



Gyroscopic Chaplygin Systems and Integrable Magnetic Flows on Spheres

Vladimir Dragović^{1,2} · Borislav Gajić² · Božidar Jovanović² 

Received: 24 December 2021 / Accepted: 14 February 2023 / Published online: 10 March 2023

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Abstract

We introduce and study the Chaplygin systems with gyroscopic forces. This natural class of nonholonomic systems has not been treated before. We put a special emphasis on the important subclass of such systems with magnetic forces. The existence of an invariant measure and the problem of Hamiltonization are studied, both within the Lagrangian and the almost-Hamiltonian framework. In addition, we introduce problems of rolling of a ball with the gyroscope without slipping and twisting over a plane and over a sphere in \mathbb{R}^n as examples of gyroscopic $SO(n)$ -Chaplygin systems. We describe an invariant measure and provide examples of $SO(n-2)$ -symmetric systems (ball with gyroscope) that allow the Chaplygin Hamiltonization. In the case of additional $SO(2)$ -symmetry, we prove that the obtained magnetic geodesic flows on the sphere S^{n-1} are integrable. In particular, we introduce the generalized Demchenko case in \mathbb{R}^n , where the inertia operator of the system is proportional to the identity operator. The reduced systems are automatically Hamiltonian and represent the magnetic geodesic flows on the spheres S^{n-1} endowed with the round-sphere metric, under the influence of a homogeneous magnetic field. The magnetic geodesic flow problem on the two-dimensional sphere is well known, but for $n > 3$ was not studied before. We

Communicated by Paul Newton.

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

✉ Božidar Jovanović
bozaj@mi.sanu.ac.rs

Vladimir Dragović
Vladimir.Dragovic@utdallas.edu

Borislav Gajić
gajab@mi.sanu.ac.rs

¹ Department of Mathematical Sciences, University of Texas at Dallas, Richardson, TX, USA

² Mathematical Institute, Serbian Academy of Sciences and Arts, Belgrade, Serbia

perform explicit integrations in elliptic functions of the systems for $n = 3$ and $n = 4$ and provide the case study of the solutions in both situations.

Keywords Nonholonomic dynamics · Rolling without sliding and twisting · Voronec equations · Chaplygin systems · Hamiltonization · Invariant measure · Magnetic geodesic flows on a sphere · Gyroscopic forces · Gyroscope · Integrability · Elliptic functions

Mathematics Subject Classification 37J60 · 37J35 · 33E05 · 70E45 · 70E05 · 70F25 · 53Z05

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1 Introduction

1.1 Nonholonomic Lagrangian Systems with Gyroscopic Forces

The main aim of this paper is to introduce and study a general setting for Chaplygin systems with gyroscopic forces, with a special emphasis on the important subclass of the Chaplygin systems with magnetic forces. This class of nonholonomic systems, although quite natural, has not been treated before.

In his first PhD thesis, Vasilije Demchenko (Demchenko (1924); Dragović et al. (2023)), studied the rolling of a ball with a gyroscope without slipping over a sphere in \mathbb{R}^3 , by using the Voronec equations (Voronec 1901; Woronetz 1911, 1912). Inspired by this thesis, we consider the rolling of a ball with a gyroscope without slipping and twisting over a sphere in \mathbb{R}^n . This will provide us with examples of gyroscopic $SO(n)$ -Chaplygin systems that reduce to integrable magnetic geodesic flows on a sphere S^{n-1} .

Let (Q, \mathbf{G}) be a Riemannian manifold. Consider a Lagrangian nonholonomic system (Q, L_1, \mathcal{D}) , where the constraints define a nonintegrable distribution \mathcal{D} on Q . The constraints are homogeneous and do not depend on time. The Lagrangian, along with the difference of the kinetic and potential energy, contains an additional term, which is linear in velocities:

$$L_1(q, \dot{q}) = \frac{1}{2}(\mathbf{G}(\dot{q}), \dot{q}) + (\mathbf{A}, \dot{q}) - V(q).$$

Here and throughout the text, (\cdot, \cdot) denotes the pairing between appropriate dual spaces, while \mathbf{A} is a one-form on Q . The metric \mathbf{G} is also considered as a mapping $TQ \rightarrow T^*Q$.

A smooth path $q(t) \in Q$, $t \in \Delta$ is called *admissible* if the velocity $\dot{q}(t)$ belongs to $\mathcal{D}_{q(t)}$ for all $t \in \Delta$. An admissible path $q(t)$ is a *motion of the natural mechanical nonholonomic system* (Q, L_1, \mathcal{D}) if it satisfies the Lagrange-d'Alembert equations

$$\delta L_1 = \left(\frac{\partial L_1}{\partial q} - \frac{d}{dt} \frac{\partial L_1}{\partial \dot{q}}, \delta q \right) = 0, \quad \text{for all } \delta q \in \mathcal{D}_q. \quad (1.1)$$

Equation (1.1) are equivalent to the equations

$$\delta L = \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}, \delta q \right) = \mathbf{F}(\dot{q}, \delta q), \quad \text{for all } \delta q \in \mathcal{D}_q, \quad (1.2)$$

where L is the part of the Lagrangian L_1 which does not contain the term linear in velocities:

$$L(q, \dot{q}) = \frac{1}{2}(\mathbf{G}(\dot{q}), \dot{q}) - V(q).$$

Here the additional force $\mathbf{F}(\dot{q}, \delta q)$ is defined as the exact two-form

$$\mathbf{F} = d\mathbf{A},$$

where \mathbf{A} is the one-form from the linear in velocities term of the Lagrangian L_1 . We will subsequently consider a more general class of systems where an additional force is given as a two-form which is neither exact nor even closed.

Systems with an additional force defined by a closed two-form \mathbf{F} and without nonholonomic constraints are very well studied. The corresponding Hamiltonian flows are usually called magnetic flows or twisted flows. For the problem of integrability of magnetic flows, see, e.g., Bolotin and Kozlov (2017), Bolsinov and Jovanović (2008), Taimanov (2006), Magazev et al. (2008) and Saksida (2002). Following tradition, we introduce

Definition 1.1 Let \mathbf{F} be a 2-form on Q . We refer to a system $(Q, L, \mathbf{F}, \mathcal{D})$ as a *natural mechanical nonholonomic system with gyroscopic forces*. The additional gyroscopic force $\mathbf{F}(\dot{q}, \delta q)$ is called *magnetic* if the form \mathbf{F} is closed,

$$d\mathbf{F} = 0,$$

and in this case we say that the system $(Q, L, \mathbf{F}, \mathcal{D})$ is a *natural mechanical nonholonomic system with a magnetic force*.

The equations of motion of a natural mechanical nonholonomic system with a gyroscopic force $(Q, L, \mathbf{F}, \mathcal{D})$ are given in (1.2).

Starting from the notion of G -Chaplygin systems for nonholonomic systems with gyroscopic forces [see Bakša (1975), Stanchenko (1989), Koiller (1992), Bloch et al. (1996), Cantrijn et al. (2002), Garcia-Naranjo and Marrero (2020)], we introduce the following.

Definition 1.2 Assume that Q is a principal bundle over S with respect to a free action of a Lie group G , $\pi: Q \rightarrow S = Q/G$, and that L and \mathbf{F} are G -invariant. Suppose that \mathcal{D} is a principal connection, that is, \mathcal{D} is G -invariant, transverse to the orbits of the G -action, and $\text{rank } \mathcal{D} = \dim S$. Then we refer to $(Q, L, \mathcal{D}, G, \mathbf{F})$ as a *gyroscopic G -Chaplygin system*.

Obviously, a gyroscopic G -Chaplygin system $(Q, L, \mathcal{D}, G, \mathbf{F})$ is G -invariant and reduces to the tangent space of the base-manifold $S = Q/G$.

1.2 Outline and Results of the Paper

In Sect. 2 we consider gyroscopic nonholonomic systems on fiber spaces. In Sect. 3 we employ them to describe a reduction procedure for the gyroscopic G -Chaplygin systems (Theorem 3.1). The Chaplygin systems have a natural geometrical framework as connections on principal bundles [see Koiller (1992)]. On the other hand, nonholonomic systems were incorporated into the geometrical framework of the Ehresmann connections on fiber spaces in Bloch et al. (1996). In this paper, we combine the approach of Bloch et al. (1996) with the Voronec nonholonomic equations, see Voronec (1901).

In Sect. 4 we derive the equations of motion of the reduced gyroscopic G -Chaplygin systems in an almost-Hamiltonian form and study the existence of an invariant measure (Theorem 4.1). A closely related problem is the Hamiltonization of nonholonomic systems [see Chaplygin (1911), Stanchenko (1989), Borisov and Mamaev (2001), Borisov and Mamaev (2008), Balseiro and Garcia-Naranjo (2012), Bolsinov et al. (2011), Borisov et al. (2014), Bolsinov et al. (2015), Ehlers et al. (2005), Cantrijn et al. (2002), Fedorov and Jovanović (2004), Jovanović (2019), Jovanović (2018)]. In Sect. 5 we consider the Chaplygin reducing multiplier and the time reparametrization of magnetic Chaplygin systems, both within the Lagrangian and the Hamiltonian framework (see Theorem 5.1).

In Sect. 6 we briefly review the results about integrable nonholonomic problems of rolling of a ball with the gyroscope, without slipping and twisting, over a plane and over a sphere in the three-dimensional space. In particular, we present the Demchenko integrable case (Demchenko 1924) and the Zhukovskiy condition for the system (Zhukovskiy 1893).

In Sect. 7 we introduce the problems of rolling of a ball with a gyroscope, without slipping and twisting, over a plane and over a sphere in \mathbb{R}^n . We describe the reduction (Propositions 7.1, 7.2) and an invariant measure (Proposition 7.3) of these new systems. The obtained systems are examples of gyroscopic $SO(n)$ -Chaplygin systems that reduce to magnetic flows.

In Sect. 8 we provide examples of $SO(n-2)$ -symmetric systems (ball with gyroscope) that allow the Chaplygin Hamiltonization (Theorem 8.1). We also prove the

integrability of the obtained magnetic geodesic flows on a sphere in \mathbb{R}^n , $n \geq 3$ in the case of $SO(2) \times SO(n-2)$ -symmetry (Theorem 8.2). Note that the phase space of a nonholonomic system that is integrable after the Chaplygin Hamiltonization is foliated by d -dimensional invariant tori, where the system is subject to a nonuniform quasi-periodic motion of the form

$$\dot{\varphi}_1 = \omega_1/\Phi(\varphi_1, \dots, \varphi_d), \dots, \dot{\varphi}_d = \omega_d/\Phi(\varphi_1, \dots, \varphi_d), \quad \Phi > 0, \quad (1.3)$$

with some d , $d \leq n$. In Theorem 8.2 we present two examples of such systems, one with $d = 2$ and $n = 3$ and another one with $d = 3$ and any $n > 3$.

Finally, in Sect. 9 we consider the case when the inertia operator for systems is $SO(n)$ -invariant, i.e., it satisfies the Zhukovskiy condition in \mathbb{R}^n with an additional nontwisting condition. We will refer to such systems as *the generalized Demchenko case without twisting in \mathbb{R}^n* . The reduced systems are automatically Hamiltonian. They represent the magnetic geodesic flow on a sphere S^{n-1} endowed with the round-sphere metric, under a influence of the homogeneous magnetic field placed in the ambient space \mathbb{R}^n . The magnetic geodesic flow problem on a two-dimensional sphere is well known [see Saksida (2002)]. However, the magnetic geodesic flow problems for $n > 3$ have not been studied before. We prove the complete integrability of the system on the three-dimensional sphere (Theorem 9.3). We conclude the paper with a detailed analysis of the motion of the generalized Demchenko systems without twisting for $n = 3$ and $n = 4$ in terms of elliptic functions.

2 Nonholonomic Systems with Gyroscopic Forces on Fibrated Spaces

2.1 The Voronec Equations

Following Demchenko¹ (Demchenko 1924; Dragović et al. 2023); we recall the Voronec equations for nonholonomic systems (Voronec 1901). We will then employ them to formulate the reduced equations of gyroscopic Chaplygin systems. Here we assume that the constraints may be time-dependent and nonhomogeneous.

Let $q = (q_1, \dots, q_{n+k})$ be local coordinates of the configuration space \mathcal{Q} . Consider a nonholonomic system with kinetic energy $T = T(t, q, \dot{q})$, generalized forces $Q_s = Q_s(t, q, \dot{q})$ that correspond to coordinates q_s , and time-dependent nonhomogeneous nonholonomic constraints

$$\dot{q}_{n+\nu} = \sum_{i=1}^n a_{\nu i}(q, t)\dot{q}_i + a_\nu(q, t), \quad \nu = 1, 2, \dots, k. \quad (2.1)$$

Let T_c be the kinetic energy T after imposing the constraints (2.1). Let K_ν be the partial derivatives of the kinetic energy T with respect to \dot{q}_ν , $\nu = 1, 2, \dots, k$, restricted

¹ Demchenko's PhD advisor, Anton Bilimović (1879–1970), was a distinguished student of Peter Vasilevich Voronec (1871–1923) and one of the founders of Belgrade's Mathematical Institute. We note that some recent results [see Borisov and Tsiganov (2020); Borisov et al. (2021)] are inspired by Bilimović's work in nonholonomic mechanics (Bilimovitch 1913a, b, 1914; Bilimovic 1915; Bilimovich 1916).

to the constrained subspace. We assume that the constraints (2.1) are imposed after the differentiation and get:

$$T_c(t, q_1, \dots, q_{n+k}, \dot{q}_1, \dots, \dot{q}_n) = T(t, q, \dot{q})|_{\dot{q}_{n+v} = \sum_{i=1}^n a_{vi}(q,t)\dot{q}_i + a_v(q,t)},$$

$$K_v(t, q_1, \dots, q_{n+k}, \dot{q}_1, \dots, \dot{q}_n) = \frac{\partial T}{\partial \dot{q}_{n+v}}(t, q, \dot{q})|_{\dot{q}_{n+v} = \sum_{i=1}^n a_{vi}(q,t)\dot{q}_i + a_v(q,t)}.$$

The equations of motion of the given nonholonomic system can be presented in a form which does not use the Lagrange multipliers:

$$\frac{d}{dt} \frac{\partial T_c}{\partial \dot{q}_i} = \frac{\partial T_c}{\partial q_i} + Q_i + \sum_{\nu=1}^k a_{\nu i} \left(\frac{\partial T_c}{\partial q_{n+\nu}} + Q_{n+\nu} \right) + \sum_{\nu=1}^k K_\nu \left(\sum_{j=1}^n A_{ij}^{(\nu)} \dot{q}_j + A_j^{(\nu)} \right). \tag{2.2}$$

The derivation of these equations is based on the Lagrange–d’Alembert principle and follows Voronec (1901). Here $i = 1, \dots, n$. The components $A_{ij}^{(\nu)}$ and $A_i^{(\nu)}$ are functions of the time t and the coordinates q_1, \dots, q_{n+k} given by

$$A_{ij}^{(\nu)} = \left(\frac{\partial a_{\nu i}}{\partial q_j} + \sum_{\mu=1}^k a_{\mu j} \frac{\partial a_{\nu i}}{\partial q_{n+\mu}} \right) - \left(\frac{\partial a_{\nu j}}{\partial q_i} + \sum_{\mu=1}^k a_{\mu i} \frac{\partial a_{\nu j}}{\partial q_{n+\mu}} \right),$$

$$A_i^{(\nu)} = \left(\frac{\partial a_{\nu i}}{\partial t} + \sum_{\mu=1}^k a_{\mu} \frac{\partial a_{\nu i}}{\partial q_{n+\mu}} \right) - \left(\frac{\partial a_\nu}{\partial q_i} + \sum_{\mu=1}^k a_{\mu i} \frac{\partial a_\nu}{\partial q_{n+\mu}} \right).$$

When all considered objects do not depend on the variables q_{n+v} , $\nu = 1, 2, \dots, k$, we have a *Chaplygin system*. Then Eq. (2.2) are called the *Chaplygin equations*. The Voronec and the Chaplygin equations, along with the equations of nonholonomic systems written in terms of quasi-velocities, known as *the Euler–Poincaré–Chetayev–Hamel equations*, form core tools in the study of nonholonomic mechanics [see Neimark and Fufaev (1972), Bloch et al. (1996), de León (2012), Ehlers et al. (2005), Ehlers and Koiller (2019), Zenkov (2016)].

2.2 The Ehresmann Connections and Systems with Gyroscopic Forces

Consider a natural mechanical nonholonomic system with a gyroscopic force $(Q, L, \mathbf{F}, \mathcal{D})$. After Bloch et al. (1996), we assume that Q has a structure of a fiber bundle $\pi : Q \rightarrow S$ over a base manifold S and that the distribution \mathcal{D} is transverse to the fibers of π :

$$T_q Q = \mathcal{D}_q \oplus \mathcal{V}_q, \quad \mathcal{V}_q = \ker d\pi(q).$$

The space \mathcal{V}_q is called the *vertical space* at q . The distribution \mathcal{D} can be seen as the kernel of a vector-valued one-form A on Q , which defines the *Ehresmann connection*, that satisfies

- (i) $A_q : T_q Q \rightarrow \mathcal{V}_q$ is a linear mapping, $q \in Q$;
- (ii) A is a projection: $A(X_q) = X_q$, for all $X_q \in \mathcal{V}_q$.

The distribution \mathcal{D} is called *the horizontal space* of the Ehresmann connection A . By X^h and X^v we denote the horizontal and the vertical component of the vector field $X \in \mathfrak{X}(Q)$. The *curvature* B of the connection A is a vertical vector-valued two-form defined by

$$B(X, Y) = -A([X^h, Y^h]).$$

Let $\dim Q = n + k$ and $\dim S = n$. There exist local “adapted” coordinates $q = (q_1, \dots, q_{n+k})$ on Q , such that the projection $\pi : Q \rightarrow S$ and the constraints defining \mathcal{D} are given by

$$\begin{aligned} \pi : (q_1, \dots, q_n, q_{n+1}, \dots, q_{n+k}) &\longmapsto (q_1, \dots, q_n), \\ \dot{q}_{n+v} &= \sum_{i=1}^n a_{vi}(q)\dot{q}_i, \quad v = 1, \dots, k. \end{aligned}$$

Here (q_1, \dots, q_n) are the local coordinates on S . Then, locally, we also have

$$\begin{aligned} A &= \sum_{v=1}^k \omega^v \frac{\partial}{\partial q_{n+v}}, \quad \omega^v = dq_{n+v} - \sum_{i=1}^n a_{vi} dq_i, \\ X^h &= \left(\sum_{l=1}^{n+k} X_l \frac{\partial}{\partial q_l} \right)^h = \sum_{i=1}^n X_i \frac{\partial}{\partial q_i} + \sum_{v=1}^k \sum_{i=1}^n a_{vi} X_i \frac{\partial}{\partial q_{n+v}}, \\ X^v &= \left(\sum_{l=1}^{n+k} X_l \frac{\partial}{\partial q_l} \right)^v = \sum_{v=1}^k \left(X_{n+v} - \sum_{i=1}^n a_{vi} X_i \right) \frac{\partial}{\partial q_{n+v}}, \\ B \left(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j} \right) &= \sum_{v=1}^k B_{ij}^v \frac{\partial}{\partial q_{n+v}}, \quad \mathbf{F} = \sum_{1 \leq s < l \leq n+k} F_{sl} dq_s \wedge dq_l. \end{aligned}$$

Here $B_{ij}^v(q) = A_{ij}^{(v)}(q)$, where $A_{ij}^{(v)}(q)$ come from the Voronec equations (2.2) with homogeneous constraints, which do not depend on time. The generalized forces $Q_s = Q_s(q, \dot{q})$, $s = 1, \dots, n + k$ are the sums of the potential and the gyroscopic forces

$$Q_s = Q_s^V + Q_s^F, \quad Q_s^V = -\partial V / \partial q_s, \quad Q_s^F = \sum_{l=1}^{n+k} F_{sl} \dot{q}_l.$$

The Voronec equations (2.2) take the form:

$$\frac{d}{dt} \frac{\partial L_c}{\partial \dot{q}_i} = \frac{\partial L_c}{\partial q_i} + \sum_{v=1}^k a_{vi} \frac{\partial L_c}{\partial q_{n+v}} + \sum_{v=1}^k \sum_{j=1}^n \frac{\partial L}{\partial \dot{q}_{n+v}} B_{ij}^v \dot{q}_j + Q_i^F + \sum_{v=1}^k a_{vi} Q_{n+v}^F, \tag{2.3}$$

($i = 1, \dots, n$), where L_c is the constrained Lagrangian $L_c = L(q, \dot{q}^h) = T_c - V$. In a compact form, the equations can be expressed as²:

$$\delta L_c = \mathbb{F}L(q, \dot{q})(B(\dot{q}, \delta q)) + \mathbf{F}(\dot{q}, \delta q) \tag{2.4}$$

for all virtual displacements

$$\delta q = \sum_{s=1}^{n+k} \delta q_s \frac{\partial}{\partial q_s} \in \mathcal{D}_q.$$

Here δL_c is the variational derivative of the constrained Lagrangian along the variation δq and $\mathbb{F}L$ is the fiber derivative of L :

$$\begin{aligned} \delta L_c &= \left(\frac{\partial L_c}{\partial q} - \frac{d}{dt} \frac{\partial L_c}{\partial \dot{q}}, \delta q \right) = \sum_{s=1}^{n+k} \left(\frac{\partial L_c}{\partial q_s} - \frac{d}{dt} \frac{\partial L_c}{\partial \dot{q}_s} \right) \delta q_s, \\ \mathbb{F}L(q, X)(Y) &= \frac{d}{ds} \Big|_{s=0} L(q, X + sY), \quad X, Y \in T_q Q, \\ \mathbb{F}L(q, \dot{q})(B(\dot{q}, \delta q)) &= \sum_{v=1}^k \frac{\partial L}{\partial \dot{q}_{n+v}}(q, \dot{q}) B^v(\dot{q}, \delta q). \end{aligned}$$

See Bloch et al. (1996) for the case without gyroscopic two-form \mathbf{F} .

