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Internalization and enrichment via spans and matrices in a tricategory

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Abstract
We introduce categories $\mathcal{M}$ and $\mathcal{S}$ internal in the tricategory Bicat_3 of bicategories, pseudofunctors, pseudonatural transformations and modifications, for matrices and spans in a 1-strict tricategory V. Their horizontal tricategories are the tricategories of matrices and spans in V. Both the internal and the enriched constructions are tricategorifications of the corresponding constructions in 1-categories. Following Fiore et al. (J Pure Appl Algebra 215(6):1174–1197, 2011), we introduce monads and their vertical morphisms in categories internal in tricategories. We prove an equivalent condition for when the internal categories for matrices $\mathcal{M}$ and spans $\mathcal{S}$ in a 1-strict tricategory V are equivalent, and deduce that in that case their corresponding categories of (strict) monads and vertical monad morphisms are equivalent, too. We prove that the latter categories are isomorphic to those of categories enriched and discretely internal in V, respectively. As a by-product of our tricategorical constructions, we recover some results from Femić (Enrichment and internalization in tricategories, the case of tensor categories and alternative notion to intercategories. arXiv:2101.01460v2). Truncating to 1-categories, we recover results from Cottrell et al. (Tbilisi Math J 10(3):239–254, 2017) and Ehresmann and Ehresmann (Cah Topol Géom Differ Catég 19/4:387–443, 1978) on the equivalence of enriched and discretely internal 1-categories.

Keywords Bicategory · Tricategory · Double category · Internal and enriched category · Monads

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1 Introduction

When defining a (small) category, we can equivalently require a set of objects and a set of arrows, or instead require a set of objects and, for every pair of objects, a set of arrows between them. The first notion is that of small category which generalizes to the notion of internal category, and indeed a small category is an internal category in Set, the category of sets and functions. The second notion resembles that of locally small category (although in this case even the collection of objects is small) which generalizes to the notion of enriched category, and indeed a locally small category is a category enriched in Set (with the Cartesian monoidal structure). Clearly, though, the two above definitions of a category are equivalent, so it is to be expected that the notions of internal and enriched categories must be strictly related as well.

As recalled in Sect. 4, for a Cartesian closed category $\mathcal{V}$ with finite limits and small coproducts, we can construct the bicategory $\mathcal{V}\text{-}Mat$ of matrices on $\mathcal{V}$ and the bicategory $Span_d(\mathcal{V})$ of discrete spans in $\mathcal{V}$, i.e., of spans over discrete objects of $\mathcal{V}$, where an object is discrete if it is a set-indexed coproduct of the terminal object. Then, there is an adjunction between $Span_d(\mathcal{V})$ and $\mathcal{V}\text{-}Mat$, which becomes a biequivalence when $\mathcal{V}$ is extensive. Observe that internal categories are monads in $Span(\mathcal{V})$, and internal categories with a discrete object of objects are monads in $Span_d(\mathcal{V})$. Moreover, enriched categories are monads in $\mathcal{V}\text{-}Mat$. This fact is hinted at in [6], where it is observed that the monad morphisms between monads on the bicategories $Span_d(\mathcal{V})$ and $\mathcal{V}\text{-}Mat$ are not, respectively, functors of internal categories in $\mathcal{V}$ or functors of enriched $\mathcal{V}$-categories. The authors observe that for that to be the case, one would have to use 2-categorical structures. Though they do not pursue this direction, in favor of a 1-categorical approach.

In [11], it was observed that many distinct mathematical structures can be considered as monads in appropriate 2-categories, though their morphisms are not monad morphisms. This was the motivation for the authors to switch to the *double category* instead of a 2-category and then to consider the *double category of monads in that double category*, rather than the well-known 2-category of monads in a 2-category. Besides, it is clear that oftentimes in bicategories the existence of certain additional structures is assumed, but it is only in the corresponding pseudo-double category that these additional data are indeed contained (think of the bicategory of algebras and their bimodules). As argued in [7, 16], these additional data should not be neglected, and it is often more convenient to consider the setting of internal categories.

In view of these ideas, our goal in the present article is twofold. In the first place, we carry out the construction hinted at in [6] to make a 2-categorical proof of the characterization of equivalences of bicategories of matrices and spans in a 1-category $\mathcal{V}$, on the one hand, and of the categories enriched and discretely internal in $\mathcal{V}$, on the other. We do this in Proposition 4.6 and Corollary 4.8, using constructions of [11]. Namely, in [11, Example 2.1] a pseudo-double category $\mathrm{Span}(\mathcal{V})$ of spans in $\mathcal{V}$ was introduced, whose horizontal bicategory is precisely the bicategory $Span(\mathcal{V})$. Then, a double category $\mathrm{Mnd}(\mathbb{D})$ of monads in a double category $\mathbb{D}$ is introduced in [11, Definition 2.4], so that when $\mathbb{D} = \mathrm{Span}(\mathcal{V})$, the vertical 1-cells in $\mathrm{Mnd}(\mathrm{Span}(\mathcal{V}))$ are morphisms of internal categories in $\mathcal{V}$. Inspired by this, we define the pseudo-double category $\mathrm{Span}_d(\mathcal{V})$ by modifying accordingly $\mathrm{Span}(\mathcal{V})$ and introduce a pseudo-double

category $\mathcal{V}$-Mat that allows us to extend the biequivalence of bicategories from [6] to an equivalence of pseudo-double categories. Then, consequently the equivalence of categories discretely internal and enriched in $\mathcal{V}$ is recovered, under the corresponding conditions.

Our second challenge was to generalize the latter construction to tricategories, whose 1-cells obey strict associativity and unitality laws. We call such tricategories *1-strict*. For a 1-strict tricategory V, we first set up the notions of tricategorical limits, via the weighted 3-limits, Sect. 3.1. For V having 3-pullbacks, 3-products and 3-coproducts, we then construct tricategorical analog of the pseudo-double categories of matrices and spans $\mathcal{V}$-Mat and $\mathrm{Span}_d(\mathcal{V})$ from above. In the terminology of [16], they are (1×2)-categories, respectively $\mathcal{M}$ and $\mathcal{S}$. This means that they are internal (bi)categories in a 1-strict tricategory, in our case the tricategory Bicat_3 of bicategories, pseudofunctors, pseudonatural transformations and modifications. (Internal categories in 1-strict tricategories were introduced in [10].) The (vertical) bicategory of objects in both cases is Cat_2, the 2-category of categories, while the bicategories of morphisms are suitable ones made for matrices and spans in V. We also construct a lax and a colax internal functor in Bicat_3 between these two (1×2)-categories $\mathcal{M}$ and $\mathcal{S}$.

Following the idea of [11], we introduce monads in a tricategory, and then we define a monad in a (1×2)-category $\mathbb{V}$ as a monad in the horizontal tricategory $\mathcal{H}(\mathbb{V})$ of $\mathbb{V}$. We deduce, analogously to the classical 1-categorical case, that (strict) monads in the (1×2)-categories of matrices $\mathcal{M}$ and spans $\mathcal{S}$ are categories enriched and discretely internal in V, in the sense of [10, Definitions 8.1 and 6.2], respectively. We then introduce *vertical morphisms* of monads in $\mathbb{V}$ and the corresponding category of monads $\mathrm{Mnd}(\mathbb{V})$. We prove that if (1×2)-categories $\mathbb{V}_1$ and $\mathbb{V}_2$ are equivalent, then their respective categories of strict monads $\mathrm{Mnd}(\mathbb{V}_1)$ and $\mathrm{Mnd}(\mathbb{V}_2)$ are equivalent, too (Proposition 8.5). We prove equivalence conditions for the (1×2)-categories $\mathcal{M}$ and $\mathcal{S}$ in the spirit of [6] (Corollary 8.11) and deduce that under those conditions their categories of monads $\mathrm{Mnd}(\mathcal{M})$ and $\mathrm{Mnd}(\mathcal{S})$ are equivalent, too (Corollary 8.14). Consequently, we obtain the equivalence of categories discretely internal and enriched in V, under those conditions. On the other hand, we also prove a sufficient condition to have a functor from the category of enriched to that of internal categories in V (Proposition 8.13). This recovers [10, Proposition 8.4] which here we obtain as a consequence. Truncating to 1-categories, our results recover those from [6, Section 4] and [9, Appendix].

We point out that although in our Corollary 8.11 an equivalent condition for the (1×2)-categories $\mathcal{M}$ and $\mathcal{S}$ to be equivalent is stated through a triequivalence trifunctor $\coprod : V^{\underline{\mathcal{D}}} \to V/(\underline{\mathcal{D}} \bullet 1)$, by the construction of $\coprod$ and its (co)domain tricategories it is essentially a two-dimensional functor between sub-bicategories of the bicategories of morphisms D_1 and C_1 constituting internal categories $\mathcal{M}$ and $\mathcal{S}$. Thus, we indeed relate a tricategory triequivalence with a bicategory biequivalence. In this sense, it is a proper generalization to a higher dimension of one of the two main results in [6] (recalled in our Proposition 4.3), by which the bicategories of matrices $\mathcal{V}$-Mat and discrete spans $Span_d(\mathcal{V})$ are biequivalent if and only if a one-dimensional functor $\mathcal{V}^I \xrightarrow{\coprod} \mathcal{V}/(I \bullet 1)$ is an equivalence. Mind that although we use the same notation,

our pseudofunctors Int and En act between bicategories D_1 and C_1 (different than the bicategories $\mathcal{V}$-Mat and $Span_d(\mathcal{V})$!) as parts of internal functors in Bicat_3.

