_0 such that:

$$[p, q] \in B([p_1, q_1], \delta), \ |n| \geq n_0 \quad \Rightarrow \quad C(f)^n([p, q]) \in B(\{a\}, \varepsilon).$$

We can decrease δ if necessary to obtain:

$$[p, q] \in B([p_1, q_1], \delta), \ |n| \leq n_0 \quad \Rightarrow \quad C(f)^n([p, q]) \notin B([p_2, q_2], \varepsilon).$$

We conclude that the sets

$$U := B([p_1, q_1], \delta), \quad V := B([p_2, q_2], \varepsilon)$$

have the desired properties, hence $[p_1, q_1]$ and $[p_2, q_2]$ are not mutually singular.

(b) If $[p_1, q_1]$ and $[p_2, q_2]$ are in the same orbit of $C(f)$, it is enough to show that there exist neighbourhoods $U \ni [p_1, q_1]$ and $V \ni [p_2, q_2]$, $U, V \subset Y \backslash NW(C(f))$ and $M > 0$, such that:

$$C(f)^n(U) \cap V \neq \emptyset \;\Rightarrow\; |n| < M, \quad \text{for all } n \in \mathbb{Z}. \tag{3}$$

Indeed, it follows from (3) that U and V are not mutually singular, so neither are $[p_1, q_1]$ and $[p_2, q_2]$. So take $U \subset W^s(\{a\}) \cap Y$, V and U' to be any three balls centered at $[p_1, q_1]$, $[p_2, q_2]$ and $\{a\}$ respectively, such that $V \cap U' = \emptyset$. There exists a non-negative integer n_0 such that, for all $n \in \mathbb{Z}$, $|n| \geq n_0$, $C(f)^n(U) \subset U'$. We see that for $|n| \geq n_0$ it holds $C(f)^n(U) \cap V = \emptyset$.

To show that $[a, p]$, for $p \neq a$, and $[q, a]$, for $q \neq a$, are the only two possible mutually singular points, we should consider the following four possibilities (recall that singular points are, by definition, necessarily wandering points):

(i) $[a, p]$ and $[p_1, q_1]$, for $p \neq a$, $p_1 \neq a$ and $q_1 \neq a$, but possibly $p_1 = q_1$
(ii) $[q, a]$ and $[p_1, q_1]$, for $q \neq a$, $p_1 \neq a$ and $q_1 \neq a$, but possibly $p_1 = q_1$
(iii) $[a, p_1]$ and $[a, p_2]$, for $p_i \neq a$ and $p_1 \neq p_2$
(iv) $[q_1, a]$ and $[q_2, a]$, for $q_i \neq a$ and $q_1 \neq q_2$.

The cases (i) and (ii) can be treated the same way as the case **(a)** above; the cases (iii) and (iv) are the same as the case **(b)**.

It remains to prove that for every two mutually singular points of the form $[a, p]$ and $[q, a]$, where $p, q \neq a$ and every open $Y_1 \ni [a, p]$, $Y_2 \ni [q, a]$, $Y_j \subset P \backslash NW(C(f))$, there exists a positive integer L such that for all $n \geq L$ it holds $C(f)^n(Y_1) \cap Y_2 \neq \emptyset$. Then we are able to apply Proposition 4 and finish the proof.

Fix $\varepsilon > 0$ and set $Y_1 := B([a, p], \varepsilon) \subset P \backslash NW(C(f))$ and $Y_2 := B([q, a], \varepsilon) \subset P \backslash NW(C(f))$. Consider the line

$$l := \{[x, p] \mid d(x, a) < \varepsilon\} \subset Y_1.$$

Notice that

$$C(f)^n(l) = \{[f^n(x), f^n(p)] \mid x \in B(a, \varepsilon)\}.$$

Since $f^n(p) \to a$ and $f^n(B(a, \varepsilon)) \to S^1$, when $n \to \infty$, there exists n_0 such that for all $n \geq n_0$ both $d(f^n(p), a) < \varepsilon$ and $f^n(B(a, \varepsilon)) \ni q$ hold. Denote by $x_1 \in B(a, \varepsilon)$ such that $f^n(x_1) = q$. We conclude that $d_H([f^n(p), f^n(x_1)], [q, a]) < \varepsilon$, therefore $C(f)^n(Y_1) \cap Y_2 \neq \emptyset$.

The dynamics of $C(f)|_Q$ is the following:

- $x \neq a$ and $y \neq a \Rightarrow C(f)^n([x, y]) \overset{n\to\pm\infty}{\longrightarrow} S^1$
- $x = a$ and $y \neq a \Rightarrow C(f)^n([a, y]) \overset{n\to\infty}{\longrightarrow} S^1$ and $C(f)^n([a, y]) \overset{n\to-\infty}{\longrightarrow} \{a\}$
- $[x, y] = \{a\} \Rightarrow C(f)^n([x, y]) \overset{n\to\pm\infty}{\longrightarrow} \{a\}$
- $y = a$ and $x \neq a \Rightarrow C(f)^n([x, y]) \overset{n\to\infty}{\longrightarrow} \{a\}$ and $C(f)^n([x, y]) \overset{n\to-\infty}{\longrightarrow} S^1$

- $[x,y]=S^1 \Rightarrow C(f)^n([x,y]) \overset{n\to\pm\infty}{\longrightarrow} S^1$.

The same reasoning applies to $C(f)|_Q$, so the proof of Case (1) is done.