Note that, even in the case when the two form \mathbf{F} is exact $\mathbf{F} = d\mathbf{A}$, it is convenient to use the Lagrangian L and the form of Eq. (2.4), rather than the Lagrangian L_1 with the term linear in velocities.

Remark 2.1 In the case when the constraints are nonhomogeneous and time dependent (2.1), the coefficients $A_{ij}^{(v)}, A_i^{(v)}$ can be also interpreted as the components of the curvature of the Ehresmann connection of the fiber bundle $\pi : Q \times \mathbb{R} \rightarrow S \times \mathbb{R}$ [see Bakša (2012)].

3 The Gyroscopic Chaplygin Systems

In addition to the assumptions from Sect. 2.2, we now assume that the fibration $\pi : Q \rightarrow S$ is determined by a free action of a k -dimensional Lie group G on Q , so that $S = Q/G$ and that the constraint distribution \mathcal{D} , the gyroscopic two-form \mathbf{F} and the Lagrangian $L = T - V$ are G -invariant. Then A is a principal connection and the nonholonomic system (2.4) is G -invariant and reduces to the tangent bundle of the base manifold S by the identification $TS = \mathcal{D}/G$. More precisely, we use the following definition.

² One can compare the form of Eq. (2.4) with the compact form of the Voronec equations obtained from the Voronec principle, see, e.g., Dragović et al. (2023).

Definition 3.1 Let \mathbf{G} , V , and \mathbf{F} be a G -invariant metric, a potential and a two-form on Q . The *reduced metric* \mathbf{g} , the *reduced potential* v , and the *reduced two-form* \mathbf{f} on S are defined by:

$$\mathbf{g}(X, Y)|_x = \mathbf{G}(X^h, Y^h)|_q, \quad v(x) = V(q), \quad \mathbf{f}(X, Y)|_x = \mathbf{F}(X^h, Y^h)|_q.$$

Here X^h, Y^h are the horizontal lifts of X, Y at a point $q \in \pi^{-1}(x)$ defined by

$$d\pi|_q(X^h) = X, \quad d\pi|_q(Y^h) = Y, \quad X^h, Y^h \in \mathcal{D}_q.$$

Note that we do not impose any additional assumptions on \mathbf{F} . In particular, \mathbf{F} does not need to be of the form $\mathbf{F} = \pi^*\mathbf{w}$, where \mathbf{w} is a 2-form on the base manifold S .

Equation (2.4) are G -invariant and they reduce to TS

$$\delta l = \left(\frac{\partial l}{\partial x} - \frac{d}{dt} \frac{\partial l}{\partial \dot{x}}, \delta x \right) = \mathbf{JK}(\dot{x}, \delta x) + \mathbf{f}(\dot{x}, \delta x) \quad \text{for all } \delta x \in T_x S, \quad (3.1)$$

where

$$l = \frac{1}{2}(\mathbf{g}(\dot{x}), \dot{x}) - v(x)$$

is the reduced Lagrangian and the term³ $\mathbf{JK}(\cdot, \cdot)$ depends on the metric and the curvature of the connection, induced by $\mathbb{F}L(B(\cdot, \cdot))$. The term $\mathbf{JK}(\cdot, \cdot)$ can be described as follows. Consider the (0,3)-tensor field Σ on S defined by

$$\Sigma(X, Y, Z)|_x = \mathbb{F}L(q, X^h)(B(Y^h, Z^h))|_q, \quad q \in \pi^{-1}(x), \quad (3.2)$$

where X^h, Y^h, Z^h are the horizontal lifts of vector fields X, Y, Z on S . Then Σ is skew-symmetric with respect to the second and the third argument, and

$$\mathbf{JK}(X, Y)|_{(x, \dot{x})} = \Sigma(\dot{x}, X, Y). \quad (3.3)$$

Remark 3.1 Let us explain the notation for the \mathbf{JK} -term. It is obtained from the natural pairing of the momentum mapping of the G -action $J : TQ \rightarrow \mathfrak{g}^*$ and the curvature $K : TQ \times TQ \rightarrow \mathfrak{g}$ of the principal connection A , where \mathfrak{g} is the Lie algebra of the Lie group G . Namely, we have a canonical identification of the vertical space \mathcal{V}_q with the Lie algebra \mathfrak{g} . Then the curvature of the Ehresmann connection B is \mathfrak{g} -valued and coincides with the curvature K of the principal connection. Also, within this identification, the fiber derivative $\mathbb{F}L(q, \dot{q})$ in the direction of the vertical vector $\xi \in \mathfrak{g} \cong \mathcal{V}_q$ becomes the value of the momentum mapping J of the G -action evaluated at ξ . In this way the expression (3.2), as the natural pairing of the tangent bundle momentum mapping J and the curvature two-form K , defines a (0, 3)-tensor field Σ on S . On the other hand, the \mathbf{JK} -term defined by (3.3) is a semi-basic 2-form on TS .

³ Let us note that in Ehlers et al. (2005), the term "JK" is used for the associated semi-basic two-form σ on T^*S given below.

Definition 3.2 We refer to $(S, l, \mathbf{JK}, \mathbf{f})$ as a *reduced gyroscopic G-Chaplygin system*. In the case when \mathbf{f} is a closed form, we call it a *reduced magnetic G-Chaplygin system*.

The equations of motion of the reduced gyroscopic G-Chaplygin system $(S, l, \mathbf{JK}, \mathbf{f})$ are described in (3.1).

We summarize the above considerations in the following statement.

Theorem 3.1 *The solutions of the gyroscopic G-Chaplygin system $(Q, L, \mathcal{D}, G, \mathbf{F})$ project to solutions of the reduced gyroscopic G-Chaplygin system $(S, l, \mathbf{JK}, \mathbf{f})$. Let $x(t)$ be a solution of the reduced system (3.1) with the initial conditions $x(0) = x_0, \dot{x}(0) = X_0 \in T_{x_0}S$ and let $q_0 \in \pi^{-1}(x_0)$. Then the horizontal lift $q(t)$ of $x(t)$ through q_0 is the solution of the original system (1.2), i.e., (2.4), with the initial conditions $q(0) = x_0, \dot{q}(0) = X_0^h \in \mathcal{D}_{q_0}$.*

Remark 3.2 If \mathbf{f} is an exact magnetic form, e.g., $\mathbf{f} = d\mathbf{a}$, then Eq. (3.1) are equivalent to

$$\delta l_1 = \left(\frac{\partial l_1}{\partial x} - \frac{d}{dt} \frac{\partial l_1}{\partial \dot{x}}, \delta x \right) = \mathbf{JK}(\dot{x}, \delta x) \quad \text{for all } \delta x \in T_x S, \quad (3.4)$$

where the Lagrangian l_1 , given by

$$l_1 = \frac{1}{2}(\mathbf{g}(\dot{x}), \dot{x}) + (\mathbf{a}, \dot{x}) - v(x),$$

has the linear term (\mathbf{a}, \dot{x}) .

Remark 3.3 Within the affine connection approach to the Chaplygin reduction, it is convenient to introduce (1, 2)-tensor fields \mathbf{B} and \mathbf{C} defined by [see Koiller (1992), Cantrijn et al. 2002]

$$\Sigma(X, Y, Z) = \mathbf{g}(\mathbf{B}(X, Y), Z) = \mathbf{g}(X, \mathbf{C}(Y, Z)).$$

Gajić and Jovanović (2019a), the tensor field \mathbf{B} was used, while here we work with the skew-symmetric tensor \mathbf{C} . Note that \mathbf{C} is equal to the negative gyroscopic tensor \mathcal{T} defined by Garcia-Naranjo (2019a, b).

Note that if \mathbf{F} is magnetic, then \mathbf{f} is not necessarily magnetic. Indeed, we have

Proposition 3.1 *Assume that the form \mathbf{F} is closed. Then the reduced form \mathbf{f} is closed if and only if*

$$\begin{aligned} & \mathbf{F}([X^h, Y^h] - [X, Y]^h, Z^h) + \mathbf{F}([Z^h, X^h] - [Z, X]^h, Y^h) \\ & + \mathbf{F}([Y^h, Z^h] - [Y, Z]^h, X^h) = 0, \end{aligned} \quad (3.5)$$

for all vector fields X, Y, Z on S . In the adapted coordinates $q = (q_1, \dots, q_{n+k})$ on Q described in Sect. 2.2, the condition (3.5) is equivalent to the equations

$$\sum_{v=1}^k \left(B_{ij}^v F_{p,n+v} + B_{pi}^v F_{j,n+v} + B_{jp}^v F_{i,n+v} \right) = 0, \quad 1 \leq i, j, p \leq n. \quad (3.6)$$

In particular, if the curvature B of the Ehresmann connection vanishes (equivalently, the curvature K of the principal connection vanishes), then \mathbf{f} is closed.

Proof Since \mathbf{F} is magnetic, we have

$$d\mathbf{F}(X', Y', Z') = X'\mathbf{F}(Y', Z') + Y'\mathbf{F}(Z', X') + Z'\mathbf{F}(X', Y') - \mathbf{F}([X', Y'], Z') - \mathbf{F}([Z', X'], Y') - \mathbf{F}([Y', Z'], X') = 0,$$

for arbitrary vector fields X', Y', Z' on Q . On the other hand, by using the above relation and the definition of \mathbf{f} that depends on the horizontal distribution \mathcal{D} , we get

$$\begin{aligned} d\mathbf{f}(X, Y, Z)|_x &= \left(X\mathbf{f}(Y, Z) + Y\mathbf{f}(Z, X) + Z\mathbf{f}(X, Y) - \mathbf{f}([X, Y], Z) - \mathbf{f}([Z, X], Y) - \mathbf{f}([Y, Z], X) \right) \Big|_x \\ &= \left(X^h\mathbf{F}(Y^h, Z^h) + Y^h\mathbf{F}(Z^h, X^h) + Z^h\mathbf{F}(X^h, Y^h) - \mathbf{F}([X, Y]^h, Z^h) - \mathbf{F}([Z, X]^h, Y^h) - \mathbf{F}([Y, Z]^h, X^h) \right) \Big|_q \\ &= \left(\mathbf{F}([X^h, Y^h], Z^h) + \mathbf{F}([Z^h, X^h], Y^h) + \mathbf{F}([Y^h, Z^h], X^h) - \mathbf{F}([X, Y]^h, Z^h) - \mathbf{F}([Z, X]^h, Y^h) - \mathbf{F}([Y, Z]^h, X^h) \right) \Big|_q, \end{aligned}$$

where X^h, Y^h, Z^h are the horizontal lifts of the vector fields X, Y, Z on S , $q \in \pi^{-1}(x)$ is arbitrary. Thus, $d\mathbf{f} = 0$ if and only if (3.5) is satisfied. Consider the adapted coordinates $q = (q_1, \dots, q_{n+k})$ on Q described in Sect. 2.2 and take

$$X = \frac{\partial}{\partial q_i}, \quad Y = \frac{\partial}{\partial q_j}, \quad Z = \frac{\partial}{\partial q_p}, \quad 1 \leq i, j, p \leq n.$$

Then the equation $d\mathbf{f}(X, Y, Z) = 0$ takes the form (3.6). □

Remark 3.4 In the special case, when $\mathbf{F} = \pi^*\mathbf{w}$, where \mathbf{w} is a two-form on the base manifold S , Eq. (3.6) are automatically satisfied ($F_{i,n+v} = 0, 1 \leq i \leq n, 1 \leq v \leq k$). In this special case $\mathbf{f} = \mathbf{w}$, and $d\mathbf{F} = 0$ if and only if $d\mathbf{f} = 0$.

4 Almost Hamiltonian Description and an Invariant Measure

4.1 Almost Symplectic Manifolds

Recall that an *almost symplectic structure* is a pair (M, ω) of a manifold M and a nondegenerate 2-form ω [see Libermann and Marle (1987)]. Here we do not assume

that the form ω is closed, in contrast to the symplectic case. As in the symplectic case, since ω is nondegenerate, to a given function H one can associate the *almost Hamiltonian vector field* X_H by the identity

$$i_{X_H}\omega(\cdot) = \omega(X_H, \cdot) = -dH(\cdot).$$

The almost symplectic structure (M, ω) is *locally conformally symplectic*, if in a neighborhood of each point x on M , there exists a function f different from zero such that $f\omega$ is closed. If f is defined globally, then (M, ω) is *conformally symplectic* (Libermann and Marle 1987).

4.2 Reduced Flows on Cotangent Bundles

Let (x_1, \dots, x_n) be local coordinates on S in which the metric \mathbf{g} is given by the quadratic form $\sum_{ij} g_{ij} dx_i \otimes dx_j$ and the components of the (1,2)-tensor \mathbf{C} are C_{ij}^k (see Remark 3.3). Then the Lagrangian, the gyroscopic two-form and the **JK**-term read as follows

$$l(x, \dot{x}) = \frac{1}{2} \sum g_{ij} \dot{x}_i \dot{x}_j - v(x), \quad \mathbf{f} = \sum_{i < j} f_{ij} dx_i \wedge dx_j,$$

$$\mathbf{JK}(X, Y)|_{(x, \dot{x})} = \mathbf{g}(\dot{x}, \mathbf{C}(X, Y)) = \sum_{k,l,i,j} g_{kl} C_{ij}^k X_i Y_j \dot{x}_l.$$

We also introduce the Hamiltonian function

$$h(x, p) = \frac{1}{2}(p, \mathbf{g}^{-1}(p)) + v(x) = \frac{1}{2} \sum g^{ij} p_i p_j + v(x),$$

as the usual Legendre transformation of l . Here $(p_1, \dots, p_n, x_1, \dots, x_n)$ are the canonical coordinates of the cotangent bundle T^*S ,

$$p_i = \partial l / \partial \dot{x}_i = \sum_j g_{ij} \dot{x}_j,$$

and $\{g^{ij}\}$ is the inverse of the metric matrix $\{g_{ij}\}$. For simplicity, the same symbol denotes a function on the base manifold $f : S \rightarrow \mathbb{R}$ and its lift to the cotangent bundle $\rho^* f = f \circ \rho : T^*S \rightarrow \mathbb{R}$, where $\rho : T^*S \rightarrow S$ is the canonical projection.

In canonical coordinates Eq. (3.1) takes the form

$$\dot{x}_i = \frac{\partial h}{\partial p_i} = \sum_{j=1}^n g^{ij} p_j, \tag{4.1}$$

$$\dot{p}_i = -\frac{\partial h}{\partial x_i} + \Pi_i(x, p) + \sum_{j=1}^n f_{ij}(x) \frac{\partial h}{\partial p_j}. \tag{4.2}$$

Here, the **JK**-term is given in the form

$$\begin{aligned} \Pi_i(x, p) &= \mathbf{JK} \left(\frac{\partial}{\partial x_i}, \dot{x} \right) \Big|_{\dot{x}=\mathbf{g}^{-1}(p)} = \sum_{k,l,j=1}^n g_{kl} \dot{x}_l C_{ij}^k(x) \dot{x}_j \Big|_{\dot{x}=\mathbf{g}^{-1}(p)} \\ &= \sum_{j,k=1}^n C_{ij}^k(x) p_k \frac{\partial h}{\partial p_j}. \end{aligned} \tag{4.3}$$

Let $z = (x, p)$. The reduced Eqs. (4.1), (4.2) on the cotangent bundle T^*S can be written in the almost Hamiltonian form

$$\dot{z} = X_{red}, \quad i_{X_{red}}(\Omega + \sigma + \rho^*\mathbf{f}) = -dh, \tag{4.4}$$

where Ω is the canonical symplectic form on T^*S , σ is a semi-basic form defined by the **JK** term [see Cantrijn et al. (2002), Stanchenko (1989)]:

$$\Omega = dp_1 \wedge dx_1 + \dots + dp_n \wedge dx_n, \tag{4.5}$$

$$\sigma = \sum_{1 \leq i < j \leq n} \sum_{k=1}^n C_{ij}^k(x) p_k dx_i \wedge dx_j. \tag{4.6}$$

4.3 Invariant Measure

The existence of an invariant measure for nonholomic problems is well studied [see Fedorov (1988), Veselov and Veselova (1988), Kozlov (1988), Fedorov and Kozlov (1995), Zenkov and Bloch (2003), Fasso et al. (2019), Jovanović (2015), Fedorov et al. (2015)]. We will consider smooth measures of the form $\mu = \nu \Omega^n$, where Ω^n [see (4.5)] is the standard measure on the cotangent bundle T^*S and ν is a nonvanishing smooth function, called *the density* of the measure μ .

In absence of potential and gyroscopic forces, it was proved in Cantrijn et al. (2002) that Eqs. (4.1), (4.2) have an invariant measure if and only if its density is basic, i.e., $\nu = \nu(x)$. Then the system with a potential force $v(x)$ also preserves the same measure [see Stanchenko (1989), Cantrijn et al. (2002)].

For $\mathbf{f} = 0$, the existence of the basic density $\nu = \nu(x)$ is equivalent to the condition that the one-form

$$\Theta = \sum_{i,j} C_{ij}^j(x) dx_i, \quad \text{i.e.,} \quad \Theta(X)|_x = \text{tr } \mathbf{C}(X, \cdot)|_x, \quad X \in T_x S, \tag{4.7}$$

is exact: there exists a function λ such that $\Theta = d\lambda$. Then the function $\nu(x) = \exp(\lambda(x))$ is the density of an invariant measure [see Cantrijn et al. (2002), Garcia-Naranjo and Marrero (2020)]. The statement formulated in terms of the tensor field **B** is given in Cantrijn et al. (2002), while in Garcia-Naranjo and Marrero (2020) it is formulated in terms of the gyroscopic tensor $\mathcal{T} = -\mathbf{C}$. An example of a system with a

potential force and with an invariant nonbasic measure is also given in Garcia-Naranjo and Marrero (2020).

In the presence of the gyroscopic form \mathbf{f} we have a similar situation.