The paper is structured as follows. In the second section, we set up conventions for notations in tricategories. In the third section, we introduce weighted 3-limits and specialize to 3-pullbacks and 3-(co)products. In Sect. 4, we present the characterization from [6] of the equivalence of bicategories of spans and matrices in 1-categories in terms of extensivity, we extend it to their respective double categories, and considering the double categories of monads in the latter double categories, we deduce that if the former double categories are equivalent, then so are the latter. In Sects. 5 and 6, we construct the (1×2)-categories $\mathcal{M}$ and $\mathcal{S}$ of matrices and spans in V, respectively, and in Sect. 7 we define the (co)lax functors between them, defining pseudofunctors between their respective bicategories of morphisms D_1 and C_1. In the last section, we introduce monads in tricategories and the category of monads and vertical monad morphisms in (1×2)-categories. We also give an equivalent condition for $\mathcal{M}$ and $\mathcal{S}$ to be equivalent; consequently, their categories of monads $\mathrm{Mnd}(\mathcal{M})$ and $\mathrm{Mnd}(\mathcal{S})$ and categories enriched and discretely internal in V are equivalent under those conditions.

2 Preliminaries: notational conventions and computing with 2-cells

We assume that the reader is familiar with the definition of a tricategory. For the reference, we recommend [13, 14]. By a *trifunctor* we mean a trihomomorphism from [13, Section 3]. We also assume that the reader is familiar with internal category theory. To clarify the terminology, we use equivalently "internal category (in V)" and "category internal in V" where V is the ambient (weak) n-category, for $n = 1, 2, 3$. We do the same regarding enrichment. Given an internal category $\mathcal{C}$ (in any of the mentioned dimensions), we use the standard notations of C_0 for the "object of objects" and C_1 for the "object of morphisms."

Throughout V will be a *1-strict* tricategory, meaning that its 1-cells obey strict associativity and unitality laws, and also *2v-strict*, meaning that the associativity and unitality laws for the vertical composition of 2-cells will hold strictly. Composition of 1-cells and consequently horizontal composition of 2- and 3-cells we denote by $\otimes$, where $y \otimes x$ means that first x is applied, then y. For the horizontal composition, we will also use more intuitive notation: $[x|y] = y \otimes x$. Vertical composition of 2- and 3-cells we denote by $\frac{\alpha}{\beta}$, read from top to bottom. Transversal composition of 3-cells we denote by $\cdot$ and read it from right to left.

We are going to use diagrammatic and formulaic notation. When we write 2-cells in the form of square diagrams, we will usually read them as in the first diagram below, but sometimes also as in the right one:

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{\alpha}"] & \cdot \arrow[d, "p"] \arrow[dl, phantom, "\Swarrow"] \\ \cdot \arrow[r, "a'"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{\overline{\alpha}}"] & \cdot \arrow[d, "p"] \arrow[dl, phantom, "\Nwarrow"] \\ \cdot \arrow[r, "a'"'] & \cdot \end{tikzcd}$$

This means that $\alpha : pa \Rightarrow a'm$ and $\overline{\alpha} : a'm \Rightarrow pa$. Usually, we will suppress the double arrow labels, and we will write explicitly which order of mapping we mean.

Such written 2-cells can be concatenated both horizontally and vertically. Concatenating them horizontally with the two directions of mapping corresponds to the formulaic notations as follows:

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{\alpha}\ \Swarrow"] & \cdot \arrow[r, "b"] \arrow[d, "p"'] \arrow[dr, phantom, "\boxed{\beta}\ \Swarrow"] & \cdot \arrow[d, "q"] \\ \cdot \arrow[r, "a'"'] & \cdot \arrow[r, "b'"'] & \cdot \end{tikzcd} = \frac{\beta \otimes \mathrm{Id}_a}{\mathrm{Id}_{b'} \otimes \alpha} \qquad \begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{\overline{\alpha}}\ \Nearrow"] & \cdot \arrow[r, "b"] \arrow[d, "p"'] \arrow[dr, phantom, "\boxed{\overline{\beta}}\ \Nearrow"] & \cdot \arrow[d, "q"] \\ \cdot \arrow[r, "a'"'] & \cdot \arrow[r, "b'"'] & \cdot \end{tikzcd} = \frac{\mathrm{Id}_{b'} \otimes \overline{\alpha}}{\overline{\beta} \otimes \mathrm{Id}_a} \tag{1}$$

and for vertical concatenation we have:

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{\alpha}\ \Swarrow"] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, "a'"] \arrow[d, "n"'] \arrow[dr, phantom, "\boxed{\gamma}\ \Swarrow"] & \cdot \arrow[d, "r"] \\ \cdot \arrow[r, "a''"'] & \cdot \end{tikzcd} = \frac{\mathrm{Id}_r \otimes \alpha}{\gamma \otimes \mathrm{Id}_m} \qquad \begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{\overline{\alpha}}\ \Nearrow"] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, "a'"] \arrow[d, "n"'] \arrow[dr, phantom, "\boxed{\gamma}\ \Nearrow"] & \cdot \arrow[d, "r"] \\ \cdot \arrow[r, "a''"'] & \cdot \end{tikzcd} = \frac{\overline{\gamma} \otimes \mathrm{Id}_m}{\mathrm{Id}_r \otimes \overline{\alpha}}. \tag{2}$$

The two ways of considering 2-cells with respect to the direction of mapping, are clearly inverse to each other. Nevertheless, since depending on an occasion we will use both of them, we record these properties. The following simple rule can now easily be deduced, it will be useful to us in further computations.

Lemma 2.1 *Let V be a 1-strict tricategory. Given equivalence 2-cells α, α' with their quasi-inverses $\alpha^{-1}, (\alpha')^{-1}$ and consider them as in the diagrams:* $\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{\alpha}\ \Swarrow"] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, "a'"'] & \cdot \end{tikzcd}$ *and* $\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{\alpha^{-1}}\ \Nearrow"] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, "a'"'] & \cdot \end{tikzcd}$ *and similarly for $\alpha' : p'a \Rightarrow a'm'$. Moreover, suppose that we are given 2-cells $\lambda : m \Rightarrow m'$ and $\rho : p \Rightarrow p'$. Then, to give a 3-cell*

$$\begin{tikzcd} \cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "m'"'] \arrow[dr, phantom, "\boxed{\lambda}\ \Swarrow"] & \cdot \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{\alpha}\ \Swarrow"] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, "\mathrm{Id}"'] & \cdot \arrow[r, "a'"'] & \cdot \end{tikzcd} \overset{\Sigma}{\Rrightarrow} \begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "m'"'] \arrow[dr, phantom, "\boxed{\alpha'}\ \Swarrow"] & \cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "p'"'] \arrow[dr, phantom, "\boxed{\rho}\ \Swarrow"] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, "a'"'] & \cdot \arrow[r, "\mathrm{Id}"'] & \cdot \end{tikzcd}$$

is equivalent to giving a 3-cell

$$\begin{tikzcd} \cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{\lambda}\, m' \Swarrow"] & \cdot \arrow[r, "a"] \arrow[d, "m'"] \arrow[dr, phantom, "\boxed{\alpha'^{-1}} \Swarrow"] & \cdot \arrow[d, "p'"] \\ \cdot \arrow[r, "\mathrm{Id}"'] & \cdot \arrow[r, "a'"'] & \cdot \end{tikzcd} \overset{\tilde{\Sigma}}{\Rrightarrow} \begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{\alpha^{-1}} \Swarrow"] & \cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "p"] \arrow[dr, phantom, "\boxed{\rho} \Swarrow"] & \cdot \arrow[d, "p'"] \\ \cdot \arrow[r, "a'"'] & \cdot \arrow[r, "\mathrm{Id}"'] & \cdot \end{tikzcd}.$$

In formulas: to give a 3-cell $\Sigma : \dfrac{[\mathrm{Id}\,|\alpha]}{[\lambda|\,\mathrm{Id}_{a'}]} \Rrightarrow \dfrac{[\mathrm{Id}_a\,|\rho]}{[\alpha'|\,\mathrm{Id}]}$ *is equivalent to giving a 3-cell* $\tilde{\Sigma} : \dfrac{[\lambda|\,\mathrm{Id}_{a'}]}{[\mathrm{Id}\,|\alpha'^{-1}]} \Rrightarrow \dfrac{[\alpha^{-1}|\,\mathrm{Id}]}{[\mathrm{Id}_a\,|\rho]}$.

When the direction of mapping of 2-cells written in (square) diagrams is fixed, to shorten notation we will also denote horizontal concatenation as in (1) by $(\alpha|\beta)$, respectively $(\overline{\alpha}|\overline{\beta})$.

3 3-Limits

In this section, we study limits and colimits in 3-categories. In the first subsection, we develop the notion of weighted 3-limits and apply it to deduce tricategorical pullbacks, which we will simply call 3-pullbacks. In later subsections, we will introduce 3-(co)products and deduce their properties that will be crucial for operating throughout in our proofs.

3.1 Weighted 3-limits and 3-pullbacks

We will need 3-natural transformations among trifunctors; we define them here.