Case (2). Suppose $Fix(f)=\{a_1,\dots,a_m\}$, and there are no fixed points between the points a_j and a_{j+1}. Denote by $C_j := [a_j, a_{j+1}]$. It is obvious that the sets C_j are f-invariant, therefore the sets

$$D_{ij} := \{[x,y] \mid x\in C_i,\, y\in C_j\}$$

are $C(f)$-invariant. It is also easy to see that all D_{ij} are closed as well as $C(S^1)=\bigcup_{i,j} D_{i,j}$. The proof of B is completed if we prove that $h_{pol}(C(f)|_{D_{ij}})=2$ for all i,j.

We see that $D_{i,j}$ can be identified with $[a_i,a_{i+1}]\times[a_j,a_{j+1}]$ and since $f([x,y])=[f(x),f(y)]$, $C(f)$ is conjugated to $f\times f$. Therefore we reduce the problem to the of

$$\begin{aligned} f\times f &: [a_i,a_{i+1}]\times[a_j,a_{j+1}] \to [a_i,a_{i+1}]\times[a_j,a_{j+1}],\\ f\times f(x,y) &= (f(x),f(y)). \end{aligned}$$

Since $h_{pol}(f\times g)=h_{pol}(f)+h_{pol}(g)$, we have $h_{pol}(C(f)|_{D_{ij}})=h_{pol}(f|_{a_i,a_{i+1}]})+h_{pol}(f|_{[a_j,a_{j+1}]})$. Similarly as in the proof of A, we have $h_{pol}(f|_{[a_i,a_{i+1}]}) = h_{pol}(f|_{[a_j,a_{j+1}]})=1$.

Case (3). Suppose a_1 and a_2 are the only two fixed points. Then either f maps both $[a_1,a_2]$ and $[a_2,a_1]$ to themselves, or one to another. If the latter is the case, then f^2 maps both arcs to itself, so we can assume that this is true (since $h_{pol}(C(f^2)) = h_{pol}(C(f)^2)=h_{pol}(C(f)))$, and apply the same argument as in Case (2). Namely, $C_1=[a_1,a_2]$ and $C_2=[a_2,a_1]$ are f-invariant, therefore the sets D_{12} and D_{21} are $C(f)$-invariant. Following the calculations in Case (2), we obtain once again that $h_{pol}(C(f))=2$.

Now we prove the statement for f^{*k}. Recall first that $h_{pol}(f)=1$. This can be proved using the coding methods (it is easy to see that f possesses no two mutually singular points), or, alternatively, by refering to Theorem 2 in [17], which states that the polynomial entropy of a circle homeomorphism f is 1 if and only if f is not conjugated to a rotation.

Define a relation $\leq$ on S^1 by identifying S^1 with $[0,1)$. We can again assume that f is orientation preserving, since if not, f^2 is and

$$h_{pol}\left(\left(f^2)^{*k}\right)\right) = h_{pol}\left(\left(f^{*k}\right)^2\right) = h_{pol}\left(f^{*k}\right).$$

So we can consider f as an increasing homeomorphism of $[0,1]$ with $f(0)=0$ and $f(1)=1$ (we can assume that 0 is a fixed point of f). As before, by considering a semi-conjugacy $\pi : \left(S^1\right)^k \to \left(S^1\right)^{*k}$, we derive $h_{pol}\left(f^{*k}\right)\leq k$. For $k=2$, because the aforemention identification of S^1 with $[0,1)$, we can apply the same reasoning as in the proof of A and conclude that $h_{pol}\left(f^{*2}\right) = h_{pol}\left(f^{\times 2}|_{A(2)}\right) = 2$ (see page 10 for more details).

If f possesses at least two fixed points, a and b, we can define B as a subset of $\left(S^1\right)^{*k}$ consisting of sets of points from the interval $[a, b]$ and identify the map $f^{*k}|_B$ with $\left(f|_{[a,b]}\right)^{*k}$. Therefore we have

$$h_{pol}\left(f^{*k}\right) \geq h_{pol}\left(f^{*k}|_B\right) = h_{pol}\left(\left(f|_{[a,b]}\right)^{*k}\right) = k$$

(the last equality follows from the proof of A), so the proof is finished.

If f has only one fixed point, 0, we can define the sets $\tilde{A}(k)$ and $\hat{A}(k)$ as in the proof of A, conclude that $h_{pol}\left(f^{\times k}; \tilde{A}(k)\right) = k$, and then prove that $h_{pol}\left(f^{*k}; \hat{A}(k)\right) = k$, as in the proof of Lemma 6.

The last statement, $h_{pol}\left(2^f\right) = \infty$, follows in the same way as in the case of an interval. $\square$

Remark 8 If (X, f) and (Y, g) are two dynamical systems and there exists a semi-conjugacy $\pi : X \to Y$ that is uniformly finite-to-one (meaning that there exists C such for any $y \in Y$ it holds $\sharp\left(\pi^{-1}(y)\right) \leq C$), then the topological entropy $h_{top}(f)$ and $h_{top}(g)$ coincide (see, for example, Theorem 1.8 on p. 340 in [22]). It easily follows from this that $h_{top}(f^{*k}) = h_{top}(f^{\times k})$ (see Lemma 5 in [14]). An analogous formula for the polynomial entropy is still not proved or disproved, so we had to use the inequality relation, as a particularity of one-dimensional sets.

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Declarations

Conflict of interest The authors declare no competing interests.

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