Theorem 4.1 *The reduced gyroscopic Chaplygin equations (4.1), (4.2) have an invariant measure $\mu = v \Omega^n$ with a basic density $v(x)$ if and only if the one-form (4.7) is exact $\Theta = d\lambda$. Then the function $v = \exp(\lambda(x))$ is the density of the invariant measure.*

In other words, according to Cantrijn et al. (2002), Garcia-Naranjo and Marrero (2020), a Chaplygin system with a gyroscopic term possesses a basic invariant measure if and only if the same Chaplygin system without gyroscopic term preserves the same basic invariant measure.

Proof The Lie derivative $\mathcal{L}_{X_{red}}(\mu)$ vanishes if and only if the divergence of the vector field vX_{red} with respect to the canonical measure equals to zero. By using the identities

$$\frac{\partial}{\partial p_i} \frac{\partial h}{\partial p_j} = g^{ji}, \quad \sum_{i,j} f_{ij} g^{ji} = 0, \quad \sum_{ij} C_{ij}^k g^{ji} = 0,$$

we get:

$$\begin{aligned} \operatorname{div}(vX_{red}) &= \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(v \frac{\partial h}{\partial p_i} \right) \\ &+ \sum_i \frac{\partial}{\partial p_i} \left(v \left(-\frac{\partial h}{\partial x_i} + \sum_{k,j=1}^n C_{ij}^k(x) p_k \frac{\partial h}{\partial p_j} + \sum_{j=1}^n f_{ij}(x) \frac{\partial h}{\partial p_j} \right) \right) \\ &= \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i} - v \sum_{j=1}^n C_{ij}^j(x) \right) \frac{\partial h}{\partial p_i}. \end{aligned}$$

Since $\dot{x}_i = \frac{\partial h}{\partial p_i}$ is arbitrary for each fixed x , the vector field X_{red} preserves the measure $v \Omega^n$ if and only if

$$v^{-1} \frac{\partial v}{\partial x_i} = \sum_{j=1}^n C_{ij}^j(x), \quad i = 1, \dots, n,$$

that is, if and only if

$$d \ln v = \sum_{i=1}^n v^{-1} \frac{\partial v}{\partial x_i} dx_i = \sum_{i,j=1}^n C_{ij}^j(x) dx_i = \Theta.$$

Note that, although the proof is derived in local coordinates, all considered objects are global and the identity $d \ln v = \Theta$ holds globally. □

5 Chaplygin Hamiltonization for Systems with Magnetic Forces

5.1 Chaplygin Multipliers in the Lagrangian Framework

We consider the reduced Chaplygin systems (3.1) and study the question of their transformation into a Lagrangian system after a time reparametrization.

Let us consider a time substitution $d\tau = \mathcal{N}(x)dt$, where $\mathcal{N}(x)$ is a differentiable nonvanishing function on S . Denote $x' = dx/d\tau = \mathcal{N}^{-1}\dot{x}$.

We first treat the exact case: $\mathbf{f} = d\mathbf{a}$ (see Remark 3.2). Locally, the one-form \mathbf{a} is given by $\mathbf{a} = \sum_i a_i(x)dx_i$ and

$$l_1(x, \dot{x}) = \frac{1}{2} \sum g_{ij} \dot{x}_i \dot{x}_j + \sum_i a_i \dot{x}_i - v(x).$$

The Lagrangians l and l_1 in the coordinates (x, x') are denoted l^* and l_1^* respectively and take the form

$$l^*(x, x') = \frac{1}{2} \sum \mathcal{N}^2 g_{ij} x'_i x'_j - v(x), \tag{5.1}$$

$$l_1^*(x, x') = \frac{1}{2} \sum \mathcal{N}^2 g_{ij} x'_i x'_j + \sum_i \mathcal{N} a_i(x) x'_i - v(x). \tag{5.2}$$

Following Chaplygin (1911), we are looking for a nowhere vanishing function $\mathcal{N}(x)$, called a *Chaplygin reducing multiplier* such that the reduced Chaplygin system (3.4)

$$\frac{d}{dt} \frac{\partial l_1}{\partial \dot{x}_i} = \frac{\partial l_1}{\partial x_i} + \sum_{k,l,j=1}^n C_{ij}^k(x) g_{kl} \dot{x}_l \dot{x}_j \tag{5.3}$$

after a time reparametrization $d\tau = \mathcal{N}(x)dt$ becomes the Lagrangian system

$$\frac{d}{d\tau} \frac{\partial l_1^*}{\partial x'_i} = \frac{\partial l_1^*}{\partial x_i}, \quad i = 1, \dots, n. \tag{5.4}$$

Equivalently, we can use the Lagrangians l and l^* . Let

$$\begin{aligned} \mathbf{f} = d\mathbf{a} &= \sum_{i < j} f_{ij} dx_i \wedge dx_j, \quad f_{ij} = \frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j}, \\ \mathbf{f}^* = d(\mathcal{N}\mathbf{a}) &= \sum_{i < j} f_{ij}^* dx_i \wedge dx_j, \quad f_{ij}^* = \mathcal{N} f_{ij} + a_j \frac{\partial \mathcal{N}}{\partial x_i} - a_i \frac{\partial \mathcal{N}}{\partial x_j}. \end{aligned}$$

Then, we are looking for a nowhere vanishing function $\mathcal{N}(x)$, such that the reduced Chaplygin system

$$\frac{d}{dt} \frac{\partial l}{\partial \dot{x}_i} = \frac{\partial l}{\partial x_i} + \sum_{k,l,j=1}^n C_{ij}^k(x) g_{kl} \dot{x}_l \dot{x}_j + \sum_{j=1}^n f_{ij}(x) \dot{x}_j \quad (5.5)$$

after a time reparametrization $d\tau = \mathcal{N}(x)dt$ becomes the Lagrangian system with magnetic forces

$$\frac{d}{d\tau} \frac{\partial l^*}{\partial \dot{x}'_i} = \frac{\partial l^*}{\partial x_i} + \sum_{j=1}^n f_{ij}^* \dot{x}'_j, \quad i = 1, \dots, n. \quad (5.6)$$

Proposition 5.1 *Suppose that \mathbf{f} is exact: $\mathbf{f} = d\mathbf{a}$. The reduced equations of the Chaplygin system with a linear term in velocities (5.3) after a time reparametrization $d\tau = \mathcal{N}(x)dt$ becomes the Lagrangian system (5.4) if and only if the corresponding system without the linear term allows the Chaplygin multiplier $\mathcal{N}(x)$ and $d\mathcal{N} \wedge \mathbf{a} = 0$, that is, if*

$$a_j \frac{\partial \mathcal{N}}{\partial x_i} = a_i \frac{\partial \mathcal{N}}{\partial x_j}. \quad (5.7)$$

Note that conditions (5.7) imply that

$$\mathbf{f}^* = d(\mathcal{N}\mathbf{a}) = \mathcal{N}d\mathbf{a} + d\mathcal{N} \wedge \mathbf{a} = \mathcal{N}\mathbf{f}$$

and

$$d(\mathcal{N}\mathbf{f}) = d\mathcal{N} \wedge \mathbf{f} = 0. \quad (5.8)$$

Let us now turn to the nonexact case. Thus, we assume now \mathbf{f} is not exact. In this case we set

$$\mathbf{f}^* = \mathcal{N}\mathbf{f}. \quad (5.9)$$

Proposition 5.2 *Suppose that \mathbf{f} is not exact. The equations of motion of the reduced gyroscopic Chaplygin system (5.5) after a time reparametrization $d\tau = \mathcal{N}dt$ become the Lagrangian equations with gyroscopic forces (5.6), where f^* is given by (5.9) if and only if the corresponding system without gyroscopic forces allows the Chaplygin multiplier $\mathcal{N}(x)$.*

Propositions 5.1 and 5.2 follow from the derivation given below for the Hamiltonian setting as indicated in Remark 5.1.

Note that the gyroscopic system (5.6) is magnetic if the form (5.9) is closed. In particular, if \mathbf{f} is closed, but not exact, then the Lagrangian system (5.6) is magnetic only if the condition (5.8) holds. The condition (5.8) is always satisfied when $n = 2$. This is a rather strong condition for $n \geq 3$. When $n = 3$, condition (5.8) reduces to

the partial differential equation

$$f_{23} \frac{\partial \mathcal{N}}{\partial x_1} + f_{31} \frac{\partial \mathcal{N}}{\partial x_2} + f_{12} \frac{\partial \mathcal{N}}{\partial x_3} = 0.$$

Finally, it is important to note that even if we consider the exact case $\mathbf{f} = d\mathbf{a}$ and the Lagrangians that are linear in velocities, instead of Eqs. (5.3) and (5.4) and the gyroscopic form defined by $\mathbf{f}^* = d(\mathcal{N}\mathbf{a})$ it is more natural to consider Eqs. (5.5) and (5.6) with \mathbf{f}^* defined as $\mathbf{f}^* = \mathcal{N}\mathbf{f} = \mathcal{N}d\mathbf{a}$. In the latter case, for $n = 2$, the form \mathbf{f}^* is magnetic regardless of (5.7).

5.2 Conformally Symplectic Structures

The existence of an invariant measure is closely related to the Hamiltonization problem for magnetic G -Chaplygin systems. We first consider G -Chaplygin systems without the gyroscopic term, see Cantrijn et al. (2002), Stanchenko (1989) and Ehlers et al. (2005). For $\mathbf{f} \equiv 0$, the reduced system (4.4) takes the form

$$\dot{z} = X_{red}^0, \quad i_{X_{red}^0}(\Omega + \sigma) = -dh. \tag{5.10}$$

Suppose that the form $\Omega + \sigma$ is conformally symplectic, i.e., there exists a nonvanishing function \mathcal{N} , such that $d(\mathcal{N}(\Omega + \sigma)) = 0$. Since $d\Omega = 0$, the last relation can be rewritten as:

$$d\mathcal{N} \wedge \Omega + d\mathcal{N} \wedge \sigma + \mathcal{N}d\sigma = 0. \tag{5.11}$$

After the time rescaling $d\tau = \mathcal{N}dt$, Eq. (5.10) reads

$$z' = \mathcal{N}^{-1}\dot{z} = \mathcal{N}^{-1}X_{red}^0 =: \tilde{X}_{red}^0.$$

The last relation introduces the rescaled vector field \tilde{X}_{red}^0 , which is Hamiltonian:

$$i_{\tilde{X}_{red}^0} \mathcal{N}(\Omega + \sigma) = -dh.$$

Therefore, the system in the new time becomes the Hamiltonian system with respect to the symplectic form $\mathcal{N}(\Omega + \sigma)$. Then, according to the Liouville theorem (Arnold 1974), the Hamiltonian vector field \tilde{X}_{red}^0 preserves the standard measure $\mathcal{N}^n(\Omega + \sigma)^n = \mathcal{N}^n \Omega^n$,

$$\mathcal{L}_{\tilde{X}_{red}^0}(\mathcal{N}^n \Omega^n) = d(i_{\tilde{X}_{red}^0}(\mathcal{N}^n \Omega^n)) = 0.$$

Thus, for the almost Hamiltonian vector field $X_{red}^0 = \mathcal{N}\tilde{X}_{red}^0$ we have

$$\mathcal{L}_{X_{red}^0}(\mathcal{N}^{n-1} \Omega^n) = d(i_{X_{red}^0}(\mathcal{N}^{n-1} \Omega^n)) = d(i_{\tilde{X}_{red}^0}(\mathcal{N}^n \Omega^n)) = 0,$$

and the flow of X_{red}^0 preserves the measure $\mathcal{N}^{n-1}\Omega^n$.

Now, we consider G -Chaplygin systems with a gyroscopic term.

Proposition 5.3 *The function $\mathcal{N} = \mathcal{N}(x)$ is a conformal factor for the almost symplectic form $\Omega + \sigma + \rho^*\mathbf{f}$ if and only if it is a conformal factor for the almost symplectic form $\Omega + \sigma$ and the form $\mathbf{f}^* = \mathcal{N}\mathbf{f}$ is magnetic.*

Proof The form $\Omega + \sigma + \rho^*\mathbf{f}$ is conformally symplectic with a conformal factor \mathcal{N} if and only if

$$d\mathcal{N} \wedge \Omega + d\mathcal{N} \wedge \sigma + \mathcal{N}d\sigma + d\mathcal{N} \wedge \rho^*\mathbf{f} + \mathcal{N}\rho^*d\mathbf{f} = 0. \tag{5.12}$$

Assume that $\mathcal{N} = \mathcal{N}(x)$ is basic. Since only two last terms are basic, equation (5.12) is satisfied if and only if $\mathcal{N}(x)$ satisfies (5.11) and $\mathbf{f}^* = \mathcal{N}\mathbf{f}$ is closed. \square

Consider the reduced gyroscopic Chaplygin system (4.4). If $\mathcal{N} = \mathcal{N}(x)$ is a conformal factor for $\Omega + \sigma + \rho^*\mathbf{f}$, as above we have that the rescaled vector field $\tilde{X}_{red} = \mathcal{N}^{-1}X_{red}$ is Hamiltonian and preserves the measure $\mathcal{N}^n(\Omega + \sigma + \rho^*\mathbf{f})^n = \mathcal{N}^n\Omega^n$. Thus, the reduced gyroscopic Chaplygin system $\dot{z} = X_{red}$ preserves the same measure as in the case of the absence of gyroscopic forces. This is in accordance with Theorem 4.1.

The existence of a basic conformal factor, as we will see in Sect. 5.3, is equivalent to the condition that \mathcal{N} is the classical Chaplygin multiplier in the Lagrangian framework described above.

5.3 Chaplygin Multipliers: From the Lagrangian to the Hamiltonian Framework

In the study of nonholonomic rigid body systems in \mathbb{R}^n [see Fedorov and Jovanović (2004), Jovanović (2010), Jovanović (2018), Jovanović (2019)], the Chaplygin time reparametrization of Lagrangian systems was transported into the Hamiltonian framework via the Legendre transformation. Similarly, consider the time substitution $d\tau = \mathcal{N}(x)dt$ and the Lagrangian function $l^*(x, x')$ given in (5.1). Then the conjugate momenta are

$$\tilde{p}_i = \partial l^* / \partial x'_i = \mathcal{N}^2 \sum_j g_{ij} x'_j,$$

and the corresponding Hamiltonian is

$$h^*(x, \tilde{p}) = \frac{1}{2} \sum \frac{1}{\mathcal{N}^2} g^{ij} \tilde{p}_i \tilde{p}_j + v(x).$$

The following diagram commutes:

$$\begin{array}{ccc}
 TS\{x, \dot{x}\} & \xrightarrow{x'=\mathcal{N}^{-1}\dot{x}} & TS\{x, x'\} \\
 p=\mathbf{g}(\dot{x}) \downarrow & & \downarrow \tilde{p}=\mathcal{N}^2\mathbf{g}(x') \\
 T^*S\{x, p\} & \xrightarrow{\tilde{p}=\mathcal{N}p} & T^*S\{x, \tilde{p}\}.
 \end{array} \tag{5.13}$$

Let $\tilde{\Omega}$ be the canonical symplectic form on T^*S with respect to the coordinates (x, \tilde{p}) . Then

$$\tilde{\Omega} = \sum_i d\tilde{p}_i \wedge dx_i = \mathcal{N}\Omega + d\mathcal{N} \wedge \theta, \quad \theta = p_1dx_1 + \dots + p_ndx_n, \quad \Omega = d\theta. \tag{5.14}$$

Thus, h and h^* represent the same Hamiltonian function on T^*S written in two coordinate systems. These coordinate systems are related by the noncanonical change of variables

$$(x, p) \longmapsto (x, \tilde{p}) = (x, \mathcal{N}p). \tag{5.15}$$

Assume that the two-form $\mathbf{f}^* = \mathcal{N}\mathbf{f}$ is closed on S .

By using the commutative diagram (5.13), we get that the function \mathcal{N} is a Chaplygin reducing multiplier for the reduced gyroscopic Chaplygin system (5.5) (see Sect. 5.1) if and only if the almost Hamiltonian equations (4.1), (4.2), after the time reparametrization $d\tau = \mathcal{N}(x)dt$ and the coordinate transformation (5.15) become the Hamiltonian equations

$$x'_i = \frac{\partial h^*}{\partial \tilde{p}_i}(x, \tilde{p}), \quad \tilde{p}'_i = -\frac{\partial h^*}{\partial x_i}(x, \tilde{p}) + \mathcal{N} \sum_j f_{ij}(x) \frac{\partial h^*}{\partial \tilde{p}_j}(x, \tilde{p}) \tag{5.16}$$

with respect to the twisted symplectic form

$$\tilde{\Omega} + \rho^*\mathbf{f}^* = \sum_i d\tilde{p}_i \wedge dx_i + \mathcal{N} \sum_{i<j} f_{ij}dx_i \wedge dx_j. \tag{5.17}$$

Let \mathcal{N} be a nonvanishing function and consider the time reparametrization $d\tau = \mathcal{N}(x)dt$. Eq. (5.16) in the original time t after the coordinate transformation (5.15) takes the form

$$\dot{x}_i = \mathcal{N} \frac{\partial h^*}{\partial \tilde{p}_i}(x, \tilde{p}) = \mathcal{N} \mathcal{N}^{-2} \sum_j g^{ij} \tilde{p}_j = \sum_j g^{ij} p_j, \tag{5.18}$$

$$\begin{aligned} \dot{\tilde{p}}_i &= -\mathcal{N} \frac{\partial h^*}{\partial x_i}(x, \tilde{p}) + \mathcal{N}^2 \sum_j f_{ij}(x) \frac{\partial h^*}{\partial \tilde{p}_j}(x, \tilde{p}) \tag{5.19} \\ &= -\mathcal{N} \left(\frac{1}{2\mathcal{N}^2} \sum_{j,k} \frac{\partial g^{jk}}{\partial x_i} \tilde{p}_j \tilde{p}_k - \frac{1}{\mathcal{N}^3} \frac{\partial \mathcal{N}}{\partial x_i} \sum_{j,k} g^{jk} \tilde{p}_j \tilde{p}_k + \frac{\partial v}{\partial x_i} - \sum_{j,k} f_{ij} g^{jk} p_k \right) \\ &= -\mathcal{N} \left(\frac{1}{2} \sum_{j,k} \frac{\partial g^{jk}}{\partial x_i} p_j p_k - \frac{1}{\mathcal{N}} \frac{\partial \mathcal{N}}{\partial x_i} \sum_{j,k} g^{jk} p_j p_k + \frac{\partial v}{\partial x_i} - \sum_{j,k} f_{ij} g^{jk} p_k \right) \end{aligned}$$

Eqs. (4.1) and (5.18) coincide. From $\tilde{p}_i = \mathcal{N} p_i$, we get $\dot{\tilde{p}}_i = \mathcal{N} \dot{p}_i + \dot{\mathcal{N}} p_i$, that is $\dot{p}_i = \mathcal{N}^{-1}(\dot{\tilde{p}}_i - \dot{\mathcal{N}} p_i)$. Therefore, using Eq. (5.19), we obtain

$$\begin{aligned} \dot{p}_i &= -\frac{1}{2} \sum_{j,k} \frac{\partial g^{jk}}{\partial x_i} p_j p_k + \frac{1}{\mathcal{N}} \frac{\partial \mathcal{N}}{\partial x_i} \sum_{j,k} g^{jk} p_j p_k \tag{5.20} \\ &\quad - \frac{1}{\mathcal{N}} \sum_j \frac{\partial \mathcal{N}}{\partial x_j} \dot{x}_j p_i - \frac{\partial v}{\partial x_i} + \sum_{j,k} f_{ij} g^{jk} p_k \\ &= -\frac{\partial h}{\partial x_i}(x, p) + \frac{1}{\mathcal{N}} \frac{\partial \mathcal{N}}{\partial x_i} \sum_{j,k} g^{jk} p_j p_k - \frac{1}{\mathcal{N}} \sum_{j,k} \frac{\partial \mathcal{N}}{\partial x_j} g^{jk} p_k p_i + \sum_{j,k} f_{ij} g^{jk} p_k \\ &= -\frac{\partial h}{\partial x_i}(x, \tilde{p}) + \sum_{j,k,l=1}^n \mathcal{N}^{-1} \left(\delta_j^k \frac{\partial \mathcal{N}}{\partial x_i} - \delta_i^k \frac{\partial \mathcal{N}}{\partial x_j} \right) g^{jl} p_k p_l + \sum_{j=1}^n f_{ij}(x) \frac{\partial h}{\partial p_j}, \end{aligned}$$

Equations (4.2), (4.3), and (5.20) imply that the reduced gyroscopic Chaplygin system (4.1), (4.2) after the time reparametrization $d\tau = \mathcal{N}(x)dt$ and the change of variables (5.15) takes the twisted canonical form (5.16) if and only if we have the equality of the quadratic forms in momenta:

$$\sum_{j,k,l=1}^n C_{ij}^k(x) g^{jl} p_k p_l = \sum_{j,k,l=1}^n \mathcal{N}^{-1} \left(\delta_j^k \frac{\partial \mathcal{N}}{\partial x_i} - \delta_i^k \frac{\partial \mathcal{N}}{\partial x_j} \right) g^{jl} p_k p_l, \quad i = 1, \dots, n. \tag{5.21}$$

In the invariant form, (5.21) can be written as the condition on **JK** force term (4.3):

$$\Pi(x, p) = \mathcal{N}^{-1}(p, \mathbf{g}^{-1}(p))d\mathcal{N} - \mathcal{N}^{-1}(d\mathcal{N}, \mathbf{g}^{-1}(p))p, \tag{5.22}$$

Remark 5.1 Note that Eqs. (5.16)–(5.20) are valid without assumption that the form $\mathbf{f}^* = \mathcal{N}\mathbf{f}$ is closed, i.e., when $\tilde{\Omega} + \rho^*\mathbf{f}^*$ [see (5.17)] is an almost symplectic form as well. In this way, according to the commutative diagram (5.13), they imply Propositions 5.1 and 5.2.