Definition 3.1 For trifunctors $F, G : \mathcal{C} \to \mathcal{D}$ of 3-categories, a 3-natural transformation $\alpha : F \Rightarrow G$ is given by

- For each A in $\mathcal{C}$, a 1-cell $\alpha_A : F_0 A \to G_0 A$;
- For each $f : A \to B$ in $\mathcal{C}$ an equivalence 2-cell α_f

$$\begin{tikzcd} F_0 A \arrow[r, "F_1 f"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\overset{\alpha_f}{\nearrow}"] & F_0 B \arrow[d, "\alpha_B"] \\ G_0 A \arrow[r, "G_1 f"'] & G_0 B \end{tikzcd}$$

such that

– For each A and B in $\mathcal{C}$, the α_f are the components of a 2-natural transformation (equivalence)

$$\alpha_{A,B} : (\alpha_A)^* \circ G_{A,B} \Rightarrow (\alpha_B)_* \circ F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(F_0 A, G_0 B)$$

– the assignment $f \mapsto \alpha_f$ is well behaved with respect to identity and composition.

The above definition implies that, in particular, for each $\phi\colon f \Rightarrow g\colon A \to B$ in $\mathcal{C}$ there is an isomorphism 3-cell α_ϕ

$$\begin{tikzcd}
\alpha_B \circ F_1(f) \arrow[r, Rightarrow, "\alpha_f"] \arrow[d, Rightarrow, "\mathrm{Id}_{\alpha_B} \otimes F(\phi)"'] \arrow[dr, phantom, "\overset{\alpha_\phi}{\Rrightarrow}"] & G_1(f) \circ \alpha_A \arrow[d, Rightarrow, "G(\phi)\otimes \mathrm{Id}_{\alpha_A}"] \\
\alpha_B \circ F_1(g) \arrow[r, Rightarrow, "\alpha_g"'] & G_1(g) \circ \alpha_A
\end{tikzcd}$$

such that, for each A, B and $f, g\colon A \to B$ in $\mathcal{C}$, the α_ϕ are the components of a 1-natural transformation (isomorphism)

$$\alpha_{f,g}\colon (\alpha_f)^* \circ (G_{A,B})_{f,g} \Rightarrow (\alpha_g)_* \circ (F_{A,B})_{f,g}\colon \mathcal{C}(A, B)(f, g) \to \mathcal{D}(F_0A, G_0B)(F_1f, G_1g)$$

Moreover, the assignment $\phi \mapsto \alpha_\phi$ is well behaved with respect to identity and composition.

Let Bicat_3 be the tricategory of (small) bicategories, pseudofunctors, pseudonatural transformations, and modifications between pseudonatural transformations. Observe that it is both 1- and 2v-strict.

Let $\mathcal{D} = 1 \to 0 \leftarrow 2$ be the cospan graph, seen as a tricategory, and $J\colon \mathcal{D} \xrightarrow{!} \mathbf{1} \xrightarrow{\mathbf{1}} \mathrm{Bicat}_3$ the constantly valued trifunctor on the terminal bicategory. Let $\mathcal{K}$ be a tricategory and $F\colon \mathcal{D} \to \mathcal{K}$ a diagram on $\mathcal{K}$ with image the cospan $A \xrightarrow{f} C \xleftarrow{g} B$. The bicategory $[\mathcal{D}, \mathcal{K}](\Delta_X, F)$ is given by:

0-cells 3-natural transformations $\Delta_X \Rightarrow F$. By definition, that amounts to 1-cells $p_1\colon \Delta_X(1) \to F_0(1)$ and $p_2\colon \Delta_X(2) \to F_0(2)$, i.e., $p_1\colon X \to A$ and $p_2\colon X \to B$, such that the square

$$\begin{tikzcd}
X \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\
A \arrow[r, "f"] & C
\end{tikzcd}$$

commutes up to an equivalence 2-cell $\omega\colon fp_1 \Rightarrow gp_2$. (The third component, $p_0\colon \Delta_X(0) \to F_0(0)$, is determined up to equivalence by either p_1 or p_2 via composition with f or g, respectively.) To sum up, a 0-cell of $[\mathcal{D}, \mathcal{K}](\Delta_X, F)$ is given by a triple

$$(p_1\colon X \to A, p_2\colon X \to B, \omega\colon fp_1 \xrightarrow{\equiv} gp_2)$$

1-cells given 0-cells (p_1, p_2, ω) and (q_1, q_2, σ), a 1-cell $(p_1, p_2, \omega) \to (q_1, q_2, \sigma)$ is a modification of 3-natural transformations, i.e., 2-cells $\alpha_1\colon p_1 \Rightarrow q_1$ and

$\alpha_2\colon p_2 \Rightarrow q_2$ of $\mathcal{K}$ such that there is an invertible 3-cell Γ

$$\begin{tikzcd} fp_1 \arrow[r, Rightarrow, "\omega"] \arrow[d, Rightarrow, "\mathrm{Id}_f \otimes \alpha_1"'] \arrow[dr, phantom, "\overset{\Gamma}{\nearrow}"] & gp_2 \arrow[d, Rightarrow, "\mathrm{Id}_g \otimes \alpha_2"] \\ fq_1 \arrow[r, Rightarrow, "\sigma"] & gq_2 \end{tikzcd}$$

2-cells given 0-cells (p_1, p_2, ω) and (q_1, q_2, σ), and 1-cells $(\alpha_1, \alpha_2, \Gamma)\colon (p_1, p_2, \omega) \to (q_1, q_2, \sigma)$ and $(\beta_1, \beta_2, \Gamma')\colon (p_1, p_2, \omega) \to (q_1, q_2, \sigma)$, a 2-cell $(\alpha_1, \alpha_2, \Gamma) \to (\beta_1, \beta_2, \Gamma')$ is a perturbation, i.e., 3-cells $\Theta_1\colon \alpha_1 \Rrightarrow \beta_1$ and $\Theta_2\colon \alpha_2 \Rrightarrow \beta_2$ in $\mathcal{K}$ such that

$$\begin{tikzcd} \frac{\mathrm{Id}_f \otimes \alpha_1}{\sigma} \arrow[r, Rightarrow, "\Gamma"] \arrow[d, Rightarrow, "\frac{\mathrm{Id}_{\mathrm{Id}_f} \otimes \Theta_1}{\mathrm{Id}_\sigma}"'] & \frac{\omega}{\mathrm{Id}_g \otimes \alpha_2} \arrow[d, Rightarrow, "\frac{\mathrm{Id}_\omega}{\mathrm{Id}_{\mathrm{Id}_g} \otimes \Theta_2}"] \\ \frac{\mathrm{Id}_f \otimes \beta_1}{\sigma} \arrow[r, Rightarrow, "\Gamma'"] & \frac{\omega}{\mathrm{Id}_g \otimes \beta_2} \end{tikzcd}$$

commutes.

Two 0-cells (p_1, p_2, ω) and (q_1, q_2, θ) are equivalent if there exist 1-cells

$$(\alpha_1, \alpha_2, \Gamma)\colon (p_1, p_2, \omega) \to (q_1, q_2, \theta) \tag{3}$$

$$(\beta_1, \beta_2, \Omega)\colon (q_1, q_2, \theta) \to (p_1, p_2, \omega) \tag{4}$$

and invertible 2-cells

$$(\Theta_1, \Theta_2)\colon \left(\beta_1\alpha_1, \beta_2\alpha_2, \frac{(\mathrm{Id}_g \otimes \beta_2) \otimes \Gamma}{\Omega \otimes (\mathrm{Id}_f \otimes \alpha_1)}\right) \cong \mathrm{Id}_{(p_1, p_2, \omega)} \tag{5}$$

$$(\Omega_1, \Omega_2)\colon \left(\alpha_1\beta_1, \alpha_2\beta_2, \frac{(\mathrm{Id}_g \otimes \alpha_2) \otimes \Omega}{\Gamma \otimes (\mathrm{Id}_f \otimes \beta_1)}\right) \cong \mathrm{Id}_{(q_1, q_2, \sigma)} \tag{6}$$

meaning that $p_1 \equiv q_1$ in $\mathcal{K}(X, A)$ and $p_2 \equiv q_2$ in $\mathcal{K}(X, B)$ and the invertible 3-cells Γ and Ω exist.

Every 1-cell $h\colon Y \to X$ induces a pseudofunctor

$$[\mathcal{D}, \mathcal{K}](\Delta_h, F)\colon [\mathcal{D}, \mathcal{K}](\Delta_X, F) \to [\mathcal{D}, \mathcal{K}](\Delta_Y, F)$$

given by precomposition. Likewise, every 2-cell $\phi\colon h \Rightarrow k\colon Y \to X$ induces a pseudonatural transformation

$$[\mathcal{D}, \mathcal{K}](\Delta_\phi, F)\colon [\mathcal{D}, \mathcal{K}](\Delta_h, F) \Rightarrow [\mathcal{D}, \mathcal{K}](\Delta_k, F)$$

defined by

$$[\mathcal{D}, \mathcal{K}](\Delta_\phi, F)_{(p_1, p_2, \omega)} = (\mathrm{Id}_{p_1} \otimes \phi, \mathrm{Id}_{p_2} \otimes \phi, \epsilon_{\omega, \phi}) \tag{7}$$

$$[\mathcal{D}, \mathcal{K}](\Delta_\phi, F)_{(\alpha_1, \alpha_2, \Gamma)} = (\epsilon_{\alpha_1, \phi}, \epsilon_{\alpha_2, \phi}) \tag{8}$$

for every 0-cell (p_1, p_2, ω), and 1-cell $(\alpha_1, \alpha_2, \Gamma)\colon (p_1, p_2, \omega) \to (q_1, q_2, \theta)$, and where ϵ is the interchange 3-isomorphism; in particular, $\epsilon_{\omega,\phi}\colon \frac{\mathrm{Id}_{fp_1} \otimes \phi}{\omega \otimes \mathrm{Id}_k} \Rrightarrow \frac{\omega \otimes \mathrm{Id}_h}{\mathrm{Id}_{gp_2} \otimes \phi}$ and $\epsilon_{\alpha_i,\phi}\colon \frac{\alpha_i \otimes \mathrm{Id}_h}{\mathrm{Id}_{q_i} \otimes \phi} \Rrightarrow \frac{\mathrm{Id}_{p_i} \otimes \phi}{\alpha_i \otimes \mathrm{Id}_k}$.

Finally, every 3-cell $\Gamma\colon \phi \Rrightarrow \psi\colon h \Rightarrow k\colon Y \to X$ induces a modification

$$[\mathcal{D}, \mathcal{K}](\Delta_\Gamma, F)\colon [\mathcal{D}, \mathcal{K}](\Delta_\phi, F) \Rightarrow [\mathcal{D}, \mathcal{K}](\Delta_\psi, F)$$

defined by

$$[\mathcal{D}, \mathcal{K}](\Delta_\Gamma, F)(p_1, p_2, \omega) = (\mathrm{Id}_{\mathrm{Id}_{p_1}} \otimes \Gamma, \mathrm{Id}_{\mathrm{Id}_{p_2}} \otimes \Gamma).$$

Recall that a biequivalence $F\colon \mathcal{B}_1 \to \mathcal{B}_2$ of bicategories can be characterized as essentially surjective, fully faithful pseudofunctor (see *e.g.,* [12, Definition 2.4.9]), meaning:

Essentially surjective surjective on equivalence classes of objects.
Fully faithful for each pair of objects A and B in $\mathcal{B}_1$, the component functor

$$F_{A,B}\colon \mathcal{B}_1(A, B) \to \mathcal{B}_2\big(F_0(A), F_0(B)\big)$$

is an equivalence of hom-categories, so an essentially surjective, fully faithful functor itself.

Definition 3.2 Let $\mathcal{K}$ and $\mathcal{D}$ be tricategories, and $J\colon \mathcal{D} \to \mathrm{Bicat}_3$ and $F\colon \mathcal{D} \to \mathcal{K}$ be trifunctors. A J-weighted 3-limit of F is an object L of $\mathcal{K}$ equipped with a 3-natural biequivalence

$$\epsilon_X\colon \mathcal{K}(X, L) \equiv [\mathcal{D}, \mathrm{Bicat}_3](J, \mathcal{K}(X, F-))$$

where $[\mathcal{D}, \mathrm{Bicat}_3](J, \mathcal{K}(X, F-))$ is the bicategory of 3-natural transformations between J and $\mathcal{K}(X, F-)$, modifications between them and their perturbations.

Proposition 3.3 *Set $\mathcal{D} = (1 \to 0 \leftarrow 2)$ and let $A \xrightarrow{f} C \xleftarrow{g} B$ be the image of F in $\mathcal{K}$. Also let $J\colon \mathcal{D} \xrightarrow{!} \mathbf{1} \xrightarrow{\mathbf{1}} \mathrm{Bicat}_3$ be the constantly valued trifunctor on the terminal bicategory. A 3-natural transformation $J \Rightarrow \mathcal{K}(X, F-)$ amounts to a 3-natural transformation $\Delta_X \Rightarrow F$.*

Proof A 3-natural transformation $\alpha\colon J \Rightarrow \mathcal{K}(X, F-)$ is given by a 3-natural family of pseudofunctors $\alpha_D\colon J_0(D) \to \mathcal{K}(X, F_0(D))$, i.e., $\alpha_D\colon 1 \to \mathcal{K}(X, F_0(D))$, such that the naturality squares

$$\begin{tikzcd}
1 \arrow[r, "\alpha_D"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\overset{\alpha_f}{\Nearrow}"] & \mathcal{K}(X, F_0(D)) \arrow[d, "\mathcal{K}(X, F_1(f))"] \\
1 \arrow[r, "\alpha_{D'}"'] & \mathcal{K}(X, F_0(D'))
\end{tikzcd}$$

for $f\colon D \to D'$ commute up to a 2-natural equivalence α_f. That amounts to a family of 1-cells $\alpha_D\colon X \to F_0(D)$ in $\mathcal{K}$ and 2-natural equivalence $\alpha_f\colon F_1(f)\alpha_D \equiv \alpha_{D'}$, i.e., a 3-natural transformation $\Delta_X \Rightarrow F$. □

Since 3-natural transformations $J \Rightarrow \mathcal{K}(X, F-)$ are the same as 3-natural transformations $\Delta_X \Rightarrow F$, then the 3-pullback of g along f is an object $A \times_C B$ of $\mathcal{K}$ equipped with a 3-natural biequivalence of bicategories

$$\epsilon_X\colon \mathcal{K}(X, A \times_C B) \equiv [\mathcal{D}, \mathcal{K}](\Delta_X, F)$$

(i.e., a 3-natural transformation ϵ whose components ϵ_X are biequivalence of bicategories). This means that, in particular, there is a biequivalence

$$\epsilon_{A\times_C B}\colon \mathcal{K}(A \times_C B, A \times_C B) \equiv [\mathcal{D}, \mathcal{K}](\Delta_{A\times_C B}, F)$$

making the identity on $A \times_C B$ correspond to a 3-natural transformation $\Delta_{A\times_C B} \Rightarrow F$, i.e., to 1-cells $p_1\colon A \times_C B \to A$ and $p_2\colon A \times_C B \to B$ and to an equivalence 2-cell $\omega\colon fp_1 \equiv gp_2$. So, let $\epsilon_{A\times_C B}(\mathrm{Id}_{A\times_C B}) = (p_1, p_2, \omega)$.

For every 0-cell X in $\mathcal{K}$, we know that ϵ_X is a biequivalence of bicategories, i.e., by the characterization of biequivalences, an essentially surjective, fully faithful pseudo-functor.

Consider a 0-cell X in $\mathcal{K}$ and a 0-cell

$$(q_1\colon X \to A, q_2\colon X \to B, \sigma\colon fq_1 \equiv gq_2)$$

in $[\mathcal{D}, \mathcal{K}](\Delta_X, F)$. Since ϵ_X is essentially surjective, there exists a 0-cell $u\colon X \to A \times_C B$ in $\mathcal{K}(X, A \times_C B)$ such that $\epsilon_X(u) \equiv (q_1, q_2, \sigma)$. By 3-naturality of ϵ_X in X, the naturality square

$$\begin{tikzcd}
\mathcal{K}(A \times_C B, A \times_C B) \arrow[r, "\epsilon_{A\times_C B}"] \arrow[d, "\mathcal{K}(u, A\times_C B)"'] \arrow[dr, phantom, "\Rrightarrow \epsilon_u"] & {[\mathcal{D}, \mathcal{K}](\Delta_{A\times_C B}, F)} \arrow[d, "{[\mathcal{D},\mathcal{K}](\Delta_u, F)}"] \\
\mathcal{K}(X, A \times_C B) \arrow[r, "\epsilon_X"] & {[\mathcal{D}, \mathcal{K}](\Delta_X, F)}
\end{tikzcd}$$

commutes up to a 2-natural equivalence ϵ_u. Thus,

$$\begin{aligned}
\epsilon_X(u) &= \epsilon_X(\mathcal{K}(u, A \times_C B)(\mathrm{Id}_{A\times_C B})) \\
&\equiv ([\mathcal{D}, \mathcal{K}](\Delta_u, F))(\epsilon_{A\times_C B}(\mathrm{Id}_{A\times_C B})) \\
&= ([\mathcal{D}, \mathcal{K}](\Delta_u, F))(p_1, p_2, \omega) \\
&= (p_1u, p_2u, \omega \otimes \mathrm{Id}_u)
\end{aligned}$$

and thus $(q_1, q_2, \sigma) \equiv (p_1u, p_2u, \omega \otimes \mathrm{Id}_u)$, i.e., there exist equivalence 2-cells $\zeta_1\colon p_1u \Rightarrow q_1$ and $\zeta_2\colon p_2u \Rightarrow q_2$, and an invertible 3-cell $\frac{\mathrm{Id}_f \otimes \zeta_1}{\sigma} \Rrightarrow \frac{\omega\otimes \mathrm{Id}_u}{\mathrm{Id}_g \otimes \zeta_2}$.

Moreover, since ϵ_X is fully faithful, for each pair of objects u and v in $\mathcal{K}(X, A \times_C B)$, the component functor

$$(\epsilon_X)_{u,v} \colon \mathcal{K}(X, A \times_C B)(u, v) \to [\mathcal{D}, \mathcal{K}](\Delta_X, F)(\epsilon_X(u), \epsilon_X(v))$$

is an equivalence of hom-categories, i.e., it is an essentially surjective, fully faithful functor.

Since $(\epsilon_X)_{u,v}$ is essentially surjective, for each object

$$(\alpha_1 \colon p_1 u \Rightarrow p_1 v, \alpha_2 \colon p_2 u \Rightarrow p_2 v, \kappa \colon \frac{\mathrm{Id}_f \otimes \alpha_1}{\omega \otimes \mathrm{Id}_v} \overset{\cong}{\Rightarrow} \frac{\omega \otimes \mathrm{Id}_u}{\mathrm{Id}_g \otimes \alpha_2})$$

in $[\mathcal{D}, \mathcal{K}](\Delta_X, F)(\epsilon_X(u), \epsilon_X(v))$, i.e., $[\mathcal{D}, \mathcal{K}](\Delta_X, F)\big((p_1 u, p_2 u, \omega \otimes \mathrm{Id}_u), (p_1 v, p_2 v, \omega \otimes \mathrm{Id}_v)\big)$, there exists a 0-cell $\gamma \colon u \Rightarrow v$ in $\mathcal{K}(X, A \times_C B)(u, v)$ (i.e., a 2-cell in $\mathcal{K}$) such that $(\epsilon_X)_{u,v}(\gamma) \cong (\alpha_1, \alpha_2, \kappa)$.