It is clear that the sufficient conditions for the identities (5.21) are:

$$C_{ij}^k(x) = \mathcal{N}^{-1} \left(\delta_j^k \frac{\partial \mathcal{N}}{\partial x_i} - \delta_i^k \frac{\partial \mathcal{N}}{\partial x_j} \right), \quad i, j, k = 1, \dots, n. \tag{5.23}$$

Thus, if the (1, 2)-tensor field \mathbf{C} defined in Remark 3.3 satisfies (5.23), \mathcal{N} is a Chaplygin reducing multiplier for the reduced gyroscopic G -Chaplygin system (4.1), (4.2), e.g., (5.5). Then the (1, 2)-tensor \mathbf{C} and the two-form σ in the invariant form can be written as

$$\mathbf{C}(X, Y) = \mathcal{N}^{-1} X(\mathcal{N})Y - \mathcal{N}^{-1} Y(\mathcal{N})X, \tag{5.24}$$

$$\sigma = \mathcal{N}^{-1} d\mathcal{N} \wedge \theta. \tag{5.25}$$

Moreover, from (5.14), (5.17), and (5.25), we obtain that the form $\Omega + \sigma + \rho^*\mathbf{f}$ is conformally symplectic with \mathcal{N} a conformal factor being a Chaplygin reducing multiplier:

$$\tilde{\Omega} + \rho^*\mathbf{f}^* = \mathcal{N}(\Omega + \sigma + \rho^*\mathbf{f}).$$

In the terminology of Garcia-Naranjo (2019a, b), Eqs. (5.23) and (5.24) mean that the gyroscopic tensor $\mathcal{T} = -\mathbf{C}$ is ϕ -simple, where $\phi = \ln \mathcal{N}$. Following Garcia-Naranjo, we say that a (1, 2)-tensor \mathbf{C} is $\ln \mathcal{N}$ -simple if (5.24) holds.

Garcia-Naranjo and Marrero (2020) the following inverse statement is proved: if a two-form $\Omega + \sigma$ is conformally symplectic with a basic conformal factor $\mathcal{N}(x)$, then the gyroscopic tensor \mathcal{T} is $\ln \mathcal{N}$ -simple. Now, based on the above considerations, we can reformulate and extend Theorem 3.21 from Garcia-Naranjo and Marrero (2020) on ϕ -simple Chaplygin systems as follows:

Theorem 5.1 (i) *Assume that two-form $\mathbf{f}^* = \mathcal{N}\mathbf{f}$ is closed on S . The conditions (a)–(c) listed below are equivalent. The conditions (d) and (e) are equivalent, while (e) implies (c):*

- (a) *the reduced gyroscopic Chaplygin system (5.5) after the time reparametrization $d\tau = \mathcal{N}(x)dt$ takes the form of the magnetic Lagrangian system (5.6);*
- (b) *the reduced gyroscopic Chaplygin system (4.1), (4.2) after the time reparametrization $d\tau = \mathcal{N}(x)dt$ and the change of variables (5.15) takes the twisted canonical form (5.16);*
- (c) *the \mathbf{JK} force term (4.3) on T^*S has the form (5.22);*
- (d) *the almost symplectic form $\Omega + \sigma + \rho^*\mathbf{f}$ is conformally symplectic with the base conformal factor $\mathcal{N}(x)$ and σ is given by (5.25);*
- (e) *the (1, 2)-tensor \mathbf{C} is $\ln \mathcal{N}$ -simple, that is, it is given by (5.24).*

(ii) *If $\mathcal{N}(x)$ is a Chaplygin multiplier, then the reduced equations of motion (4.1), (4.2) possess the base invariant measure*

$$\mathcal{N}^{n-1} \Omega^n. \tag{5.26}$$

(iii) If $n = 2$, then the statement (ii) can be inverted: if the reduced equations of motion (4.1), (4.2) possess the base invariant measure

$$\mathcal{N}(x)dp_1 \wedge dp_2 \wedge dx_1 \wedge dx_2,$$

then, after the time reparametrization $d\tau = \mathcal{N}(x)dt$ the reduced equations become the usual Hamiltonian equations on T^*S with respect to the twisted symplectic form (5.17). For $n = 2$, all items (a)–(e) are equivalent.

Theorem 5.1 relates the classical Chaplygin Hamiltonization [items (a)–(c) see Chaplygin (1911), Fedorov and Jovanović (2004)] and the Chaplygin Hamiltonization within the framework of almost symplectic forms and the gyroscopic tensor field \mathbf{C} [items (d) and (e), see Cantrijn et al. (2002), Garcia-Naranjo and Marrero (2020)].

For the Veselova problem on $SO(n)$ [see Fedorov and Jovanović (2004)], it is proved in Garcia-Naranjo and Marrero (2020) that (c) implies (d) as well. A similar statement can be proved for the nonholonomic problem of a ball rolling over a sphere considered in Jovanović (2018).

Remark 5.2 Note that (5.21) implies that the symmetric parts of the tensors $\sum_{j=1}^n C_{ij}^k(x)g^{jl}$ and $\sum_{j=1}^n \mathcal{N}^{-1} \left(\delta_j^k \frac{\partial \mathcal{N}}{\partial x_i} - \delta_i^k \frac{\partial \mathcal{N}}{\partial x_j} \right) g^{jl}$ are equal, but the conditions (5.21) and (5.23), i.e., the items (c) and (e) of Theorem 5.1 do not need to be equivalent. For example, one can have \mathbf{C} and σ different from zero, but with $i_{X_{red}}\sigma = 0$. Then $\Pi = 0$ and X_{red} is a Hamiltonian vector field with respect to the magnetic symplectic form $\Omega + \rho^*\mathbf{f}$. Thus, the constant $\mathcal{N} = 1$ can be chosen as a Chaplygin multiplier. As a result, the right-hand side of (5.23) is zero, while the left-hand side of (5.23) is different from zero.

Further, from Theorem 5.1 it follows that if a Chaplygin system without gyroscopic force allows Hamiltonization with a basic multiplier \mathcal{N} , and if $\mathcal{N}\mathbf{f}$ is closed, then the system with reduced gyroscopic force \mathbf{f} also allows Hamiltonization and vice versa: if a Chaplygin system with gyroscopic force \mathbf{f} allows Hamiltonization with a basic multiplier \mathcal{N} (either in the sense that \mathcal{N} is a conformal factor for the almost symplectic form $\Omega + \sigma + \rho^*\mathbf{f}$ and according to Proposition 5.3 $\mathcal{N}\mathbf{f}$ is closed, or in the sense of the classical Hamiltonization where $\mathcal{N}\mathbf{f}$ is also closed) then the system without the gyroscopic force \mathbf{f} allows Hamiltonization as well.

For $n = 2$, Eq. (5.5) are

$$\frac{d}{dt} \frac{\partial l}{\partial \dot{x}_1} = \frac{\partial l}{\partial x_1} + S(x)\dot{x}_2, \tag{5.27}$$

$$\frac{d}{dt} \frac{\partial l}{\partial \dot{x}_2} = \frac{\partial l}{\partial x_2} - S(x)\dot{x}_1, \quad S(x) = \sum_{k,l=1}^2 C_{12}^k(x)g_{kl}\dot{x}_l + f_{12}(x). \tag{5.28}$$

Item (iii) of Theorem 5.1 is given in Borisov et al. (2005) and Bolsinov et al. (2015), where the Lagrangian systems of the form (5.27), (5.28), for $f_{12}(x) \neq 0$ are called *generalized Chaplygin systems*.

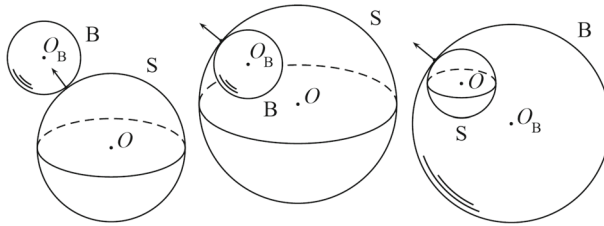


Fig. 1 Rolling of the ball \mathbf{B} with center $O_{\mathbf{B}}$ over the sphere \mathbf{S} with center O : three scenarios

6 Chaplygin Ball with a Gyroscope Rolling Over a Plane and Over a Sphere

6.1 Chaplygin Ball with a Gyroscope Rolling without Slipping

One of the most famous solvable problems in nonholonomic mechanics describes rolling without slipping of a balanced, dynamically nonsymmetric ball over a horizontal plane (Chaplygin 1903). After Chaplygin (1903), a balanced, dynamically nonsymmetric ball is called the *Chaplygin ball*, see Kozlov (2002), Arnold et al. (1989), Borisov et al. (2005), Borisov and Mamaev (2008), Borisov and Mamaev (2001), Balseiro and Garcia-Naranjo (2012), Borisov et al. (2014) and Bolsinov et al. (2015).

Let $O_{\mathbf{B}}, a, m, \mathbb{I} = \text{diag}(A, B, C)$, be the center, radius, mass and the inertia operator of a ball \mathbf{B} . There are three possible configurations in the problem of rolling without slipping of the Chaplygin ball \mathbf{B} over a fixed sphere \mathbf{S} of the radius b :

- (i) rolling of \mathbf{B} over the outer surface of \mathbf{S} and \mathbf{S} is outside \mathbf{B} (see the leftmost part of Fig. 1);
- (ii) rolling of \mathbf{B} over the inner surface of \mathbf{S} ($b > a$) (see the central part of Fig. 1);
- (iii) rolling of \mathbf{B} over the outer surface of \mathbf{S} and \mathbf{S} is within \mathbf{B} ; in this case $b < a$ and the rolling ball \mathbf{B} is a spherical shell (see the rightmost part of Fig. 1).

Let $\varepsilon = b/(b \pm a)$, where we take “+” for the case (i) and “−” in the cases (ii) and (iii) and let $D = ma^2$. The equations of motion in the frame attached to the ball can be written in the form

$$\dot{\mathbf{k}} = \mathbf{k} \times \vec{\omega}, \quad \dot{\vec{\gamma}} = \varepsilon \vec{\gamma} \times \vec{\omega}, \tag{6.1}$$

where ω is the angular velocity of the ball, $\mathbf{k} = \mathbb{I}\vec{\omega} + D\vec{\omega} - D\langle \vec{\omega}, \vec{\gamma} \rangle \vec{\gamma}$ is the angular momentum of the ball with respect to the point of contact, and $\vec{\gamma}$ is the unit normal to the sphere \mathbf{S} at the contact point.

When b tends to infinity, then ε tends to 1 and $\vec{\gamma}$ tends to the unit vector that is constant in the fixed reference frame. This way, for $\varepsilon = 1$, we obtain the equations of motion of the Chaplygin ball rolling over the plane orthogonal to $\vec{\gamma}$.

An invariant measure of the system was derived by Chaplygin for $\varepsilon = 1$ (Chaplygin 1903), and by Yaroshchuk for $\varepsilon \neq 1$ (Yaroshchuk 1992). Remarkably, for $\varepsilon = -1$,

which is the case (iii) above with $a = 2b$, the problem is integrable [see Borisov and Fedorov (1995), Borisov et al. (2008), Borisov and Mamaev (2013)].

Next, we assume that a gyroscope is placed in a ball \mathbf{B} such that the mass center of the system coincides with the geometric center $O_{\mathbf{B}}$ of the ball. The addition of a gyroscope to the problem is equivalent to the addition of a constant angular momentum $\vec{\kappa}$ directed along the axis of the gyroscope to $\vec{\mathbf{k}}$ (Bobilev 1892; Zhukovskiy 1893):

$$\frac{d}{dt}(\vec{\mathbf{k}} + \vec{\kappa}) = (\vec{\mathbf{k}} + \vec{\kappa}) \times \vec{\omega}, \quad \dot{\vec{\gamma}} = \varepsilon \vec{\gamma} \times \vec{\omega}. \tag{6.2}$$

As above, $\vec{\mathbf{k}} = \mathbb{I}\vec{\omega} + D\vec{\omega} - D(\vec{\omega}, \vec{\gamma})\vec{\gamma}$, where $D = a^2 m$, m is the mass of the system (ball with gyroscope), \mathbb{I} is a new inertia operator that is described below [see (6.3)] together with the momentum $\vec{\kappa}$ for the Bobilev symmetric case.

Markeev proved that the equations of motion for the rolling over the plane ($\varepsilon = 1$) can be resolved in quadratures (Markeev 1985). The analysis of the bifurcation diagram and the topology of the phase space of the Markeev case are studied in Moskvín (2009) and Zhila (2020), respectively.

There are two famous classical cases of the system (6.2) for $\varepsilon = 1$ where the quadratures are given in elliptic functions. These cases were studied by Bobilev (1892) and Zhukovskiy (1893).

In the Bobilev case the central ellipsoid of inertia of the ball \mathbf{B} is rotationally symmetric and the gyroscope axis coincides to the axis of symmetry. Let $O_{\mathbf{B}}\vec{\mathbf{e}}_1\vec{\mathbf{e}}_2\vec{\mathbf{e}}_3$ and $O_{\mathbf{B}}\vec{\mathbf{e}}'_1\vec{\mathbf{e}}'_2\vec{\mathbf{e}}'_3$ be the moving frames attached to the ball \mathbf{B} and the gyroscope in which the inertia operator has the forms $\mathbb{I}_1 = (A_1, A_1, C_1)$ and $\mathbb{I}_2 = (A_2, A_2, C_2)$, respectively. It is assumed that the axis of the gyroscope is fixed with respect to the ball and coincides with the axis of symmetry of the inertia ellipsoid of the ball ($\vec{\mathbf{e}}_3 = \vec{\mathbf{e}}'_3$) and that the forces applied to the gyroscope do not induce torque about the axis of the gyroscope. Thus, the gyroscope rotates with a constant angular velocity ω'_3 about the axis of symmetry. Then the operator \mathbb{I} and the momentum $\vec{\kappa}$ in (6.2) for the Bobilev case are given by:

$$\mathbb{I} = \text{diag}(A, A, C) = \text{diag}(A_1 + A_2, A_1 + A_2, C_1) \quad \text{and} \quad \vec{\kappa} = C_2\omega'_3\vec{\mathbf{e}}_3. \tag{6.3}$$

In the Zhukovskiy case there is an additional assumption, (called the Zhukovskiy condition):

$$C_1 = A_1 + A_2, \tag{6.4}$$

that is, it is assumed that \mathbb{I} is proportional to the identity matrix $\mathbb{E} = \text{diag}(1, 1, 1)$.

Demchenko used the Zhukovskiy condition to integrate the problem of rolling of the gyroscopic ball over a sphere (Demchenko 1924) [see also Dragović et al. (2023)]. The integrability of the problem of rolling of the gyroscopic ball over a sphere with the Bobilev conditions (6.3) can be found in Borisov et al. (2005). The question about the existence of an integrable case for a dynamically nonsymmetric ball with a gyroscope rolling over a sphere is still open. Another natural extension of the problem of the ball rolling over a sphere is recently given in Dragović et al. (2023, 2022).

6.2 Chaplygin Ball with a Gyroscope Rolling without Slipping and Twisting

One can consider the additional nonholonomic constraint $\langle \vec{\omega}, \vec{\gamma} \rangle = 0$ describing *no-twisting* condition: the ball \mathbf{B} does not rotate around the normal at the contact point and is called a *rubber Chaplygin ball*. Then the momentum with respect to the contact point can be expressed as $\mathbf{k} = \mathbf{I}\vec{\omega}$, $\mathbf{I} = \mathbb{I} + D\mathbb{E}$. The gyroscopic equations take the form

$$\frac{d}{dt}(\vec{\mathbf{k}} + \vec{\kappa}) = (\vec{\mathbf{k}} + \vec{\kappa}) \times \vec{\omega} + \lambda \vec{\gamma}, \quad \dot{\vec{\gamma}} = \varepsilon \vec{\gamma} \times \vec{\omega}, \tag{6.5}$$

where the Lagrange multiplier is given by $\lambda = -\langle \vec{\gamma}, \mathbf{I}^{-1}((\vec{\mathbf{k}} + \vec{\kappa}) \times \vec{\omega}) \rangle / \langle \vec{\gamma}, \mathbf{I}^{-1}\vec{\gamma} \rangle$.

The system has an invariant measure with the same density as in the absence of a gyroscope [see Ehlers et al. (2005) for $\varepsilon = 1$ and Ehlers and Koiller (2007) for $\varepsilon \neq 1$]. As in the Markeev integrable case, for $\varepsilon = 1$, the system is integrable according to the Euler-Jacobi theorem. This is proved in Borisov et al. (2005) for the Veselova problem with a gyroscope, which is described by the same system of equations. Borisov, Bizyaev, and Mamaev also pointed out the integrability of Eq. (6.5) for $\varepsilon \neq 1$ in the case of the dynamical symmetry $A = B$ if the gyroscope is oriented in the direction of the axis of the dynamical symmetry, which gives the Bobilev conditions (6.3) [see Table 2 in Borisov et al. (2013)]. Borisov and Mamaev proved the integrability of the problem without the gyroscope, for $\varepsilon = -1$ (Borisov and Mamaev 2007), providing analogy with the nonrubber rolling.

The system of a Chaplygin ball with a gyroscope rolling without slipping and twisting over a sphere deserves to be studied in more detail. In order to describe its reduction and Hamiltonization, we will consider a general problem in \mathbb{R}^n .

7 The Rolling of a Gyroscopic Ball without Slipping and Twisting in \mathbb{R}^n

7.1 Rolling of a Ball without Slipping and Twisting Over a Sphere

The aim of this Section is to generalize the considerations from Sect. 6 from \mathbb{R}^3 to \mathbb{R}^n , for any $n > 3$. We start with the situation without gyroscopic or magnetic forces, following Jovanović (2018) and Gajić and Jovanović (2019a, b). We consider in this Subsection the rolling without slipping and twisting of an n -dimensional ball \mathbf{B} of radius a over the $(n - 1)$ -dimensional fixed sphere \mathbf{S} of radius b . There are three possible scenarios, in a full analogy with the three configurations described at the beginning of Sect. 6.1 for $n = 3$, recall Fig. 1.