By 2-naturality of ϵ_u in u, there is a naturality square

$$\begin{tikzcd}
\epsilon_X \circ \mathcal{K}(u, A \times_C B) \arrow[r, Rightarrow, "\epsilon_u"] \arrow[d, Rightarrow, "\mathrm{Id}_{\epsilon_X} \otimes \mathcal{K}(\gamma, A \times_C B)"'] \arrow[dr, phantom, "\Rrightarrow \epsilon_\gamma"] & [\mathcal{D}, \mathcal{K}](\Delta_u, F) \circ \epsilon_{A \times_C B} \arrow[d, Rightarrow, "{[\mathcal{D}, \mathcal{K}](\Delta_\gamma, F) \otimes \mathrm{Id}_{\epsilon_{A \times_C B}}}"] \\
\epsilon_X \circ \mathcal{K}(v, A \times_C B) \arrow[r, Rightarrow, "\epsilon_v"] & [\mathcal{D}, \mathcal{K}](\Delta_v, F) \circ \epsilon_{A \times_C B}
\end{tikzcd}$$

commuting up to an invertible 3-cell ϵ_γ in Bicat_3, i.e., a modification. (Observe that in the vertices we have composition of pseudofunctors, and in the edges we have horizontal composition of 2-natural transformations.) Then, the component of ϵ_γ at $\mathrm{Id}_{A \times_C B}$ is an invertible 2-cell in $[\mathcal{D}, \mathcal{K}](\Delta_X, F)$,

$$\begin{tikzcd}
\epsilon_X(\mathcal{K}(u, A \times_C B)(\mathrm{Id}_{A \times_C B})) \arrow[r, "\epsilon_u(\mathrm{Id}_{A \times_C B})"] \arrow[d, "{(\mathrm{Id}_{\epsilon_X} \otimes \mathcal{K}(\gamma, A \times_C B))(\mathrm{Id}_{A \times_C B})}"'] \arrow[dr, phantom, "\Rrightarrow \epsilon_\gamma(\mathrm{Id}_{A \times_C B})"] & [\mathcal{D}, \mathcal{K}](\Delta_u, F)(\epsilon_{A \times_C B}(\mathrm{Id}_{A \times_C B})) \arrow[d, "{([\mathcal{D}, \mathcal{K}](\Delta_\gamma, F) \otimes \mathrm{Id}_{\epsilon_{A \times_C B}})(\mathrm{Id}_{A \times_C B})}"] \\
\epsilon_X(\mathcal{K}(v, A \times_C B)(\mathrm{Id}_{A \times_C B})) \arrow[r, "\epsilon_v(\mathrm{Id}_{A \times_C B})"'] & [\mathcal{D}, \mathcal{K}](\Delta_v, F)(\epsilon_{A \times_C B}(\mathrm{Id}_{A \times_C B}))
\end{tikzcd}$$

i.e., considering that

$$\begin{aligned}
([\mathcal{D}, \mathcal{K}](\Delta_\gamma, F) \otimes \mathrm{Id}_{\epsilon_{A \times_C B}})(\mathrm{Id}_{A \times_C B}) &= [\mathcal{D}, \mathcal{K}](\Delta_\gamma, F)(\epsilon_{A \times_C B}(\mathrm{Id}_{A \times_C B})) \\
&= [\mathcal{D}, \mathcal{K}](\Delta_\gamma, F)(p_1, p_2, \omega) \\
&= (\mathrm{Id}_{p_1} \otimes \gamma, \mathrm{Id}_{p_2} \otimes \gamma, \epsilon_{\omega,\gamma})
\end{aligned}$$

where $\epsilon_{\omega,\gamma} \colon \frac{\mathrm{Id}_{fp_1} \otimes \gamma}{\omega \otimes \mathrm{Id}_v} \Rrightarrow \frac{\omega \otimes \mathrm{Id}_u}{\mathrm{Id}_{gp_2} \otimes \gamma}$ is the interchange 3-isomorphism, and that

$$(\mathrm{Id}_{\epsilon_X} \otimes \mathcal{K}(\gamma, A \times_C B))(\mathrm{Id}_{A \times_C B}) = (\epsilon_X)_{u,v}(\mathcal{K}(\gamma, A \times_C B)(\mathrm{Id}_{A \times_C B})) = (\epsilon_X)_{u,v}(\gamma)$$

we have an invertible 2-cell

$$\begin{tikzcd}
\epsilon_X(u) \arrow[r, "\epsilon_u(\mathrm{Id}_{A\times_C B})"] \arrow[d, "(\epsilon_X)_{u,v}(\gamma)"'] \arrow[dr, phantom, "\Rrightarrow \epsilon_\gamma(\mathrm{Id}_{A\times_C B})"] & (p_1u,\, p_2u,\, \omega\otimes \mathrm{Id}_u) \arrow[d, "(\mathrm{Id}_{p_1}\otimes\gamma, \mathrm{Id}_{p_2}\otimes\gamma, \epsilon_{\omega,\gamma})"] \\
\epsilon_X(v) \arrow[r, "\epsilon_v(\mathrm{Id}_{A\times_C B})"'] \arrow[ur, phantom, "=="] & (p_1v,\, p_2v,\, \omega\otimes \mathrm{Id}_v)
\end{tikzcd}$$

where the 1-cells $\epsilon_u(\mathrm{Id}_{A\times_C B})$ and $\epsilon_v(\mathrm{Id}_{A\times_C B})$ are equivalences. Thus,

$$(\alpha_1, \alpha_2, \kappa) \cong (\epsilon_X)_{u,v}(\gamma) \cong (\mathrm{Id}_{p_1}\otimes\gamma, \mathrm{Id}_{p_2}\otimes\gamma, \epsilon_{\omega,\gamma} : \frac{\mathrm{Id}_{fp_1}\otimes\gamma}{\omega\otimes\mathrm{Id}_v} \Rrightarrow \frac{\omega\otimes\mathrm{Id}_u}{\mathrm{Id}_{gp_2}\otimes\gamma})$$

up to equivalence 1-cells $\epsilon_u(\mathrm{Id}_{A\times_C B})$ and $\epsilon_v(\mathrm{Id}_{A\times_C B})$. That is, there exist invertible 3-cells $\Theta_1 \colon \mathrm{Id}_{p_1}\otimes\gamma \Rrightarrow \alpha_1$ and $\Theta_2 \colon \mathrm{Id}_{p_2}\otimes\gamma \Rrightarrow \alpha_2$ such that

$$\begin{tikzcd}
\frac{\mathrm{Id}_{fp_1}\otimes\gamma}{\omega\otimes\mathrm{Id}_v} \arrow[r, Rightarrow, "\epsilon_{\omega,\gamma}"] \arrow[d, Rightarrow, "\frac{\mathrm{Id}_f\otimes\Theta_1}{\omega\otimes\mathrm{Id}_v}"'] & \frac{\omega\otimes\mathrm{Id}_u}{\mathrm{Id}_{gp_2}\otimes\gamma} \arrow[d, Rightarrow, "\frac{\omega\otimes\mathrm{Id}_u}{\mathrm{Id}_g\otimes\Theta_2}"] \\
\frac{\mathrm{Id}_f\otimes\alpha_1}{\omega\otimes\mathrm{Id}_v} \arrow[r, Rightarrow, "\kappa"] & \frac{\omega\otimes\mathrm{Id}_u}{\mathrm{Id}_g\otimes\alpha_2}
\end{tikzcd}$$

commutes.

Since $(\epsilon_X)_{u,v}$ is fully faithful, for each pair of 0-cells $\gamma, \gamma' \colon u \Rightarrow v$ in $\mathcal{K}(X, A\times_C B)(u, v)$ (i.e., 2-cells in $\mathcal{K}$) and 1-cell

$$\Big((\epsilon_X)_{u,v}(\gamma) \xrightarrow{(\chi_1,\chi_2)} (\epsilon_X)_{u,v}(\gamma')\Big) = \Big((\mathrm{Id}_{p_1}\otimes\gamma, \mathrm{Id}_{p_2}\otimes\gamma, \epsilon_{\omega,\gamma}) \to (\mathrm{Id}_{p_1}\otimes\gamma', \mathrm{Id}_{p_2}\otimes\gamma', \epsilon_{\omega,\gamma'})\Big)$$

in $[\mathcal{D}, \mathcal{K}](\Delta_X, F)\big(\epsilon_X(u), \epsilon_X(v)\big)$, where $\chi_1 \colon \mathrm{Id}_{p_1}\otimes\gamma \Rrightarrow \mathrm{Id}_{p_1}\otimes\gamma'$ and $\chi_2 \colon \mathrm{Id}_{p_2}\otimes\gamma \Rrightarrow \mathrm{Id}_{p_2}\otimes\gamma'$ such that

$$\begin{tikzcd}
\frac{\mathrm{Id}_f\otimes(\mathrm{Id}_{p_1}\otimes\gamma)}{\omega\otimes\mathrm{Id}_v} \arrow[r, Rightarrow, "\epsilon_{\omega,\gamma}"] \arrow[d, Rightarrow, "\frac{\mathrm{Id}_f\otimes\chi_1}{\omega\otimes\mathrm{Id}_v}"'] & \frac{\omega\otimes\mathrm{Id}_u}{\mathrm{Id}_g\otimes(\mathrm{Id}_{p_2}\otimes\gamma)} \arrow[d, Rightarrow, "\frac{\omega\otimes\mathrm{Id}_u}{\mathrm{Id}_g\otimes\chi_2}"] \\
\frac{\mathrm{Id}_f\otimes(\mathrm{Id}_{p_1}\otimes\gamma')}{\omega\otimes\mathrm{Id}_v} \arrow[d, Rightarrow, "\frac{\alpha_{\mathrm{Id}_f,\mathrm{Id}_{p_1},\gamma'}}{\omega\otimes\mathrm{Id}_v}"'] & \frac{\omega\otimes\mathrm{Id}_u}{\mathrm{Id}_g\otimes(\mathrm{Id}_{p_2}\otimes\gamma')} \arrow[d, Rightarrow, "\frac{\omega\otimes\mathrm{Id}_u}{\alpha_{\mathrm{Id}_g,\mathrm{Id}_{p_2},\gamma'}}"] \\
\frac{(\mathrm{Id}_f\otimes\mathrm{Id}_{p_1})\otimes\gamma'}{\omega\otimes\mathrm{Id}_v} \arrow[r, Rightarrow, "\epsilon_{\omega,\gamma'}"] & \frac{\omega\otimes\mathrm{Id}_u}{(\mathrm{Id}_g\otimes\mathrm{Id}_{p_2})\otimes\gamma'}
\end{tikzcd}$$

commutes, there exists a *unique* 1-cell $\chi \colon \gamma \Rrightarrow \gamma'$ in $\mathcal{K}(X, A\times_C B)(u, v)$ (i.e., a 3-cell in $\mathcal{K}$) such that $((\epsilon_X)_{u,v})_{\gamma,\gamma'}(\chi) = (\chi_1, \chi_2)$. We need to compute $((\epsilon_X)_{u,v})_{\gamma,\gamma'}(\chi)$.

By naturality of ϵ_γ in γ, for any 1-cell $\chi\colon \gamma \Rrightarrow \gamma'$ we have that the naturality square

$$\begin{tikzcd}
\dfrac{\mathrm{Id}_{\epsilon_X} \otimes \mathcal{K}(\gamma, A\times_C B))}{\epsilon_v} \arrow[r, Rightarrow, "\epsilon_\gamma"] \arrow[d, Rightarrow, "\dfrac{\mathrm{Id}_{\mathrm{Id}_{\epsilon_X}} \otimes \mathcal{K}(\chi, A\times_C B)}{\mathrm{Id}_{\epsilon_v}}"'] & \dfrac{\epsilon_u}{[\mathcal{D},\mathcal{K}](\Delta_\gamma, F)\otimes \mathrm{Id}_{\epsilon_{A\times_C B}}} \arrow[d, Rightarrow, "\dfrac{\mathrm{Id}_{\epsilon_u}}{[\mathcal{D},\mathcal{K}](\Delta_\chi, F)\otimes \mathrm{Id}_{\mathrm{Id}_{\epsilon_{A\times_C B}}}}"] \\
\dfrac{\mathrm{Id}_{\epsilon_X} \otimes \mathcal{K}(\gamma', A\times_C B)}{\epsilon_v} \arrow[r, Rightarrow, "\epsilon_{\gamma'}"'] & \dfrac{\epsilon_u}{[\mathcal{D},\mathcal{K}](\Delta_{\gamma'}, F)\otimes \mathrm{Id}_{\epsilon_{A\times_C B}}}
\end{tikzcd}$$

commutes. Computing the component of the square at $\mathrm{Id}_{A\times_C B}$, and considering that $(\epsilon_\gamma)_{\mathrm{Id}_{A\times_C B}}$ and $(\epsilon_{\gamma'})_{\mathrm{Id}_{A\times_C B}}$ are invertible 2-cells and $(\epsilon_u)_{\mathrm{Id}_{A\times_C B}}$ and $(\epsilon_v)_{\mathrm{Id}_{A\times_C B}}$ are equivalence 1-cells, and that

$$(\mathrm{Id}_{\mathrm{Id}_{\epsilon_X}} \otimes \mathcal{K}(\chi, A \times_C B))_{\mathrm{Id}_{A\times_C B}} = \epsilon_X(\mathcal{K}(\chi, A \times_C B)_{\mathrm{Id}_{A\times_C B}}) = \epsilon_X(\chi)$$

and

$$\begin{aligned}
([\mathcal{D}, \mathcal{K}](\Delta_\chi, F) \otimes \mathrm{Id}_{\mathrm{Id}_{\epsilon_{A\times_C B}}})_{\mathrm{Id}_{A\times_C B}} &= [\mathcal{D}, \mathcal{K}](\Delta_\chi, F)(\epsilon_{A\times_C B}(\mathrm{Id}_{A\times_C B})) \\
&= [\mathcal{D}, \mathcal{K}](\Delta_\chi, F)(p_1, p_2, \omega) \\
&= (\mathrm{Id}_{\mathrm{Id}_{p_1}} \otimes \chi, \mathrm{Id}_{\mathrm{Id}_{p_2}} \otimes \chi)
\end{aligned}$$

we get that $\epsilon_X(\chi) = (\mathrm{Id}_{\mathrm{Id}_{p_1}} \otimes \chi, \mathrm{Id}_{\mathrm{Id}_{p_2}} \otimes \chi)$, up to invertible 2-cells $(\epsilon_\gamma)_{\mathrm{Id}_{A\times_C B}}$ and $(\epsilon_{\gamma'})_{\mathrm{Id}_{A\times_C B}}$ and equivalence 1-cells $(\epsilon_u)_{\mathrm{Id}_{A\times_C B}}$ and $(\epsilon_v)_{\mathrm{Id}_{A\times_C B}}$. Thus, χ is such that $\chi_1 = \mathrm{Id}_{\mathrm{Id}_{p_1}} \otimes \chi$ and $\chi_2 = \mathrm{Id}_{\mathrm{Id}_{p_2}} \otimes \chi$.

Based on the above discussion, we characterize 3-pullbacks in Definition 3.4.

3.2 3-(co)products

In the characterization of 3-limits through biequivalence of bicategories we will also use an equivalent reformulation of the condition of essential fullness, in terms of a pair of pseudofunctors $F : \mathcal{B}_1 \to \mathcal{B}_2 : G$ and pseudonatural transformations $\Psi : GF \Rightarrow \mathrm{Id}$ and $\Phi : \mathrm{Id} \Rightarrow FG$ which are themselves equivalences. Namely, essential surjectiveness corresponds to an equivalence 2-cell $\Phi_p : p \Rightarrow FG(p)$ in V for every 0-cell $p \in \mathcal{B}_2$, which is a 1-cell in V, whereas essential fullness corresponds to the fact that for every 1-cell in $\mathcal{B}_2$, that is a 2-cell $\theta : p \Rightarrow p'$ in V, there is a family of invertible 2-cells in $\mathcal{B}_2$, that is invertible 3-cells

$$\Phi_\theta : \frac{\theta}{\Phi_{p'}} \Rrightarrow \frac{\Phi_p}{FG(\theta)} \tag{9}$$

in V, which moreover satisfy a naturality condition.

A terminal object in a tricategory V is an object 1 such that for any object X in V the bicategory $V(X, 1)$ is biequivalent to the terminal bicategory. This means that there is a particular 1-cell $! : X \to 1$ such that for every 1-cell $f : X \to 1$ there is an equivalence 2-cell $\psi :! \Rightarrow f$ unique up to a unique isomorphism, every 2-endocell

on ! is isomorphic to identity, and the only 3-endocell on the identity 2-cell on ! is the identity one.

In particular, for a terminal object 1 and any 1-cell $f : A \to B$ there is an equivalence 2-cell:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "!"'] & \boxed{\kappa_f} & B \arrow[dl, "!"] \\ & 1 & \end{tikzcd} \tag{10}$$

unique up to a unique isomorphism, which we will call *terminal 2-cell*. Moreover, due to (9), given any 2-cell $\alpha : f \Rightarrow g$ there is an isomorphism 3-cell $K : \frac{\alpha}{\kappa_g} \Rrightarrow \kappa_f$. We call them *terminal 3-cells*.

Analogous dual properties we have for an initial object 0 of V, with *initial 2- and 3-cells*.

We recall here the definition of a 3-pullback from [10].