Consider the space frame $\mathbb{R}^n(\mathbf{x})$ with the origin O at the center of the fixed sphere \mathbf{S} and the moving frame $\mathbb{R}^n(\mathbf{X})$ with the origin $O_{\mathbf{B}}$ at the center of the rolling ball \mathbf{B} . The mapping from the moving to the space frame is given by $\mathbf{x} = g\mathbf{X} + \mathbf{r}$, where $g \in SO(n)$ is a rotation matrix and $\mathbf{r} = \overrightarrow{OO_{\mathbf{B}}}$ is the position vector of the ball center $O_{\mathbf{B}}$ in the space frame. The configuration space Q is the direct product of the Lie group $SO(n)$ and the sphere $S = \{\mathbf{r} \in \mathbb{R}^n \mid (\mathbf{r}, \mathbf{r}) = (b \pm a)^2\}$.

Remark 7.1 Here and below, we take the sign “+” for the case (i) and the case “−” for the cases (ii) and (iii) of the three possible scenarios in analogy with the three cases from the beginning of Sect. 6.1.

Let $\omega = g^{-1}\dot{g}$ be the angular velocity of the ball in the moving frame, m be the mass of the ball, and $\mathbb{I} : so(n) \rightarrow so(n)$ the inertia operator. We additionally assume that the ball is balanced, i.e., its geometric center coincides with the mass center. We will call such a system a Chaplygin ball in \mathbb{R}^n . Then the Lagrangian of the system is given by

$$L(g, \mathbf{r}, \omega, \dot{\mathbf{r}}) = \frac{1}{2} \langle \mathbb{I}\omega, \omega \rangle + \frac{1}{2} m \langle \dot{\mathbf{r}}, \dot{\mathbf{r}} \rangle, \tag{7.1}$$

where now $\langle \cdot, \cdot \rangle$ is the invariant scalar product proportional to the Killing form on $so(n)$ ($\langle \cdot, \cdot \rangle = -\frac{1}{2} \text{tr}(\cdot \circ \cdot)$) and the Euclidean scalar product in \mathbb{R}^n , respectively.

The direction $\frac{\vec{OA}}{|\vec{OA}|}$ of the contact point A in the frame attached to the ball is given by the unit vector $\gamma = \frac{1}{b \pm a} g^{-1} \mathbf{r}$. It is invariant with respect to the diagonal left $SO(n)$ -action: $\tilde{g} \cdot (g, \mathbf{r}) = (\tilde{g}g, \tilde{g}\mathbf{r})$, $\tilde{g} \in SO(n)$. The action defines $SO(n)$ -bundle

$$\begin{array}{ccc} SO(n) & \longrightarrow & Q = SO(n) \times \mathcal{S} \\ & & \downarrow \pi \\ & & S^{n-1} = Q/SO(n) \end{array} \tag{7.2}$$

with the submersion π given by $\gamma = \pi(g, \mathbf{r}) = \frac{1}{b \pm a} g^{-1} \mathbf{r}$.

The contact point A of the ball in the moving frame is $\mathbf{X}_A = -(\pm a \gamma)$. The condition that the ball is rolling without slipping is that the velocity $\dot{\mathbf{x}}_A$ of the contact point in the space frame is equal to zero

$$0 = \dot{\mathbf{x}}_A = \frac{d}{dt} (g \mathbf{X}_A + \mathbf{r}) = \mp a \dot{g} \gamma + \dot{\mathbf{r}} = \mp a (\dot{g} g^{-1}) g \gamma + \dot{\mathbf{r}}.$$

This leads to the constraint $\dot{\mathbf{r}} = \pm \frac{a}{b \pm a} \Omega \mathbf{r}$, where $\Omega = \text{Ad}_g \omega = \dot{g} g^{-1}$ is the angular velocity in the space frame. On the other hand, the condition of no twisting at the contact point can be written as the condition on Ω : $\Omega \in \mathbf{r} \wedge \mathbb{R}^n$. The same condition can be written in terms of ω : $\omega \in \gamma \wedge \mathbb{R}^n$. For more details, see Jovanović (2018). The constraints determine the distribution

$$\mathcal{D}_{(g, \mathbf{r})} = \left\{ (\omega, \dot{\mathbf{r}}) \in T_{(g, \mathbf{r})} SO(n) \times \mathcal{S} \mid \dot{\mathbf{r}} = \pm \frac{a}{b \pm a} (\text{Ad}_g \omega) \mathbf{r}, \omega \in g^{-1} \mathbf{r} \wedge \mathbb{R}^n \right\}$$

of rank $(n - 1)$, a principal connection of the bundle (7.2). The Lagrangian L from (7.1) is $SO(n)$ -invariant as well. Thus, an n -dimensional Chaplygin ball rolling without slipping and twisting over a fixed sphere in \mathbb{R}^n is a $SO(n)$ -Chaplygin system. It reduces to the tangent bundle $TS^{n-1} \cong \mathcal{D}/SO(n)$.

As in the three-dimensional case, we set $\varepsilon = b/(b \pm a)$. The horizontal lift $\dot{\gamma}^h|_{(g, \mathbf{r})} = (\omega, \mathbf{V})$ is given by:

$$\begin{aligned} \omega &= \frac{1}{\varepsilon} \gamma \wedge \dot{\gamma}, \\ \mathbf{V} = \dot{\mathbf{r}} &= (b \pm a) \frac{d}{dt}(g\gamma) = (b \pm a) \left(1 - \frac{1}{\varepsilon}\right) g\dot{\gamma}. \end{aligned}$$

The reduced Lagrangian l and the $(0, 3)$ -tensor field Σ are [see Jovanović (2018)]

$$l(\gamma, \dot{\gamma}) = \frac{1}{2} \mathbf{g}(\dot{\gamma}, \dot{\gamma}) = -\frac{1}{4\varepsilon^2} \text{tr}(\mathbf{I}(\gamma \wedge \dot{\gamma}) \circ (\gamma \wedge \dot{\gamma})) = -\frac{1}{2\varepsilon^2} \langle \mathbf{I}(\gamma \wedge \dot{\gamma})\gamma, \dot{\gamma} \rangle, \tag{7.3}$$

$$\Sigma(X, Y, Z)|_\gamma = \frac{2\varepsilon - 1}{2\varepsilon^3} \text{tr}(\mathbf{I}(\gamma \wedge X) \circ (Y \wedge Z)) = \frac{2\varepsilon - 1}{\varepsilon^3} \langle \mathbf{I}(\gamma \wedge X)Y, Z \rangle, \tag{7.4}$$

where, as in the three-dimension, $\mathbf{I} = \mathbb{I} + D \cdot \text{Id}_{so(n)}$ and $D = ma^2$. We have

$$\begin{aligned} \frac{\partial l}{\partial \gamma} &= \frac{1}{\varepsilon^2} \mathbf{I}(\gamma \wedge \dot{\gamma})\dot{\gamma}, \quad \frac{\partial l}{\partial \dot{\gamma}} = -\frac{1}{\varepsilon^2} \mathbf{I}(\gamma \wedge \dot{\gamma})\gamma, \\ \mathbf{JK}(\dot{\gamma}, \delta\gamma) &= \frac{2\varepsilon - 1}{\varepsilon^3} \langle \mathbf{I}(\gamma \wedge \dot{\gamma})\dot{\gamma}, \delta\gamma \rangle \end{aligned} \tag{7.5}$$

Therefore, the reduced Chaplygin equations (3.1) without gyroscopic forces are:

$$\delta l - \mathbf{JK}(\dot{\gamma}, \delta\gamma) = \left\langle \frac{1}{\varepsilon^2} \frac{d}{dt} \left(\mathbf{I}(\gamma \wedge \dot{\gamma})\gamma \right) + \frac{1-\varepsilon}{\varepsilon^3} \mathbf{I}(\gamma \wedge \dot{\gamma})\dot{\gamma}, \delta\gamma \right\rangle = 0, \quad \delta\gamma \in T_\gamma S^{n-1}. \tag{7.6}$$

Remark 7.2 Note that if the radii of the sphere and the ball are equal, then $\varepsilon = 1/2$. Then, the curvature of \mathcal{D} vanishes and $\Sigma \equiv 0$ (Jovanović 2018). For $n = 3$, see Ehlers and Koiller (2007) and Borisov et al. (2014). Also, if \mathbb{I} is proportional to the identity operator then $\Sigma \equiv 0$. Then the \mathbf{JK} -term vanishes although the curvature of \mathcal{D} is different from zero. Under these conditions, the reduced system is Hamiltonian without any time reparametrization.

7.2 Gyroscopic Ball

Now, we want to consider the gyroscopic Chaplygin ball in \mathbb{R}^n and to study how the addition of a gyroscopic term is going to modify the reduced equations of motion (7.6). Eq. (6.5) without the gyroscope have an analog in in \mathbb{R}^n :

$$\dot{\mathbf{k}} = [\mathbf{k}, \omega] + \lambda_0, \quad \dot{\gamma} = -\varepsilon\omega\gamma. \tag{7.7}$$

Here $\mathbf{k} = \mathbf{I}\omega$ and the Lagrange multiplier $\lambda_0 \in (\mathbb{R}^n \wedge \gamma)^\perp$ is determined from the condition that $\omega \in \mathbb{R}^n \wedge \gamma$ [see Jovanović (2018)].

Let us notice that Eq. (7.6) alternatively can be derived directly by the substitution of $\omega = \frac{1}{\varepsilon}\gamma \wedge \dot{\gamma}$ in Eq. (7.7). Eq. (7.7) are also a convenient starting point for gyroscopic generalizations. With a suitable modification of \mathbb{I} for the gyroscopic ball, the analogue of the equation (6.5) in \mathbb{R}^n is

$$\dot{\mathbf{k}} = [\mathbf{k}, \omega] + [\kappa, \omega] + \lambda_0, \quad \dot{\gamma} = -\varepsilon\omega\gamma, \tag{7.8}$$

where now $\kappa \in so(n)$ is a fixed matrix, $\mathbf{k} = \mathbf{I}\omega = \mathbb{I}\omega + D\omega$, $D = a^2m$, and m is the mass of the system (ball with gyroscope).

After the substitution $\omega = \frac{1}{\varepsilon}\gamma \wedge \dot{\gamma}$, and taking the scalar product with $\frac{1}{\varepsilon}\gamma \wedge \delta\gamma$, Eq. (7.8) take the form

$$\left\langle \frac{1}{\varepsilon^2}\mathbf{I}(\gamma \wedge \ddot{\gamma}) - \frac{1}{\varepsilon^3}[\mathbf{I}(\gamma \wedge \dot{\gamma}), \gamma \wedge \dot{\gamma}], \gamma \wedge \delta\gamma \right\rangle = \frac{1}{\varepsilon^2}\langle [\kappa, \gamma \wedge \dot{\gamma}], \gamma \wedge \delta\gamma \rangle, \tag{7.9}$$

where we used that λ_0 is orthogonal to $\gamma \wedge \mathbb{R}^n$. Now, since

$$\begin{aligned} [\kappa, \gamma \wedge \dot{\gamma}] &= (\kappa\gamma) \wedge \dot{\gamma} - (\kappa\dot{\gamma}) \wedge \gamma \\ \text{and } \langle X \wedge Y, Z \wedge T \rangle &= \langle X, Z \rangle \langle Y, T \rangle - \langle X, T \rangle \langle Y, Z \rangle, \end{aligned}$$

we get the right-hand side of (7.9):

$$\begin{aligned} \text{rhs} &= \frac{1}{\varepsilon^2} \left(\langle \kappa\gamma, \gamma \rangle \langle \dot{\gamma}, \delta\gamma \rangle - \langle \kappa\gamma, \delta\gamma \rangle \langle \dot{\gamma}, \gamma \rangle - \langle \kappa\dot{\gamma}, \gamma \rangle \langle \gamma, \delta\gamma \rangle + \langle \kappa\dot{\gamma}, \delta\gamma \rangle \langle \gamma, \gamma \rangle \right) \\ &= \frac{1}{\varepsilon^2} \langle \kappa\dot{\gamma}, \delta\gamma \rangle. \end{aligned}$$

Similarly, the left-hand side of (7.9) is given by

$$\text{lhs} = \left\langle -\frac{1}{\varepsilon^2}\mathbf{I}(\gamma \wedge \ddot{\gamma})\gamma - \frac{1}{\varepsilon^3}\mathbf{I}(\gamma \wedge \dot{\gamma})\dot{\gamma}, \delta\gamma \right\rangle = -\delta l + \mathbf{JK}(\dot{\gamma}, \delta\gamma),$$

where the second equality follows from (7.6). Therefore, from (7.9) we obtain

Proposition 7.1 *The reduced equations of motion of a gyroscopic ball rolling without slipping and twisting over a sphere are given by*

$$\delta l - \mathbf{JK}(\dot{\gamma}, \delta\gamma) = \left\langle \frac{1}{\varepsilon^2}\mathbf{I}(\gamma \wedge \ddot{\gamma})\gamma - \frac{1}{\varepsilon^3}\mathbf{I}(\gamma \wedge \dot{\gamma})\dot{\gamma}, \delta\gamma \right\rangle = \mathbf{f}(\dot{\gamma}, \delta\gamma) \tag{7.10}$$

where the gyroscopic term is given by $\mathbf{f}(\dot{\gamma}, \delta\gamma) = \frac{1}{\varepsilon^2}\langle \dot{\gamma}, \kappa\delta\gamma \rangle$.

Note that the gyroscopic two-form \mathbf{f}

$$\mathbf{f} = \frac{1}{\varepsilon^2} \sum_{i < j} \kappa_{ij} d\gamma_i \wedge d\gamma_j. \tag{7.11}$$

is exact magnetic: $\mathbf{f} = d\mathbf{a}$, where

$$\mathbf{a} = \frac{1}{2\varepsilon^2} \sum_{ij} \kappa_{ij} \gamma_i d\gamma_j.$$

Thus, the reduced equations of motion of a gyroscopic ball rolling without slipping and twisting over a sphere (7.10) can be rewritten in the equivalent form (see Remark 3.2):

$$\delta l_1 = \mathbf{JK}(\dot{\gamma}, \delta\gamma),$$

where the Lagrangian l_1 is

$$l_1(\dot{\gamma}, \gamma) = \frac{1}{2\varepsilon^2} \langle \mathbf{I}(\gamma \wedge \dot{\gamma})\dot{\gamma}, \gamma \rangle + \frac{1}{2\varepsilon^2} \langle \gamma, \kappa\dot{\gamma} \rangle.$$

Remark 7.3 As in the three-dimensional case, when b tends to infinity, ε tends to 1, γ tends to the unit vector that is constant in the fixed reference frame and we obtain the equations of motion of the Chaplygin ball with a gyroscope rolling without slipping and twisting over the plane orthogonal to γ .

Remark 7.4 In addition, let us note that for $\varepsilon = 1$ the system (7.8) with $\kappa = 0$ represents also the Veselova problem with the left-invariant metric on $SO(n)$ defined by the operator \mathbf{I} [see Veselov and Veselova (1988); Fedorov and Jovanović (2004)]. In this way, the system (7.8) for $\varepsilon = 1$ can be seen as a Veselova problem with the addition of a gyroscope.

Note that the Veselova problem is an example of an LR system. These are non-holonomic systems with left-invariant metrics and right-invariant constraints on Lie groups (Veselov and Veselova 1988; Fedorov and Jovanović 2004). One can consider LR systems with gyroscopic forces and their reduction to homogeneous spaces as well. Along with the gyroscopic Chaplygin reduction, it is interesting to consider the symplectic reduction of the corresponding Hamiltonian magnetic systems on Lie groups by using a general framework for the reduction of the systems with symmetries on magnetic cotangent bundles given in Kowalzig et al. (2005). The reduction problems based on Kowalzig et al. (2005) will be consider elsewhere.

7.3 Invariant Measure

We are going to describe the reduced magnetic flow (7.10) and its invariant measure on the cotangent bundle of a sphere S^{n-1} . Consider the Legendre transformation of the Lagrangian l given by (7.3).

$$p = \frac{\partial l}{\partial \dot{\gamma}} = \mathbf{g}(\dot{\gamma}) = -\frac{1}{\varepsilon^2} \mathbf{I}(\gamma \wedge \dot{\gamma})\gamma. \tag{7.12}$$

Since $\mathbf{I}(\gamma \wedge \dot{\gamma})$ is skew-symmetric, we get $\langle \gamma, p \rangle = 0$. Thus, the point (p, γ) belongs to the cotangent bundle of a sphere realized as a symplectic submanifold in the symplectic linear space $(\mathbb{R}^{2n}\{\gamma, p\}, dp_1 \wedge d\gamma_1 + \dots + dp_n \wedge d\gamma_n)$ defined by the equations:

$$\phi_1 = \langle \gamma, \gamma \rangle = 1, \quad \phi_2 = \langle \gamma, p \rangle = 0. \tag{7.13}$$

Let $\dot{\gamma} = \mathbf{g}^{-1}(p) = X_\gamma(p, \gamma)$ be the inverse of the Legendre transformation (7.12), which is unique on the subvariety (7.13). Then

$$h(\gamma, p) = \frac{1}{2} \langle X_\gamma(\gamma, p), p \rangle \tag{7.14}$$

is the Hamiltonian function of the reduced system. From (7.6) and (7.10), we have

$$\left\langle -\dot{p} + \frac{1-\varepsilon}{\varepsilon^3} \mathbf{I}(\gamma \wedge X_\gamma) X_\gamma, \delta\gamma \right\rangle = \frac{1}{\varepsilon^2} \langle X_\gamma, \kappa\delta\gamma \rangle.$$

Therefore,

$$\dot{p} = \frac{1-\varepsilon}{\varepsilon^3} \mathbf{I}(\gamma \wedge X_\gamma) X_\gamma + \frac{1}{\varepsilon^2} \kappa X_\gamma + \mu\gamma,$$

where μ is the multiplier determined from the condition that $(\dot{\gamma}, \dot{p})$ is tangent to T^*S^{n-1} :

$$\langle \dot{\gamma}, p \rangle + \langle \gamma, \dot{p} \rangle = 0.$$

Proposition 7.2 *The reduced equations of the rolling of a ball with a gyroscope over a sphere without slipping and twisting on T^*S^{n-1} are*

$$\dot{\gamma} = X_\gamma(\gamma, p), \quad \dot{p} = X_p(\gamma, p) = \frac{1-\varepsilon}{\varepsilon^3} \mathbf{I}(\gamma \wedge X_\gamma) X_\gamma + \frac{1}{\varepsilon^2} \kappa X_\gamma + \mu\gamma, \tag{7.15}$$

where

$$\mu = \frac{(\varepsilon-1)}{\varepsilon^3} \langle (\mathbf{I}(\gamma \wedge X_\gamma)) X_\gamma, \gamma \rangle - 2h(\gamma, p) + \frac{1}{\varepsilon^2} \langle X_\gamma, \kappa\gamma \rangle. \tag{7.16}$$

Let

$$\mathbf{w} = dp_1 \wedge d\gamma_1 + \dots + dp_n \wedge d\gamma_n |_{T^*S^{n-1}} \tag{7.17}$$

be the canonical symplectic form on T^*S^{n-1} .

From Theorem 4.1 and the formula for an invariant measure without magnetic term [see Jovanović (2018)], we have:

Proposition 7.3 *The reduced equations of the rolling of a ball with a gyroscope over a sphere without slipping and twisting (7.15) have an invariant measure $\nu(\gamma)\mathbf{w}^{n-1}$, where w is from (7.17) and ν is defined by:*

$$\nu(\gamma) := (\det \mathbb{I}|_{\mathbb{R}^n \wedge \gamma})^{\frac{1}{2\varepsilon}-1}. \tag{7.18}$$

8 Hamiltonization and Integrability

8.1 Hamiltonization of the $SO(n - 2)$ -Invariant Case

As already mentioned above, the existence of an invariant measure of a nonholonomic system is closely related to the problem of its Hamiltonization. In this section we provide a class of examples of $SO(n - 2)$ -symmetric systems (ball with gyroscope) that allow a Chaplygin Hamiltonization.