Definition 3.4 A 3-pullback with respect to a cospan $M \xrightarrow{f} S \xleftarrow{g} N$ of 1-cells in a tricategory V is given by: a 0-cell P, 1-cells $p_1 : P \to M$, $p_2 : P \to N$ and an equivalence 2-cell $\omega : gp_2 \Rightarrow fp_1$ so that

1. for every 0-cell T, 1-cells $q_1 : T \to M$, $q_2 : T \to N$ and equivalence 2-cell $\sigma : gq_2 \Rightarrow fq_1$ there exist a 1-cell $u : T \to P$, equivalence 2-cells $\zeta_1 : p_1 u \Rightarrow q_1$ and $\zeta_2 : q_2 \Rightarrow p_2 u$ and an invertible 3-cell

$$\Sigma : \frac{\dfrac{\mathrm{Id}_g \otimes \zeta_2}{\omega \otimes \mathrm{Id}_u}}{\mathrm{Id}_f \otimes \zeta_1} \Rrightarrow \sigma$$

$$\begin{tikzcd} T \arrow[dr, "u"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & P \arrow[ul, Rightarrow, "\zeta_1"'] \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\equiv"] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] \arrow[ur, Rightarrow, "\omega"] & C \end{tikzcd}$$

2. for all 1-cells $u, v : T \to P$, 2-cells $\alpha : p_1 u \Rightarrow p_1 v$, $\beta : p_2 u \Rightarrow p_2 v$ and an invertible 3-cell $\kappa : \frac{\mathrm{Id}_g \otimes \beta}{\omega \otimes \mathrm{Id}_v} \Rrightarrow \frac{\omega \otimes \mathrm{Id}_u}{\mathrm{Id}_f \otimes \alpha}$ there are a 2-cell $\gamma : u \Rightarrow v$ and isomorphism 3-cells $\Gamma_1 : \mathrm{Id}_{p_1} \otimes \gamma \Rightarrow \alpha$, $\Gamma_2 : \mathrm{Id}_{p_2} \otimes \gamma \Rightarrow \beta$ such that

$$\begin{tikzcd}
\frac{(\mathrm{Id}_g \otimes \mathrm{Id}_{p_2})\otimes\gamma}{\omega\otimes\mathrm{Id}_v} \arrow[r, Rightarrow, "\epsilon_{\omega,\gamma}"] \arrow[d, Rightarrow, "\frac{a^{-1}}{\mathrm{Id}}"'] & \frac{\omega\otimes\mathrm{Id}_u}{(\mathrm{Id}_f \otimes \mathrm{Id}_{p_1})\otimes\gamma} \arrow[d, Rightarrow, "\frac{\mathrm{Id}}{a^{-1}}"] \\
\frac{\mathrm{Id}_g \otimes(\mathrm{Id}_{p_2} \otimes\gamma)}{\omega\otimes\mathrm{Id}_v} \arrow[d, Rightarrow, "\frac{\mathrm{Id}_{\mathrm{Id}_g}\otimes\Gamma_2}{\mathrm{Id}}"'] & \frac{\omega\otimes\mathrm{Id}_u}{\mathrm{Id}_f \otimes(\mathrm{Id}_{p_1} \otimes\gamma)} \arrow[d, Rightarrow, "\frac{\mathrm{Id}}{\mathrm{Id}_{\mathrm{Id}_f}\otimes\Gamma_1}"] \\
\frac{\mathrm{Id}_g \otimes\beta}{\omega\otimes\mathrm{Id}_v} \arrow[r, Rightarrow, "\kappa"] & \frac{\omega\otimes\mathrm{Id}_u}{\mathrm{Id}_f \otimes\alpha}
\end{tikzcd}$$

commutes (where $\epsilon_{\omega,\gamma}$ is the interchange 3-cell).

$$\begin{tikzcd}
T \arrow[dr, "u"] \arrow[drr, bend left, "p_2u"] \arrow[ddr, bend right, "p_1v"'] \arrow[dr, phantom, "\Swarrow\gamma \quad \Downarrow\beta"] & & \\
& P \arrow[l, Rightarrow, "v"', "\alpha"] \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\equiv"] & B \arrow[d, "g"] \arrow[dl, Rightarrow, "\omega"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

3. for all 2-cells $\gamma, \gamma' : u \Rightarrow v$ and 3-cells $\chi_1 \colon \mathrm{Id}_{p_1} \otimes\gamma \Rrightarrow \mathrm{Id}_{p_1} \otimes\gamma'$ and $\chi_2 \colon \mathrm{Id}_{p_2} \otimes\gamma \Rrightarrow \mathrm{Id}_{p_2} \otimes\gamma'$ such that

$$\begin{tikzcd}
\frac{\mathrm{Id}_g \otimes(\mathrm{Id}_{p_2} \otimes\gamma)}{\omega\otimes\mathrm{Id}_v} \arrow[r, Rightarrow, "\epsilon_{\omega,\gamma}"] \arrow[d, Rightarrow, "\frac{\mathrm{Id}_g\otimes\chi_2}{\mathrm{Id}}"'] & \frac{\omega\otimes\mathrm{Id}_u}{\mathrm{Id}_f \otimes(\mathrm{Id}_{p_1} \otimes\gamma)} \arrow[d, Rightarrow, "\frac{\mathrm{Id}}{\mathrm{Id}_f\otimes\chi_1}"] \\
\frac{\mathrm{Id}_g \otimes(\mathrm{Id}_{p_2} \otimes\gamma')}{\omega\otimes\mathrm{Id}_v} \arrow[d, Rightarrow, "\frac{a}{\mathrm{Id}}"'] & \frac{\omega\otimes\mathrm{Id}_u}{\mathrm{Id}_f \otimes(\mathrm{Id}_{p_1} \otimes\gamma')} \arrow[d, Rightarrow, "\frac{\mathrm{Id}}{a}"] \\
\frac{(\mathrm{Id}_g \otimes\mathrm{Id}_{p_2})\otimes\gamma'}{\omega\otimes\mathrm{Id}_v} \arrow[r, Rightarrow, "\epsilon_{\omega,\gamma'}"] & \frac{\omega\otimes\mathrm{Id}_u}{(\mathrm{Id}_f \otimes\mathrm{Id}_{p_1})\otimes\gamma'}
\end{tikzcd}$$

commutes, there exists a unique 3-cell $\chi \colon \gamma \Rrightarrow \gamma'$ such that $\chi_1 = \mathrm{Id}_{\mathrm{Id}_{p_1}} \otimes\chi$ and $\chi_2 = \mathrm{Id}_{\mathrm{Id}_{p_2}} \otimes\chi$.

For convenience, we write out the definition of a tricategorical product from [10] that we call 3-product for short. It is the dual of Definition 5.3 from *loc.cit.*.

Definition 3.5 A 3-product of 0-cells A and B in a tricategory V consists of: a 0-cell $A \times B$ and 1-cells $p_1 : A \times B \to A$, $p_2 : A \times B \to B$, such that

1. for every 0-cell T and 1-cells $f_1 : T \to A$, $f_2 : T \to B$ there are a 1-cell $u : T \to A \times B$ and equivalence 2-cells $\zeta_i : f_i \Rightarrow p_i u$, $i = 1, 2$;
2. for all 1-cells $u, v : T \to A \times B$ and 2-cells $\alpha : p_1 u \Rightarrow p_1 v$ and $\beta : p_2 u \Rightarrow p_2 v$, there are a 2-cell $\gamma : u \Rightarrow v$ and isomorphism 3-cells $\Gamma_1 : \alpha \Rrightarrow \mathrm{Id}_{p_1} \otimes\gamma$ and $\Gamma_2 : \beta \Rrightarrow \mathrm{Id}_{p_2} \otimes\gamma$;

3. for every two 2-cells $\gamma, \gamma' : u \Rightarrow v$ and every two 3-cells $\chi_i : \mathrm{Id}_{p_i} \otimes \gamma \Rrightarrow \mathrm{Id}_{p_i} \otimes \gamma', i = 1, 2$ there is a unique 3-cell $\Gamma : \gamma \Rrightarrow \gamma'$ such that $\chi_i = \mathrm{Id}_{\mathrm{Id}_{p_i}} \otimes \Gamma, i = 1, 2$.

Corresponding to (9) and with notations as in items 1) and 2) above, one has that for all 2-cells $(\theta_1, \theta_1) : (f_1, f_2) \Rightarrow (f_1', f_2')$ there are invertible 3-cells

$$\Omega_1 : \frac{\zeta_1}{\mathrm{Id}_{p_1} \otimes \theta} \Rrightarrow \frac{\theta_1}{\zeta_1'} \quad \text{and} \quad \Omega_2 : \frac{\zeta_2}{\mathrm{Id}_{p_2} \otimes \theta} \Rrightarrow \frac{\theta_2}{\zeta_2'} \tag{11}$$

where $\zeta_i' : f_i' \Rightarrow p_i v, i = 1, 2$ and $\theta = G(\theta_1, \theta_2)$ where $G : V(T, a) \times V(T, B) \to V(T, A \times B)$ is a biequivalence.

Remark 3.6 The 3-limits in the above two definitions are unique up to a *1-cell invertible up to a 2-equivalence*. We will call such cells *biequivalence 1-cells*. The latter means a 1-cell $f : L \to L'$ for which there exists a 1-cell $g : L' \to L$ and an equivalence 2-cell $\omega : gf \Rightarrow id_L$. Likewise, in the item 1) of the definitions the 1-cells whose existence is claimed are unique up to equivalence 2-cells, and the equivalence 2-cells whose existence is claimed are unique up to isomorphism. In the item 2) of the definitions the 2-cells γ whose existence is claimed are unique up to a unique isomorphism. We record this in the next corollary.

Corollary 3.7 *Let $A \times B$ be a 3-product in a tricategory V. Then, in reference to the items in the above definition, it is:*

1. *in the item 1) the 1-cell u is unique up to an equivalence 2-cell, and the equivalence 2-cells ζ_i are unique up to isomorphism;*
2. *if α, β as in the item 2) induce 2-cells $\gamma, \gamma' : u \Rightarrow v$, then there is a unique isomorphism 3-cell $\Gamma : \gamma \Rrightarrow \gamma'$;*
3. *given 2-cells $\gamma, \gamma' : u \Rightarrow v$ and invertible 3-cells $\chi_i : \mathrm{Id}_{p_i} \otimes \gamma \Rrightarrow \mathrm{Id}_{p_i} \otimes \gamma', i = 1, 2$, then the unique 3-cell $\Gamma : \gamma \Rrightarrow \gamma'$ from the item 3) is invertible.*

An important direct consequence of the definition is:

Lemma 3.8 *If the 2-cells $\alpha : p_1 u \Rightarrow p_1 v$ and $\beta : p_2 u \Rightarrow p_2 v$ in the part 2) in Definition 3.5 are equivalence 2-cells, then so is $\gamma : u \Rightarrow v$. Namely, quasi-inverses α^{-1} and β^{-1} induce a quasi-inverse γ^{-1}.*

Analogous results to the above two hold also for 3-pullbacks. In view of these results, for any quasi-inverse of an equivalence 2-cell ζ obtained in the context of these 3-limits we will write simply ζ^{-1}, throughout, without any further reference to a choice in the respective isomorphism class.