Consider the inertia operators

$$\mathbb{I}(\mathbf{e}_i \wedge \mathbf{e}_j) = (a_i a_j - D)\mathbf{e}_i \wedge \mathbf{e}_j \quad \text{i.e.,} \quad \mathbb{I}(X \wedge Y) = \mathbb{A}X \wedge \mathbb{A}Y, \tag{8.1}$$

parameterized by $\mathbb{A} = \text{diag}(a_1, \dots, a_n)$, where $[\mathbf{e}_1, \dots, \mathbf{e}_n]$ is the standard basis of \mathbb{R}^n . The formulas for the reduced Lagrangian l (7.3), the Hamiltonian h (7.14), and the density of an invariant measure ν (7.18) take the form:

$$l(\gamma, \dot{\gamma}) = \frac{1}{2\varepsilon^2} \left(\langle \mathbb{A}\dot{\gamma}, \dot{\gamma} \rangle \langle \mathbb{A}\gamma, \gamma \rangle - \langle \mathbb{A}\gamma, \dot{\gamma} \rangle^2 \right), \tag{8.2}$$

$$h(\gamma, p) = \frac{\varepsilon^2 \langle p, \mathbb{A}^{-1}p \rangle}{2 \langle \gamma, \mathbb{A}\gamma \rangle}, \tag{8.3}$$

$$\nu(\gamma) = \text{const} \cdot \langle \mathbb{A}\gamma, \gamma \rangle^{\frac{n-2}{2\varepsilon}+2-n}, \tag{8.4}$$

[see Jovanović (2015, 2018)]. Moreover, the function $\mathcal{N}(\gamma) = \varepsilon \langle \mathbb{A}\gamma, \gamma \rangle^{\frac{1}{2\varepsilon}-1}$ is a Chaplygin multiplier: under the time substitution $d\tau = \mathcal{N}(\gamma)dt$, the reduced system (7.6) with $\kappa = 0$ becomes the geodesic flow of the metric

$$ds_{A,\varepsilon}^2 = (\gamma, A\gamma)^{\frac{1}{\varepsilon}-2} \left((Ad\gamma, d\gamma)(A\gamma, \gamma) - (A\gamma, d\gamma)^2 \right) \tag{8.5}$$

defined by the Lagrangian [see Jovanović (2018)]

$$l^*(\gamma, \gamma') = l(\gamma, \dot{\gamma})|_{\dot{\gamma}=\mathcal{N}(\gamma)\gamma'} = \frac{1}{2} \langle \gamma, \mathbb{A}\gamma \rangle^{\frac{1}{\varepsilon}-2} \left(\langle \mathbb{A}\gamma', \gamma' \rangle \langle \mathbb{A}\gamma, \gamma \rangle - \langle \mathbb{A}\gamma, \gamma' \rangle^2 \right). \tag{8.6}$$

Remark 8.1 Note that for $n = 3$ all symmetric operators \mathbb{I} have the form (8.1) in a basis formed by its eigenvectors. Namely, after the standard identification $\mathbb{R}^3 \cong so(3)$ (Arnold 1974), for the given inertia operator $\mathbb{I} = \text{diag}(A, B, C): \mathbb{R}^3 \rightarrow \mathbb{R}^3$ for the

gyroscopic ball and the parameter $D = ma^2$, the operator $\mathbb{I} : so(3) \rightarrow so(3)$ has the form (8.1), with:

$$\mathbb{A} = \text{diag} \left(\frac{\Delta}{A + D}, \frac{\Delta}{B + D}, \frac{\Delta}{C + D} \right), \quad \Delta = \sqrt{(A + D)(B + D)(C + D)}. \tag{8.7}$$

The above Hamiltonization recovers the procedure of reduction and Hamiltonization for a three-dimensional ball without gyroscope from Ehlers and Koiller (2007). We would recall that Borisov and Mamaev proved the integrability of the three-dimensional ball without gyroscope and the spherical shell for a specific ratio between the radii: the case (iii) from Sect. 6.1, where $a = 2b$, i.e., $\varepsilon = -1$, see Borisov and Mamaev (2007). The n -dimensional reduced system of a ball without gyroscope rolling over a sphere (7.6) with the inertia operator \mathbb{I} given by (8.1) is also integrable for $\varepsilon = -1$; the integrability remains for such systems for an arbitrary ε , if the matrix \mathbb{A} has only two distinct parameters (Gajić and Jovanović 2019a, b).

Now, we turn to the systems with gyroscopic force. If

$$\begin{aligned} d(\mathcal{N}\mathbf{f}) &= \frac{2}{\varepsilon} \left(\frac{1}{2\varepsilon} - 1 \right) \langle \mathbb{A}\gamma, \gamma \rangle^{\frac{1}{2\varepsilon}-2} \left(\sum_k a_k \gamma_k d\gamma_k \right) \wedge \left(\sum_{i < j} \kappa_{ij} d\gamma_i \wedge d\gamma_j \right) \\ &= 0|_{T^*S^{n-1}} \end{aligned} \tag{8.8}$$

then the reduced gyroscopic system is Hamiltonizable as well. This follows from Theorem 5.1.

For $n = 3$, equation (8.8) is satisfied for an arbitrary gyroscopic term κ . The following statement provides a class of examples, based on the $SO(n - 2)$ -symmetry, which satisfy equation (8.8), thus are Hamiltonizable, for every $n \geq 3$.

Theorem 8.1 *Assume that the gyroscopic term \mathbf{f} from (7.11) is given by $\kappa = \kappa_{12}\mathbf{e}_1 \wedge \mathbf{e}_2$, i.e.,*

$$\mathbf{f} = \frac{\kappa_{12}}{\varepsilon^2} d\gamma_1 \wedge d\gamma_2$$

and the inertia operator of the system ball with gyroscope is given by (8.1), where $a_3 = a_4 = \dots = a_n$:

$$\mathbb{A} = \text{diag}(a_1, a_2, a_3, \dots, a_3).$$

Then the function $\mathcal{N}(\gamma) = \varepsilon A(\gamma)^{\frac{1}{2\varepsilon}-1}$, with

$$A(\gamma) = a_3 + (a_1 - a_3)\gamma_1^2 + (a_2 - a_3)\gamma_2^2, \tag{8.9}$$

is a Chaplygin multiplier. Under the time substitution $d\tau = \mathcal{N}(\gamma)dt$ and the change of momenta $\tilde{p} = \mathcal{N}(\gamma)p$, the reduced system (7.15) becomes the magnetic geodesic

flow of the metric (8.5) with respect to the twisted symplectic form given by

$$\tilde{\mathbf{w}} + \mathcal{N}\rho^*\mathbf{f} = d\tilde{p} \wedge d\gamma_1 + \dots + d\tilde{p}_n \wedge d\gamma_n + \frac{\kappa_{12}}{\varepsilon} \mathcal{A}(\gamma)^{\frac{1}{2\varepsilon}-1} d\gamma_1 \wedge d\gamma_2|_{T^*S^{n-1}}. \tag{8.10}$$

Remark 8.2 The function (8.9) satisfies $\mathcal{A}(\gamma) = \langle \mathbb{A}\gamma, \gamma \rangle$ for $\langle \gamma, \gamma \rangle = 1$. We use the function \mathcal{A} to simplify some equations below. For example, the Hamiltonian of the magnetic geodesic flow of the metric (8.5) in the coordinates (γ, \tilde{p}) can be written as

$$h^*(\gamma, \tilde{p}) = \frac{1}{2} \mathcal{A}(\gamma)^{1-\frac{1}{\varepsilon}} \langle \tilde{p}, \mathbb{A}^{-1}\tilde{p} \rangle. \tag{8.11}$$

8.2 Integrability of the $SO(2) \times SO(n - 2)$ -Invariant Case

In this section we want to impose additional symmetry with respect to $SO(n - 2)$ -symmetry considered in Sect. 8.1, and in particular in Theorem 8.1, this additional symmetry will imply integrability.

As mentioned above, the cotangent bundle T^*S^{n-1} is realized within \mathbb{R}^{2n} by the constraints (7.13). In the new coordinates $(\gamma, \tilde{p}) = (\gamma, \varepsilon \mathcal{A}(\gamma)^{\frac{1}{2\varepsilon}-1} p)$, the constraints become

$$\phi_1^* = \langle \gamma, \gamma \rangle = 1, \quad \phi_2^* = \frac{1}{\varepsilon} \mathcal{A}(\gamma)^{1-\frac{1}{2\varepsilon}} \langle \gamma, \tilde{p} \rangle = 0. \tag{8.12}$$

Instead of (8.12), we equivalently use the constraints

$$\psi_1 = \langle \gamma, \gamma \rangle = 1, \quad \psi_2 = \langle \tilde{p}, \gamma \rangle = 0. \tag{8.13}$$

The magnetic Poisson bracket on the cotangent bundle $T^*S^{n-1} \subset \mathbb{R}^{2n}\{\gamma, \tilde{p}\}$ can be described by the Dirac construction as follows:

$$\{F, G\}_d = \{F, G\}^\kappa - \frac{\{F, \psi_1\}^\kappa \{G, \psi_2\}^\kappa - \{F, \psi_2\}^\kappa \{G, \psi_1\}^\kappa}{\{\psi_1, \psi_2\}^\kappa},$$

where

$$\{F, G\}^\kappa = \{F, G\}^0 + \frac{\kappa_{12}}{\varepsilon} \mathcal{A}(\gamma)^{\frac{1}{2\varepsilon}-1} \left(\frac{\partial F}{\partial \tilde{p}_1} \frac{\partial G}{\partial \tilde{p}_2} - \frac{\partial F}{\partial \tilde{p}_2} \frac{\partial G}{\partial \tilde{p}_1} \right)$$

and

$$\{F, G\}^0 = \sum_{i=1}^n \left(\frac{\partial F}{\partial \gamma_i} \frac{\partial G}{\partial \tilde{p}_i} - \frac{\partial F}{\partial \tilde{p}_i} \frac{\partial G}{\partial \gamma_i} \right)$$

is the canonical Poisson bracket on $\mathbb{R}^{2n}\{\gamma, \tilde{p}\}$, [see Arnold et al. (1989)]. Considered on $\mathbb{R}^{2n}\{\gamma, \tilde{p}\}$ without the subset $\{\gamma = 0\}$, the bracket $\{\cdot, \cdot\}_d$ is degenerate with two

Casimir functions ψ_1 and ψ_2 . The symplectic leaf given by (8.13) is exactly the cotangent bundle T^*S^{n-1} endowed with the twisted symplectic form (8.10).

It is convenient to derive the equations of the magnetic Hamiltonian flows with respect to the Dirac bracket $\{\cdot, \cdot\}_d$ using the Lagrange multipliers and the magnetic Hamiltonian flows with respect to the magnetic bracket $\{\cdot, \cdot\}^\kappa$ [e.g., see Arnold et al. (1989)]. Let

$$H = h^* - \lambda_1\psi_1 - \lambda_2\psi_2.$$

The magnetic Hamiltonian flow generated by the Hamiltonian (8.11) with respect to the Dirac bracket $\{\cdot, \cdot\}_d$ is given by

$$\gamma' = \frac{\partial H}{\partial \tilde{p}} = \mathcal{A}(\gamma)^{1-\frac{1}{\varepsilon}} \mathbb{A}^{-1} \tilde{p} - \lambda_2 \gamma, \tag{8.14}$$

$$\begin{aligned} \tilde{p}' &= -\frac{\partial H}{\partial \gamma} + \frac{\kappa_{12}}{\varepsilon} \mathcal{A}(\gamma)^{\frac{1}{2\varepsilon}-1} \mathbf{e}_1 \wedge \mathbf{e}_2(\gamma') \\ &= \frac{1-\varepsilon}{\varepsilon} \mathcal{A}(\gamma)^{-\frac{1}{\varepsilon}} \langle \tilde{p}, \mathbb{A}^{-1} \tilde{p} \rangle \left((a_1 - a_3) \gamma_1 \mathbf{e}_1 + (a_2 - a_3) \gamma_2 \mathbf{e}_2 \right) + 2\lambda_1 \gamma + \lambda_2 \tilde{p} \\ &\quad + \frac{\kappa_{12}}{\varepsilon} \mathcal{A}(\gamma)^{\frac{1}{2\varepsilon}-1} \left((\mathcal{A}(\gamma)^{1-\frac{1}{\varepsilon}} \frac{\tilde{p}_2}{a_2} - \lambda_2 \gamma_2) \mathbf{e}_1 - (\mathcal{A}(\gamma)^{1-\frac{1}{\varepsilon}} \frac{\tilde{p}_1}{a_1} - \lambda_2 \gamma_1) \mathbf{e}_2 \right), \end{aligned} \tag{8.15}$$

where the Lagrange multipliers λ_1 and λ_2 are determined from the condition that the functions ψ_1 and ψ_2 are integrals of the flow.

From now on we consider the system (8.14), (8.15) restricted to the symplectic leaf (8.13), that is, we consider the magnetic geodesic flow of the metric (8.5).

Let us impose now the additional symmetry. Suppose: $a_1 = a_2 \neq a_3$. Both the Hamiltonian (8.11) and the magnetic two-form (8.10) are invariant with respect to the action of the group $SO(2) \times SO(n - 2)$. We first consider the case $\kappa_{12} = 0$: the corresponding first integrals are linear and given as follows:

$$\Phi_{12}^0 = \gamma_1 \tilde{p}_2 - \gamma_2 \tilde{p}_1, \quad \Phi_{ij}^0 = \gamma_i \tilde{p}_j - \gamma_j \tilde{p}_i, \quad 3 \leq i < j \leq n.$$

Such first integrals are sometimes called *Noether integrals* as their existence follow from the Emmy Noether theorem. Let us now consider a general case $\kappa_{12} \neq 0$: straightforward calculations show that $\Phi_{ij} = \Phi_{ij}^0, 3 \leq i < j \leq n$ remain to be first integrals for $\kappa_{12} \neq 0$. Moreover,

$$\frac{d}{d\tau} \Phi_{12}^0 = -\frac{\kappa_{12}}{\varepsilon} \mathcal{A}(\gamma)^{\frac{1}{2\varepsilon}-1} (\gamma_1 \gamma_1' + \gamma_2 \gamma_2') = -\frac{\kappa_{12}}{a_1 - a_3} \frac{d}{d\tau} \left(\mathcal{A}(\gamma)^{\frac{1}{2\varepsilon}} \right).$$

Thus, the first integrals for $\kappa_{12} \neq 0$ are

$$\Phi_{12} = \gamma_1 \tilde{p}_2 - \gamma_2 \tilde{p}_1 + \frac{\kappa_{12}}{a_1 - a_3} \mathcal{A}(\gamma)^{\frac{1}{2\varepsilon}}, \quad \Phi_{ij} = \gamma_i \tilde{p}_j - \gamma_j \tilde{p}_i, \quad 3 \leq i < j \leq n.$$

These first integrals are the components of the momentum mapping of the $SO(2) \times SO(n - 2)$ -action with respect to the twisted symplectic form (8.10).

Theorem 8.2 *For $a_1 = a_2 \neq a_3$ the magnetic geodesic flow of the metric $ds_{A,\varepsilon}^2$ defined by the Hamiltonian (8.11) with respect to the twisted symplectic form (8.10) is completely integrable.*

- (i) *If $n = 3$ the system is Liouville integrable. Generic invariant manifolds are two-dimensional Lagrangian tori, the common level sets of h^* and Φ_{12} .*
- (ii) *If $n = 4$ the system is Liouville integrable. Generic invariant manifolds are three-dimensional Lagrangian tori, the common level sets of h^* , Φ_{12} , and Φ_{34} .*
- (iii) *If $n \geq 5$ the system is integrable in the noncommutative sense. Generic invariant manifolds are three-dimensional isotropic tori, the common level sets of h^* , Φ_{12} , and Φ_{ij} , $3 \leq i < j \leq n$.*

Proof For $n = 3$ the statement is clear. For $n = 4$, the Hamiltonian system (8.14) possesses three independent integrals h^* , Φ_{12} , Φ_{34} , in involution:

$$\{h^*, \Phi_{12}\}_d = 0, \quad \{h^*, \Phi_{34}\}_d = 0, \quad \{\Phi_{12}, \Phi_{34}\}_d = 0.$$

Thus, the Hamiltonian system (8.14), (8.15) is completely integrable according to the Arnold–Liouville theorem.

For $n > 4$, generic common level sets of all integrals are three-dimensional tori as well. Indeed, consider the natural embedding $T^*S^3 \subset T^*S^{n-1}$ induced by the embedding $\text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\} \subset \mathbb{R}^n$. Let us set $\mathbf{P} = (\tilde{p}_3, \tilde{p}_4, \dots, \tilde{p}_n)$, $\Gamma = (\gamma_3, \gamma_4, \dots, \gamma_n)$. Then $\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \mathbf{P})$, $\gamma = (\gamma_1, \gamma_2, \Gamma)$.

The system (8.14), (8.15) is invariant with respect to the $SO(n - 2)$ -action

$$R(\gamma, \tilde{p}) = (\gamma_1, \gamma_2, R\Gamma, \tilde{p}_1, \tilde{p}_2, R\mathbf{P}), \quad R \in SO(n - 2).$$

Also, as we already mentioned, the integrals Φ_{ij} , $3 \leq i < j \leq n$ are components of the corresponding momentum mapping

$$(\gamma, \tilde{p}) \longmapsto \Gamma \wedge \mathbf{P}.$$

For any point $\mathbf{c}_0 = (\gamma_0, \tilde{p}_0) \in T^*S^{n-1}$, there exists a matrix $R_0 \in SO(n - 2)$, such that $\mathbf{d}_0 = R_0(\gamma_0, \tilde{p}_0)$ belongs to T^*S^3 . Since the system is invariant with respect to the $SO(n - 2)$ -action, the solution $\mathbf{c}(\tau) = (\gamma(\tau), \tilde{p}(\tau))$ with the initial condition $\mathbf{c}(0) = (\gamma(0), \tilde{p}(0)) = \mathbf{c}_0$ is mapped to the solution $\mathbf{d}(\tau) = R(\gamma(\tau), \tilde{p}(\tau))$ with the initial condition

$$\mathbf{d}(0) = R_0(\gamma(0), \tilde{p}(0)) = R_0(\gamma_0, \tilde{p}_0) = \mathbf{d}_0 \in T^*S^3.$$

The solutions $\mathbf{c}(\tau)$ and $\mathbf{d}(\tau)$ have the same energy, $h^*(\mathbf{c}_0) = h^*(\mathbf{d}_0)$, while the corresponding values of the momenta are different: the momentum of $\mathbf{c}(\tau)$ is transformed to the momentum of $\mathbf{d}(\tau)$ by the adjoint mapping

$$\Gamma_0 \wedge \mathbf{P}_0 \longmapsto R_0(\Gamma_0 \wedge \mathbf{P}_0)R_0^T = \Phi_{34}(\mathbf{d}_0)\mathbf{e}_3 \wedge \mathbf{e}_4,$$

where $\mathbf{c}_0 = (\gamma_{0,1}, \gamma_{0,2}, \Gamma_0, \tilde{p}_{0,1}, \tilde{p}_{0,2}, \mathbf{P}_0)$.

One can easily verify that the solution $\mathbf{d}(\tau)$ belongs to T^*S^3 , that is, it is a solution of the problem for $n = 4$. Therefore, generically, $\mathbf{d}(\tau)$ is a quasi-periodic trajectory over a three-dimensional invariant torus $\mathcal{T}_0 \subset T^*S^3$, the connected component of the level set

$$h^* = h^*(\mathbf{d}_0), \quad \Phi_{12} = \Phi_{12}(\mathbf{d}_0), \quad \Phi_{34} = \Phi_{34}(\mathbf{d}_0).$$

All other components of the momentum mapping $\Phi_{ij}, 3 \leq i < j \leq n, (i, j) \neq (3, 4)$ are equal to zero.

Note that a point $\mathbf{d} \in T^*S^{n-1}$ belongs to T^*S^3 if and only if $\Phi_{ij}(\mathbf{d}) = 0, 3 \leq i < j \leq n, (i, j) \neq (3, 4)$. Thus, the original trajectory $\mathbf{c}(\tau) = R_0^{-1}(\mathbf{d}(\tau))$ is quasi-periodic over the three-dimensional invariant torus $\mathcal{T} = R_0^{-1}(\mathcal{T}_0)$, which is also the connected component of the level set

$$h^* = h^*(\mathbf{c}_0) = h^*(\mathbf{d}_0), \quad \Phi_{12} = \Phi_{12}(\mathbf{c}_0), \quad \Phi_{ij} = \Phi_{ij}(\mathbf{c}_0), \quad 3 \leq i < j \leq n.$$

The integrability of the system is a particular example of so-called noncommutative integrability. Namely, since the common level sets of the integrals are three-dimensional, and the Hamiltonian system (8.14), (8.15) has three independent first integrals h^*, Φ_{12}^κ , and $\sum_{3 \leq i < j \leq n} (\Phi_{ij})^2$, that commute with all integrals, the system is completely integrable according the Nekhoroshev–Mishchenko–Fomenko theorem on noncommutative integrability for all $n > 4$ [e.g., see Arnold et al. (1989)].□

Note that in the original phase space $T^*S^{n-1}\{\gamma, p\}$, the first integrals have the form

$$\Phi_{12} = \varepsilon \mathcal{A}(\gamma)^{\frac{1}{2\varepsilon}-1} (\gamma_1 p_2 - \gamma_2 p_1) + \frac{\kappa_{12}}{a_1 - a_3} \mathcal{A}(\gamma)^{\frac{1}{2\varepsilon}},$$

and

$$\Phi_{ij} = \varepsilon \mathcal{A}(\gamma)^{\frac{1}{2\varepsilon}-1} (\gamma_i p_j - \gamma_j p_i), \quad 3 \leq i < j \leq n.$$

In the original time, the system over a regular invariant torus \mathcal{T} has the form (1.3), where $\Phi = \mathcal{N}^{-1}|_{\mathcal{T}}$.

Remark 8.3 For $n = 3$, within the standard isomorphism between Lie algebras $(so(3), [\cdot, \cdot])$ and (\mathbb{R}^3, \times) given by

$$a_{ij} = -\varepsilon_{ijk} a_k, \quad i, j, k = 1, 2, 3 \tag{8.16}$$

[see Arnold (1974)], Eq. (7.8) with the inertia operator defined by (8.1), $\mathbb{A} = \text{diag}(a_1, a_1, a_3)$, and $\kappa = \kappa_{12} \mathbf{e}_1 \wedge \mathbf{e}_2$ correspond to Eq. (6.5) defined by the Bobilev conditions (6.3) with $\vec{\kappa} = -\kappa_{12} \vec{\mathbf{e}}_3$ and \mathbb{I} and \mathbb{A} related by (8.7) (see Sect. 6.2 and Remark 8.1). Then, along with the Liouville integrability after the Hamiltonization described in Theorem 8.2, the system is also integrable according to the Euler-Jacobi theorem.

9 Generalized Demchenko Case without Twisting in \mathbb{R}^n

9.1 Definition of the System

As above, we will consider the rolling of a gyroscopic ball \mathbf{B} without slipping and twisting in \mathbb{R}^n , now with an additional symmetry of the system. The additional symmetry is analogous to the Zhukovskiy condition (6.4) in dimension $n = 3$. Recall that adding a gyroscopic term does not change formulas for curvature of the distribution \mathcal{D} , \mathbf{JK} term (7.5) and Σ term (7.4). For the curvature K of \mathcal{D} see Lemma 7 in Jovanović (2018):

$$K_{(g,r)}(\xi_1^h, \xi_2^h) = \frac{2\varepsilon - 1}{\varepsilon^2} \text{Ad}_g(\xi_1 \wedge \xi_2), \quad \xi_1, \xi_2 \in T_{\pi(g,r)}S^{n-1}.$$

Since the reduced gyroscopic form \mathbf{f} is exact magnetic for an arbitrary $\kappa \in so(n)$,

$$\kappa = \sum_{i < j} \kappa_{ij} \mathbf{e}_i \wedge \mathbf{e}_j, \tag{9.1}$$

if the \mathbf{JK} -term in (3.1) vanishes, then the reduced gyroscopic G -Chaplygin system (3.1) is automatically Hamiltonian without any time reparametrization.

We provide two situations when such conditions are satisfied, for the rolling of a gyroscopic Chaplygin ball without slipping and twisting over a sphere S^{n-1} (see Remark 7.2). The first situation: if the radii of the sphere and the ball are equal, which is equivalent to the condition $\varepsilon = 1/2$, then the curvature K of \mathcal{D} vanishes (the constraints are holonomic). Since the \mathbf{JK} -term is given by the coupling of the curvature K with the momentum mapping of the $SO(n)$ -action on the configuration space (7.2) (see Remark 3.1), we have $\mathbf{JK} = 0$. The second situation we get when the inertia operator \mathbb{I} of the system, that is, the modified inertia operator \mathbf{I} , is proportional to the identity operator. Then the coupling between the curvature and the momentum mapping vanishes, see (7.5), although the curvature of \mathcal{D} is different from zero. Let us remind that the curvature of the distribution measure the nonholonomicity of the constraints: it is zero if and only if the constraints are holonomic.

These two situations do not require a time reparametrization for a Hamiltonization: the reduced Eq. (7.15) are Hamiltonian with respect to the symplectic form $\mathbf{w} + \rho^*\mathbf{f}$, where \mathbf{w} is the canonical symplectic form (7.17).

For $n = 3$, the condition that the inertia operator \mathbb{I} is proportional to the identity operator is equivalent to the Zhukovskiy condition (6.4). One gets the case of motion of a gyroscopic ball considered by Demchenko (1924), see also Dragović et al. (2023) and Sect. 9.2 below, under an additional nontwisting condition. This motivates us to introduce the following definition of a generalized Demchenko case without twisting in higher dimensions.

Definition 9.1 We say that the ball with a gyroscope satisfies the *Zhukovskiy condition* if the inertia operator \mathbb{I} of the system is proportional to the identity operator. The *generalized Demchenko case without twisting in \mathbb{R}^n* , $n \geq 3$, is a system of a balanced

n -dimensional gyroscopic ball satisfying the Zhukovskiy condition, rolling without slipping and twisting over a fixed $(n - 1)$ -dimensional sphere.

As before, we consider the cotangent bundle $T^*S^{n-1} \subset \mathbb{R}^{2n}\{\gamma, p\}$ realized by the constraints (7.13), \mathbf{w} is the canonical symplectic form on T^*S^{n-1} given with (7.17) and ρ is the canonical projection $\rho : T^*S^{n-1} \rightarrow S^{n-1}$. Now, the magnetic Poisson brackets on $\mathbb{R}^{2n}\{\gamma, p\}$ without the set $\{\gamma = 0\}$ are defined by:

$$\{F, G\}_d = \{F, G\}^\kappa - \frac{\{F, \phi_1\}^\kappa \{G, \phi_2\}^\kappa - \{F, \phi_2\}^\kappa \{G, \phi_1\}^\kappa}{\{\phi_1, \phi_2\}^\kappa}, \tag{9.2}$$

where

$$\{F, G\}^\kappa = \sum_i \left(\frac{\partial F}{\partial \gamma_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial \gamma_i} \right) + \frac{1}{\varepsilon^2} \sum_{i,j} \kappa_{ij} \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial p_j}$$

and ϕ_1, ϕ_2 are given in (7.13). The symplectic leaf given by (7.13) is the cotangent bundle T^*S^{n-1} endowed with the twisted symplectic form $\mathbf{w} + \rho^*\mathbf{f}$.

Let the modified inertia operator $\mathbf{I} = \mathbb{I} + D\text{Id}_{so(n)}$ ($D = ma^2$) be equal to the identity operator on $so(n)$ multiplied by a constant τ . For example, we can take \mathbb{I} given by (8.1) with $\mathbb{A} = \text{diag}(\sqrt{\tau}, \dots, \sqrt{\tau})$. Then the reduced Hamiltonian takes the form

$$h = \frac{\varepsilon^2}{2\tau} \langle p, p \rangle. \tag{9.3}$$

By taking $H = h - \lambda_1\phi_1 - \lambda_2\phi_2$, we obtain the magnetic Hamiltonian flow of the Hamiltonian (9.3) with respect to the Dirac bracket (9.2)

$$\dot{\gamma} = \frac{\partial H}{\partial p} = \frac{\varepsilon^2}{\tau} p - \lambda_2 \gamma, \tag{9.4}$$

$$\dot{p} = -\frac{\partial H}{\partial \gamma} + \frac{1}{\varepsilon^2} \kappa \left(\frac{\partial H}{\partial p} \right) = 2\lambda_1 \gamma + \lambda_2 p + \frac{1}{\tau} \kappa p - \frac{\lambda_2}{\varepsilon^2} \kappa \gamma. \tag{9.5}$$

Here, from the condition that ϕ_1 and ϕ_2 are first integrals of the flow, the Lagrange multipliers can be calculated to get

$$\lambda_1 = \frac{\frac{1}{\tau} \langle p, \kappa \gamma \rangle - \frac{\varepsilon^2}{\tau} \langle p, p \rangle}{2 \langle \gamma, \gamma \rangle}, \quad \lambda_2 = \frac{\varepsilon^2 \langle p, \gamma \rangle}{\tau \langle \gamma, \gamma \rangle}.$$

Proposition 9.1 *The equations of motion of the n -dimensional generalized Demchenko case without twisting are:*

$$\tau \dot{\omega} = [\kappa, \omega] + \lambda_0, \quad \dot{\gamma} = -\varepsilon \omega \gamma, \tag{9.6}$$

where $\kappa \in so(n)$ is a fixed skew-symmetric matrix (9.1) and the Lagrange multiplier $\lambda_0 \in (\mathbb{R}^n \wedge \gamma)^\perp$ is determined from the condition that $\omega \in \mathbb{R}^n \wedge \gamma$. The equations of motion reduce to the magnetic geodesic flow of the Hamiltonian (9.3) with respect to the bracket (9.2)

$$\dot{\gamma} = \frac{\varepsilon^2}{\tau} p, \quad \dot{p} = \frac{1}{\tau} \kappa p + \mu \gamma, \quad \mu = \frac{1}{\tau} \langle p, \kappa \gamma \rangle - \frac{\varepsilon^2}{\tau} \langle p, p \rangle, \tag{9.7}$$

restricted to the cotangent bundle of the sphere (7.13).

The proof follows from (7.8), Eqs. (9.4) and (9.5) restricted to (7.13), and Proposition 7.2.

When $\varepsilon = 1$, we obtain the equations of motion of a gyroscopic ball rolling without slipping and twisting over the plane orthogonal to γ , such that the inertia operator \mathbb{I} of the system is proportional to the identity operator. In dimension $n = 3$ this is the Zhukovskiy problem with an additional nontwisting condition (see Sect. 6).

Let us note that integrable magnetic Hamiltonian systems on S^2 were studied in Saksida (2002), using their relation to a special Neumann system on S^3 . In particular, the reduced problem (9.7) for $n = 3$ was described there by using the Cartan model of the sphere S^2 within the group $SU(2)$. Although the systems (9.7) are quite natural as they are described by the round metric on a sphere with a magnetic field defined by a constant two-form in the ambient space, they have not been studied before for $n > 3$.

Since \mathbb{I} (and equivalently \mathbf{I}) is proportional to the identity matrix, we can consider, without loss of generality, the system in a suitable orthonormal basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of \mathbb{R}^n , such that the skew-symmetric matrix (9.1) takes the form

$$\kappa = \kappa_{12} \mathbf{e}_1 \wedge \mathbf{e}_2 + \kappa_{34} \mathbf{e}_3 \wedge \mathbf{e}_4 + \dots + \kappa_{2[n/2]-1, 2[n/2]} \mathbf{e}_{2[n/2]-1} \wedge \mathbf{e}_{2[n/2]}.$$

9.2 Three-Dimensional Demchenko Case without Twisting

In his PhD thesis (Demchenko 1924) [see also Dragović et al. (2023)] Demchenko studied the rolling of a ball with a gyroscope without slipping over a fixed sphere in \mathbb{R}^3 . He assumed that the ball is dynamically axially symmetric, that axis of gyroscope coincide with symmetry axis of the ball, and that the inertia operators of the ball and the gyroscope satisfy the Zhukovskiy condition (6.4), that is, the inertia operator of the system is proportional to the identity matrix: $\mathbb{I} = \text{diag}(A, A, A)$.

The equations of motion are [see (6.2)]

$$\dot{\mathbf{k}} = (\vec{\mathbf{k}} + \vec{\kappa}) \times \vec{\omega}, \quad \dot{\vec{\gamma}} = \varepsilon \vec{\gamma} \times \vec{\omega}, \tag{9.8}$$

where $\vec{\mathbf{k}} = (A + ma^2)\vec{\omega} - ma^2\langle \vec{\omega}, \vec{\gamma} \rangle \vec{\gamma}$. Demchenko solved the system via elliptic functions.

Now, we add the no-twisting condition on the Demchenko rolling, e.g., we additionally assume that the angular velocity $\vec{\omega}$ belongs to the common tangent plane of

the ball and the sphere in their contact point. The equations of motion are [see (6.5)]

$$\dot{\mathbf{k}} = \vec{\kappa} \times \vec{\omega} + \lambda \vec{\gamma}, \quad \dot{\vec{\gamma}} = \varepsilon \vec{\gamma} \times \vec{\omega}, \tag{9.9}$$

where $\vec{\mathbf{k}} = (A + ma^2)\vec{\omega} = ((A + ma^2)\omega_1, (A + ma^2)\omega_2, (A + ma^2)\omega_3)$ and λ is the Lagrange multiplier of the constraint $\langle \vec{\omega}, \vec{\gamma} \rangle = 0$,

$$\lambda = -\langle \vec{\gamma}, \vec{\kappa} \times \vec{\omega} \rangle.$$

After the identification (8.16), the matrix system (9.6), for $n = 3$, becomes the system (9.9) in the vector notation, where the matrix multiplier λ_0 corresponds to $\lambda \vec{\gamma}$, $\gamma \equiv \vec{\gamma}$, and the parameter τ is equal to $A + ma^2$ (see Remark 8.1).

The reduced equations of motion (9.7) on T^*S^2 , for $\kappa = \kappa_{12}\mathbf{e}_1 \wedge \mathbf{e}_2$, become

$$\begin{aligned} \dot{\gamma}_1 &= \frac{\varepsilon^2}{\tau} p_1, & \dot{p}_1 &= \frac{1}{\tau} \kappa_{12} p_2 + \mu \gamma_1, \\ \dot{\gamma}_2 &= \frac{\varepsilon^2}{\tau} p_2, & \dot{p}_2 &= -\frac{1}{\tau} \kappa_{12} p_1 + \mu \gamma_2, \\ \dot{\gamma}_3 &= \frac{\varepsilon^2}{\tau} p_3, & \dot{p}_3 &= \mu \gamma_3, \\ \mu &= \frac{\kappa_{12}}{\tau} (p_1 \gamma_2 - p_2 \gamma_1) - \frac{\varepsilon^2}{\tau} (p_1^2 + p_2^2 + p_3^2), \end{aligned} \tag{9.10}$$

They are Hamiltonian with respect to the Poisson structure (9.2) and the Hamiltonian is

$$h = \frac{\varepsilon^2}{2\tau} (p_1^2 + p_2^2 + p_3^2).$$

Theorem 9.1 *The reduced equations of the Demchenko case without twisting (9.10) are Liouville integrable on T^*S^2 with the first integrals h, Φ , where*

$$\Phi(\gamma, p) = \gamma_1 p_2 - \gamma_2 p_1 + \frac{\kappa_{12}}{2\varepsilon^2} (\gamma_1^2 + \gamma_2^2).$$

Proof follows by a direct calculation.

The reduced system (9.10) can be solved in elliptic quadratures.

Theorem 9.2 *The reduced equations of the three-dimensional Demchenko case without twisting (9.10) can be explicitly integrated via elliptic functions and their degenerations.*

Proof Instead on the cotangent bundle $T^*S^2\{\gamma, p\}$, we will equivalently integrate the system on the tangent bundle $TS^2\{\gamma, \dot{\gamma}\}$. Let us introduce polar coordinates r, φ by

$$\gamma_1 = r \cos \varphi, \quad \gamma_2 = r \sin \varphi.$$

From the condition $\langle \gamma, \gamma \rangle = 1$, it follows that $r^2 + \gamma_3^2 = 1$, while $\langle \gamma, \dot{\gamma} \rangle = 0$ is identically satisfied. By differentiating $r^2 + \gamma_3^2 = 1$ with respect to time, one gets $\dot{\gamma}_3^2 = \frac{r^2}{1-r^2} \dot{r}^2$.

In the new coordinates, using the last relation, the first integrals can be rewritten as:

$$h = \frac{\tau}{2\varepsilon^2} \left(\dot{r}^2 + r^2 \dot{\varphi}^2 + \frac{r^2 \dot{r}^2}{1-r^2} \right), \tag{9.11}$$

$$\Phi = \frac{\tau}{\varepsilon^2} r^2 \dot{\varphi} + \frac{\kappa_{12}}{2\varepsilon^2} r^2. \tag{9.12}$$

Note that $\tau > 0$. We also assume $h > 0$ since $h = 0$ corresponds to the equilibrium positions.

From (9.12), we get

$$\dot{\varphi} = \frac{2\varepsilon^2 \Phi - \kappa_{12} r^2}{2\tau r^2}, \tag{9.13}$$

and, by plugging into (9.11), it follows

$$\dot{r}^2 = \left(\frac{\varepsilon^2}{\tau^2} (2h\tau + \kappa_{12}\Phi) - \frac{\kappa_{12}^2}{4\tau^2} r^2 - \frac{\varepsilon^4 \Phi^2}{\tau^2} \frac{1}{r^2} \right) (1-r^2).$$

Introducing $u = r^2$, one derives

$$\begin{aligned} \dot{u}^2 &= Q_3(u), \\ Q_3(u) &:= \frac{\kappa_{12}^2}{\tau^2} (u-1) \left(u^2 - \frac{4\varepsilon^2}{\kappa_{12}} (2h\tau + \kappa_{12}\Phi)u + \frac{4\varepsilon^4 \Phi^2}{\kappa_{12}^2} \right) \\ &= \frac{\kappa_{12}^2}{\tau^2} (u-1)(u-u_1)(u-u_2). \end{aligned} \tag{9.14}$$

Thus, r^2 can be expressed as an elliptic function (or its degenerations) of time. Using $\gamma_3^2 = 1 - r^2$, one gets γ_3 , and from (9.13) one finds φ after an integration. \square

Notice that the polynomial Q_3 (9.14) always has $u = 1$ as a root. Observe also:

$$Q_3(0) = -\frac{4\varepsilon^4 \Phi^2}{\tau^2} < 0.$$

From Vieta’s formulas, it follows that $u_1 u_2 > 0$, or in other words, the remaining two roots u_1, u_2 of Q_3 are of the same sign. Having in mind that $0 \leq u \leq 1$, the real solutions, for $u_1 < u_2$, corresponds to the following cases:

- (A) $0 < u_1 < u_2 < 1$; Case (A) happens when the discriminant of the polynomial $Q_2(u) = (u - u_1)(u - u_2)$ is greater than zero, the minimum of $Q_2(u)$ is between

0 and 1, and $Q_2(1) > 0$. This yields conditions:

$$\begin{aligned} h\tau + \kappa_{12}\Phi &> 0, \\ 2h\tau + \kappa_{12}\Phi &< \frac{\kappa_{12}^2}{2\varepsilon^2}, \\ 2h\tau + \kappa_{12}\Phi - \varepsilon^2\Phi &< \frac{\kappa_{12}^2}{4\varepsilon^2} \end{aligned}$$

(B) $0 < u_1 < 1 < u_2$. Case (B) happens when $Q_2(1) < 0$, that is

$$2h\tau + \kappa_{12}\Phi - \varepsilon^2\Phi > \frac{\kappa_{12}^2}{4\varepsilon^2}$$

In both cases r belongs to an annulus:

$$\text{Case (A) } \sqrt{u_1} \leq r \leq \sqrt{u_2}; \quad \text{Case (B) } \sqrt{u_1} \leq r \leq 1.$$

When the discriminant of the polynomial Q_3 (9.14) vanishes, the corresponding elliptic functions degenerate. It happens if $u_1 = u_2$, or when one of the roots u_1, u_2 is equal to 1. Direct calculations show that the discriminant of the polynomial Q_3 vanishes when

$$h\tau + \kappa_{12}\Phi = 0, \quad \text{or} \quad 2h\tau + \kappa_{12}\Phi - \varepsilon^2\Phi = \frac{\kappa_{12}^2}{4\varepsilon^2}.$$

The first case corresponds to the condition that the discriminant of Q_2 is zero, and the second case corresponds to $Q_2(1) = 0$.

9.3 The Generalized Demchenko Case without Twisting in \mathbb{R}^4 . A Qualitative Analysis of the Solutions

In dimension four, the equations of motion of generalized Demchenko case without twisting reduce to Hamiltonian equations with respect to the Poisson structure (9.2) on the cotangent bundle $T^*S^3 \subset \mathbb{R}^4\{\gamma, p\}$ of the three-dimensional sphere realized by $\langle \gamma, \gamma \rangle = 1, \langle \gamma, p \rangle = 0$. Let

$$\kappa = \kappa_{12}\mathbf{e}_1 \wedge \mathbf{e}_2 + \kappa_{34}\mathbf{e}_3 \wedge \mathbf{e}_4.$$

Eq. (9.7) are:

$$\begin{aligned} \dot{\gamma}_1 &= \frac{\varepsilon^2}{\tau}p_1, & \dot{p}_1 &= \frac{1}{\tau}\kappa_{12}p_2 + \mu\gamma_1, \\ \dot{\gamma}_2 &= \frac{\varepsilon^2}{\tau}p_2, & \dot{p}_2 &= -\frac{1}{\tau}\kappa_{12}p_1 + \mu\gamma_2, \\ \dot{\gamma}_3 &= \frac{\varepsilon^2}{\tau}p_3, & \dot{p}_3 &= \frac{1}{\tau}\kappa_{34}p_4 + \mu\gamma_3, \end{aligned}$$

$$\begin{aligned} \dot{\gamma}_4 &= \frac{\varepsilon^2}{\tau} p_4, & \dot{p}_4 &= -\frac{1}{\tau} \kappa_{34} p_3 + \mu \gamma_4, \\ \mu &= \frac{1}{\tau} \left(\kappa_{12} (p_1 \gamma_2 - p_2 \gamma_1) + \kappa_{34} (p_3 \gamma_4 - p_4 \gamma_3) \right) - \frac{\varepsilon^2}{\tau} (p_1^2 + p_2^2 + p_3^2 + p_4^2). \end{aligned} \tag{9.15}$$

The Hamiltonian is

$$h = \frac{\varepsilon^2}{2\tau} (p_1^2 + p_2^2 + p_3^2 + p_4^2).$$

Theorem 9.3 *The reduced equations of generalized Demchenko case for $n = 4$ (9.15) are Liouville integrable on T^*S^3 with the three first integrals h , Φ_{12} , and Φ_{34} in involution, where*

$$\begin{aligned} \Phi_{12}(p, \gamma) &= \gamma_1 p_2 - \gamma_2 p_1 + \frac{\kappa_{12}}{2\varepsilon^2} (\gamma_1^2 + \gamma_2^2), \\ \Phi_{34}(p, \gamma) &= \gamma_3 p_4 - \gamma_4 p_3 + \frac{\kappa_{34}}{2\varepsilon^2} (\gamma_3^2 + \gamma_4^2). \end{aligned}$$

The proof follows by a direct calculation.

It is well known that the question of integrability for a Hamiltonian system is distinct from the problem of its explicit integration.

The reduced equations of generalized Demchenko case without twisting in \mathbb{R}^4 can be solved via elliptic functions by quadratures, similarly to their three-dimensional counterpart, see Theorem 9.2 above.

Theorem 9.4 *The reduced equations of generalized Demchenko case without twisting for $n = 4$ (9.15) can be explicitly integrated via elliptic functions and their degenerations.*

Proof As in dimension $n = 3$, instead on the cotangent bundle $T^*S^2\{\gamma, p\}$, we will integrate the system on the tangent bundle $TS^3\{\gamma, \dot{\gamma}\}$. Let us introduce new coordinates $\rho_1, \rho_3, \varphi_1, \varphi_3$ by

$$\gamma_1 = \rho_1 \cos \varphi_1, \quad \gamma_2 = \rho_1 \sin \varphi_1, \quad \gamma_3 = \rho_3 \cos \varphi_3, \quad \gamma_4 = \rho_3 \sin \varphi_3.$$

From the condition $\langle \gamma, \gamma \rangle = 1$ it follows that $\rho_1^2 + \rho_3^2 = 1$, while $\langle \gamma, \dot{\gamma} \rangle = 0$ is identically satisfied. In the new coordinates the first integrals become

$$\begin{aligned} h &= \frac{\tau}{2\varepsilon^2} \left(\dot{\rho}_1^2 + \rho_1^2 \dot{\varphi}_1^2 + \dot{\rho}_3^2 + \rho_3^2 \dot{\varphi}_3^2 \right), \\ \Phi_{12} &= \frac{\tau}{\varepsilon^2} \rho_1^2 \dot{\varphi}_1 + \frac{\kappa_{12}}{2\varepsilon^2} \rho_1^2, \\ \Phi_{34} &= \frac{\tau}{\varepsilon^2} \rho_3^2 \dot{\varphi}_3 + \frac{\kappa_{34}}{2\varepsilon^2} \rho_3^2. \end{aligned} \tag{9.16}$$

Since the first integrals Φ_{12} and Φ_{34} depend on $\rho_1, \dot{\varphi}_1$ and $\rho_3, \dot{\varphi}_3$ respectively, $\dot{\varphi}_1$ can be expressed as a function of ρ_1 and values of these first integrals; similarly, $\dot{\varphi}_3$ can be expressed as a function of ρ_3 and values of these first integrals:

$$\dot{\varphi}_1 = \frac{2\varepsilon^2\Phi_{12} - \kappa_{12}\rho_1^2}{2\tau\rho_1^2}, \quad \dot{\varphi}_3 = \frac{2\varepsilon^2\Phi_{34} - \kappa_{34}\rho_3^2}{2\tau\rho_3^2}. \tag{9.17}$$

By differentiating the relation $\rho_1^2 + \rho_3^2 = 1$ with respect to time, we get

$$\dot{\rho}_3^2 = \frac{\rho_1^2}{1 - \rho_1^2} \dot{\rho}_1^2.$$

Using (9.17), the last equality, and the expression for the first integral h from (9.16), one obtains

$$\dot{\rho}_1^2 = (1 - \rho_1^2) \frac{2\varepsilon^2 h}{\tau} - \frac{(2\varepsilon^2\Phi_{34} - \kappa_{34} + \kappa_{34}\rho_1^2)^2}{4\tau^2} - \frac{1 - \rho_1^2}{\rho_1^2} \frac{(2\varepsilon^2\Phi_{12} - \kappa_{12}\rho_1^2)^2}{4\tau^2}.$$

Introducing $u = \rho_1^2$, it follows

$$\dot{u}^2 = P_3(u). \tag{9.18}$$

Here, P_3 is a polynomial in u of the degree not greater than three:

$$P_3(u) := a_0u^3 + a_1u^2 + a_2u + a_3,$$

where

$$\begin{aligned} a_0 &= \frac{\kappa_{12}^2 - \kappa_{34}^2}{\tau^2}, & a_3 &= -\frac{4\varepsilon^4\Phi_{12}^2}{\tau^2}, \\ a_1 &= -\frac{8\varepsilon^2h}{\tau} - \frac{2\kappa_{34}}{\tau^2}(2\varepsilon^2\Phi_{34} - \kappa_{34}) - \frac{\kappa_{12}^2}{\tau^2} - \frac{4\varepsilon^2\kappa_{12}\Phi_{12}}{\tau^2}, \\ a_2 &= \frac{8\varepsilon^2h}{\tau} - \frac{(2\varepsilon^2\Phi_{34} - \kappa_{34})^2}{\tau^2} + \frac{4\varepsilon^2\kappa_{12}\Phi_{12}}{\tau^2} + \frac{4\varepsilon^4\Phi_{12}^2}{\tau^2}. \end{aligned}$$

Therefore, from equation (9.18), integrating, one gets ρ_1^2 as an elliptic function or a degeneration of an elliptic function, depending on the degree and composition of zeros of the polynomial $P_3(u)$. We get ρ_3 from the algebraic equation $\rho_3^2 = 1 - \rho_1^2$. Finally, the variables φ_1, φ_3 can be obtained by quadratures from (9.17). \square

Let us express the variable ρ_1^2 in terms of the Weierstrass \wp -function in a generic case: $\kappa_{12}^2 \neq \kappa_{34}^2$ and the polynomial $P_3(u)$ has all roots distinct. Introducing z such that

$$u = \frac{4}{a_0}z - \frac{a_1}{3a_0},$$

equation (9.18) takes the form

$$\dot{z}^2 = 4z^3 - g_2z - g_3, \tag{9.19}$$

where

$$g_2 = \frac{a_1^2}{12} - \frac{a_0a_2}{4}, \quad g_3 = \frac{a_0a_1a_2}{4} - \frac{a_1^3}{216} - \frac{a_0^2a_3}{16}.$$

By integration of (9.19), we get

$$\int_z^\infty \frac{d\xi}{\sqrt{4\xi^3 - g_2\xi - g_3}} - \int_{z_0}^\infty \frac{d\xi}{\sqrt{4\xi^3 - g_2\xi - g_3}} = \pm(t - t_0).$$

Finally, using the Weierstrass \wp -function [see for example Akhiezer (1990)], one obtains

$$z = \wp(A \pm (t - t_0)), \quad z_0 = \wp(A).$$

Now, we are going to provide a qualitative analysis of the solutions of the generalized Demchenko case without twisting in \mathbb{R}^4 , obtained in Theorem 9.4.

Case A. Let us consider first the case $\kappa_{12}^2 \neq \kappa_{34}^2$. Then $P_3(u)$ is a degree three polynomial. The coordinates ρ_1, φ_1 and ρ_3, φ_3 are polar coordinates on the projections of the sphere $\langle \gamma, \gamma \rangle = 1$ to the coordinate planes Oe_1e_2 and Oe_3e_4 , respectively. Hence, ρ_1 and ρ_3 , and consequently u can take values between 0 and 1.

Since

$$P_3(0) = -\frac{4\epsilon^4\Phi_{12}^2}{\tau^2} < 0,$$

and

$$P_3(1) = -\frac{4\epsilon^4\Phi_{34}^2}{\tau^2} < 0,$$

one concludes that on interval $(0, 1)$ the polynomial $P_3(u)$ has (i) no real roots; (ii) two distinct real roots; or (iii) one double real root.

- (i) If the number of real roots is zero, then the polynomial $P_3(u)$ takes negative values on the whole interval $(0, 1)$. Thus, the case (i) does not correspond to a real motion.
- (ii) In the case (ii) when the polynomial $P_3(u)$ has two distinct real roots $u_1 < u_2$ on the interval $(0, 1)$, the projection of a trajectory to the Oe_1e_2 and Oe_3e_4 planes belong, respectively, to the annuli

$$\sqrt{u_1} \leq \rho_1 \leq \sqrt{u_2} \quad \text{and} \quad \sqrt{1 - u_2} \leq \rho_3 \leq \sqrt{1 - \rho_1^2} \leq \sqrt{1 - u_1}.$$

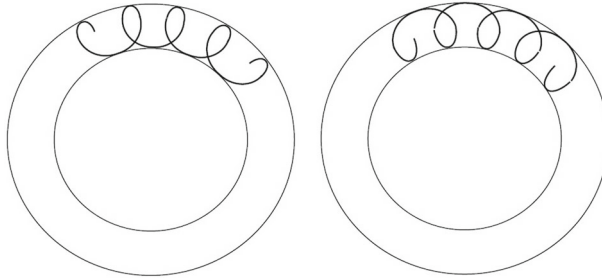


Fig. 2 The case $u_1 < \hat{u} < u_2$

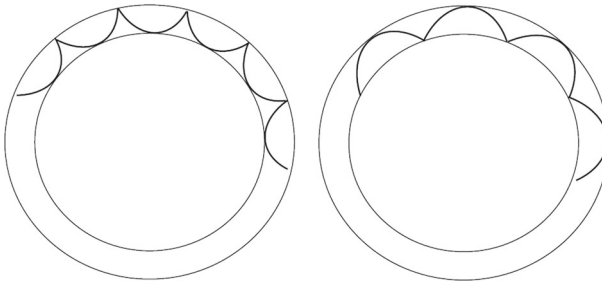
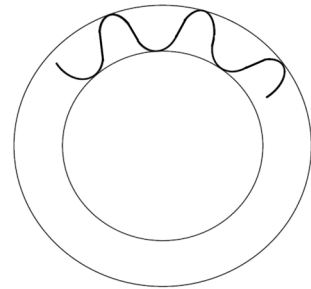


Fig. 3 The cases $\hat{u} = u_2$ (left) and $\hat{u} = u_1$ (right)

Fig. 4 The case when \hat{u} does not belong to the interval $[u_1, u_2]$



There are three types of the trajectories in this case. Let

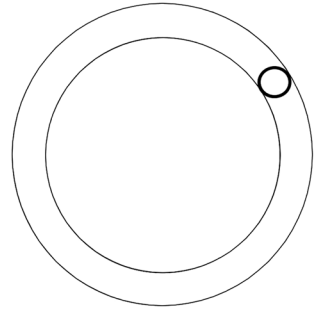
$$\hat{u} = \frac{2\varepsilon^2\Phi_{12}}{\kappa_{34}}.$$

If \hat{u} belongs to (u_1, u_2) then $\dot{\varphi}_1$ changes the sign and trajectories are presented in Fig. 2. If \hat{u} is equal to u_1 or u_2 , then the trajectories are presented in Fig. 3. Otherwise, the trajectories are presented in Fig. 4.

(iii) The case of a double root $u_1 = u_2$ corresponds to the stationary motion

$$\begin{aligned} \rho_1 &= \text{const}, & \varphi_1 &= \alpha_1 t + \varphi_{10}, \\ \rho_3 &= \sqrt{1 - \rho_1^2} = \text{const}, & \varphi_3 &= \alpha_3 t + \varphi_{20}, \end{aligned}$$

Fig. 5 A case that does not correspond to a possible motion.



where

$$\alpha_1 = \frac{2\varepsilon^2\Phi_{12} - \kappa_{12}u_1}{2\tau u_1} = \text{const}, \quad \alpha_3 = \frac{2\varepsilon^2\Phi_{34} - \kappa_{34}(1 - u_1)}{2\tau(1 - u_1)} = \text{const}.$$

From the equations of motion (9.15) it follows that the constants α_1 and α_3 should satisfy:

$$\kappa_{12}\alpha_1 - \kappa_{34}\alpha_3 + \tau(\alpha_1^2 - \alpha_3^2) = 0.$$

Since the roots u_1 and u_2 of the polynomial $P_3(u)$ coincide, the discriminant of the polynomial $P_3(u)$ is equal to zero.

As we mentioned, in the case when φ_1 changes the sign, the trajectories are presented in Fig. 2. In both cases, if we consider φ_1 as a function on the universal covering of S^1 , it is an unbounded function of time: in one case it goes to plus infinity, while in the other case it goes to minus infinity, when t goes to infinity.

We come to a natural question: *is there any case when φ_1 is a bounded or, in particular, a periodic function of time?*

In other words, are there conditions which would generate Fig. 5 as a limit case of those presented in Fig. 2. The answer is negative, as one concludes from the following:

Proposition 9.2 *If $\kappa_{12} \neq 0$, then φ_1 is unbounded function of time.*

Proof From (9.17) we have

$$\dot{\varphi}_1 = \frac{2\varepsilon^2\Phi_{12}}{2\tau u} - \frac{\kappa_{12}}{2\tau}.$$

Since $\kappa_{12} \neq 0$, the second addend is a constant, while the first one is periodic in time. So φ_1 is unbounded function of time. □

Case B. In the case $\kappa_{34} = \pm\kappa_{12}$, the coefficient of u^3 in the polynomial $P_3(u)$ is zero. Hence $P_3(u)$ is at most a quadratic polynomial in u . Qualitative pictures of the trajectories are the same as before. They are presented in Figs. 2, 3, and 4 with an important difference: now the solutions are not elliptic functions of time.

In the case when $u_1 = u_2$, the discriminant of the polynomial P_3 vanishes. This leads to the stationary motion

$$\rho_1 = \text{const}, \quad \rho_3 = \sqrt{1 - \rho_1^2} = \text{const}, \quad \varphi_1 = \alpha_1 t + \varphi_{10}, \quad \varphi_3 = \alpha_3 t + \varphi_{20}.$$

As in the case A, the constants α_1 and α_3 are not independent. If $\kappa_{12} = \kappa_{34}$ we have $\alpha_1 = \alpha_3$, or $\alpha_1 + \alpha_3 = \kappa_{12}/\tau$. When $\kappa_{34} = -\kappa_{12}$, then $\alpha_1 = -\alpha_3$ or $\alpha_1 + \alpha_3 = \kappa_{12}/\tau$.

Remark 9.1 Let us remark that in the dynamics of the Lagrange top in absence of gravity there exist a situation similar to the one mentioned before Proposition 9.2 (see Fig. 5). This system can also be seen as a symmetric Euler top. There is a stationary motion about the axis of symmetry that is in a nonvertical position. In other words, the system of equations admits the following particular solution: the nutation angle $\theta = \theta_0 \in (0, \pi/2)$ is a constant different from zero, the precession angle φ is constant, and the angle of intrinsic rotation ψ is a linear function of time. If in an initial moment of time one chooses θ close to θ_0 , then the nutation and precession will be periodic functions of time, and the axis of symmetry will uniformly rotate about the vector of angular momentum, which is fixed in the space. See Arnold (1974) for more details.

What is going on in with the Lagrange top with the presence of gravity? Can the precession angle be a periodic function on the universal covering of S^1 ?

It may look like the mentioned stationary solution exists in the presence of gravity as well. The three first integrals (the energy integral, the projection of the angular momentum on the vertical axis, the projection of the angular momentum on the axis of symmetry) are constant functions on the solution. However, from the equations of motion one gets that the stationary motion about the axis of symmetry is possible only when $\theta = 0$ or $\theta = \pi$. Based on that, one can speculate that a solution of the Lagrange top with the presence of gravity having the precession angle as a bounded or periodic function of time does not exist. A rigorous proof of that observation was provided by Hadamard (1895). Although the Lagrange top was widely studied since then, with dozens of volumes devoted to it, this Hadamard's result is very hard to find. A nice exception is a recent short note (Zubelevich and Salnikova 2018).

Acknowledgements We are very grateful to the referees for valuable remarks that significantly helped us to improve the exposition, and in particular for providing us an example which led to Remark 5.2. We thank Viswanath Ramakrishna for reading the manuscript and providing a feedback. This research has been supported by the Project No. 7744592 MEGIC "Integrability and Extremal Problems in Mechanics, Geometry and Combinatorics" of the Science Fund of Serbia, Mathematical Institute of the Serbian Academy of Sciences and Arts and the Ministry for Education, Science, and Technological Development of Serbia, and the Simons Foundation Grant No. 854861.

Declarations

Conflict of interest The authors have no conflict of interest.

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