3.3 Inducing 3-products on higher cells

In this subsection, we will induce 1-cells (2-cells) $x \times y$ for given 1-cells (2-cells) x and y. Also, given certain 3-cells $P_{\alpha,\alpha'}$ and $P_{\beta,\beta'}$, where $P_{\alpha,\alpha'}$ can be thought of as specific (transversal) prism whose bases is a 2-cell α (vertically in the back) and

the opposite face is a 2-cell α' (vertically in the front) and analogously for $P_{\beta,\beta'}$, we will induce a 3-cell $P_{\alpha\times\beta,\alpha'\times\beta'}$. In order not to make this lengthy paper even longer, we will skip the proofs in this subsection and will present only the results that we obtained.

Lemma 3.9 *Let V be a 1-strict tricategory. Given two 3-products $M \times N$ and $P \times Q$ of 0-cells in V, one has:*

a) *given objects T, S and 1-cells $q_1 : T \to M, q_2 : T \to N$ and $s_1 : S \to P, s_2 : S \to Q$, then there are 1-cells $t : T \to M \times N$ and $s : S \to P \times Q$ and equivalence 2-cells $\zeta_1 : q_1 \Rightarrow p_1 t, \zeta_2 : q_2 \Rightarrow p_2 t$ and $\theta_1 : s_1 \Rightarrow p'_1 s, \theta_2 : s_2 \Rightarrow p'_2 s$ as in the picture below;*
b) *given 1-cells $g : M \to P$ and $h : N \to Q$, then there is a 1-cell that we will denote by $g \times h$ acting between 3-products $M \times N \to P \times Q$ and there are equivalence 2-cells $\omega_1 : gp_1 \Rightarrow p'_1(g \times h)$ and $\omega_2 : hp_2 \Rightarrow p'_2(g \times h)$ as in the picture below;*
c) *additionally to the data from a) and b), given 1-cell $f : T \to S$ and 2-cells $\alpha : gq_1 \Rightarrow s_1 f$ and $\beta : hq_2 \Rightarrow s_2 f$, then there is a 2-cell $\gamma : (g \times h)t \Rightarrow sf$ and invertible 3-cells*

$$\Gamma^1_\gamma : \frac{\dfrac{\mathrm{Id}_g \otimes \zeta_1}{\omega_1 \otimes \mathrm{Id}_t}}{\mathrm{Id}_{p'_1} \otimes \gamma} \Rrightarrow \frac{\alpha}{\theta_1 \otimes \mathrm{Id}_f}, \qquad \Gamma^2_\gamma : \frac{\dfrac{\mathrm{Id}_h \otimes \zeta_2}{\omega_2 \otimes \mathrm{Id}_t}}{\mathrm{Id}_{p'_2} \otimes \gamma} \Rrightarrow \frac{\beta}{\theta_2 \otimes \mathrm{Id}_f};$$

d) *if α and β in c) are equivalence 2-cells, then so is γ.*

$$\begin{tikzcd}
 & & T \arrow[dll, "q_1"'] \arrow[d, "t"] \arrow[drr, "q_2"] \arrow[ddl, "f"] \arrow[ddr, "g \times h"] & & \\
 & M \arrow[dl, "g"'] \arrow[d, phantom, "\boxed{\omega_1}"] & M \times N \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, phantom, "\boxed{\zeta_1}\ \boxed{\zeta_2}"] & N \arrow[dr, "h"] \arrow[d, phantom, "\boxed{\omega_2}"] & \\
 & & S \arrow[dl, "s_1"'] \arrow[d, "s"] \arrow[dr, "s_2"] & & \\
P & & P \times Q \arrow[ll, "p'_1"] \arrow[rr, "p'_2"'] \arrow[d, phantom, "\boxed{\theta_1}\ \boxed{\theta_2}"] & & Q \\
 & & & &
\end{tikzcd}$$

Convention. The 3-cells Γ_1 and Γ_2 we will call informally and for short *prisms P_i from ζ_i to θ_i, $i = 1, 2$.*

Remark 3.10 Since in the item c) the 1-cells f, g, h in our drawing go transversally from back to front, we considered 2-cells α and β in the mapping order we did (from q_i to θ_i, $i = 1, 2$, so to say), though this way we get:

$$\begin{tikzcd}
 & \arrow[r, "f"] \arrow[d, "q_1"'] \arrow[dr, phantom, "\boxed{\alpha}"] & \arrow[d, "s_1"] \\
 & \arrow[r, "g"'] & \arrow[d, phantom, "\Rrightarrow"] \\
 & &
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
 & \arrow[r, "f"] \arrow[d, "q_2"'] \arrow[dr, phantom, "\boxed{\beta}"] & \arrow[d, "s_2"] \\
 & \arrow[r, "h"'] & \arrow[d, phantom, "\Rrightarrow"] \\
 & & .
\end{tikzcd}$$

However, since we usually consider 2-cells written in squares in the direction $\swarrow\!\!\!\swarrow$, we make the following remark. If one considers the 2-cells α, β in c) in the reversed order, namely: $\alpha : s_1 f \Rightarrow g q_1$ and $\beta : s_2 f \Rightarrow h q_2$, whereas ζ_i, θ_i and ω_i, $i = 1, 2$ maintain the same order, then one gets a 2-cell γ accordingly, namely $\gamma : sf \Rightarrow (g \times h)t$.

Remark 3.11 In all the three parts of Definition 3.5 the existence of the announced cells is subject to the existence of previously mentioned cells that determine them. Thus, given a 2-cell γ as in Lemma 3.9 c) it is understood that γ comes together with some 2-cells α and β that determine it.

We will next construct a prism with basis a 2-cell γ as in the part c) of Lemma 3.9 whose opposite face is an analogous 2-cell γ', out of prisms with bases α and β.

Proposition 3.12 *Let V be a 1-strict tricategory. Let $\gamma : (g \times h)t \Rightarrow sf$ be as in Lemma 3.9 c) with its assigned 2-cells $\zeta_i, \theta_i, \omega_i, i = 1, 2, \alpha$ and β, and consider another $\gamma' : (g' \times h')t \Rightarrow sf'$ induced by $\alpha' : g' q_1 \Rightarrow s_1 f'$ and $\beta' : h' q_2 \Rightarrow s_2 f'$ with $\omega'_1 : g' p_1 \Rightarrow p'_1(g' \times h')$ and $\omega'_2 : h' p_2 \Rightarrow p'_2(g' \times h')$ and the same 2-cells $\zeta_i, \theta_i, i = 1, 2$. Suppose there are 2-cells $\xi : f \Rightarrow f', \lambda : g \Rightarrow g', \rho : h \Rightarrow h'$ and 3-cells*

$$\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "f'"'] \arrow[dr, phantom, "\boxed{\xi}\ \swarrow\!\!\!\swarrow"] & \cdot \arrow[r, "q_1"] \arrow[d, "f"] \arrow[dr, phantom, "\boxed{\alpha}\ \swarrow\!\!\!\swarrow"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "\mathrm{Id}"'] & \cdot \arrow[r, "s_1"'] & \cdot
\end{tikzcd}
\overset{P_\alpha}{\Rightarrow}
\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "q_1"] \arrow[d, "f'"'] \arrow[dr, phantom, "\boxed{\alpha'}\ \swarrow\!\!\!\swarrow"] & \cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "g'"] \arrow[dr, phantom, "\boxed{\lambda}\ \swarrow\!\!\!\swarrow"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "s_1"'] & \cdot \arrow[r, "\mathrm{Id}"'] & \cdot
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "f'"'] \arrow[dr, phantom, "\boxed{\xi}\ \swarrow\!\!\!\swarrow"] & \cdot \arrow[r, "q_2"] \arrow[d, "f"] \arrow[dr, phantom, "\boxed{\beta}\ \swarrow\!\!\!\swarrow"] & \cdot \arrow[d, "h"] \\
\cdot \arrow[r, "\mathrm{Id}"'] & \cdot \arrow[r, "s_2"'] & \cdot
\end{tikzcd}
\overset{P_\beta}{\Rightarrow}
\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "q_2"] \arrow[d, "f'"'] \arrow[dr, phantom, "\boxed{\beta'}\ \swarrow\!\!\!\swarrow"] & \cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "h'"] \arrow[dr, phantom, "\boxed{\rho}\ \swarrow\!\!\!\swarrow"] & \cdot \arrow[d, "h"] \\
\cdot \arrow[r, "s_2"'] & \cdot \arrow[r, "\mathrm{Id}"'] & \cdot
\end{tikzcd}.$$

Then, there is a unique 3-cell

$$\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "f'"'] \arrow[dr, phantom, "\boxed{\xi}\ \swarrow\!\!\!\swarrow"] & \cdot \arrow[r, "t"] \arrow[d, "f"] \arrow[dr, phantom, "\boxed{\gamma}\ \swarrow\!\!\!\swarrow"] & \cdot \arrow[d, "g \times h"] \\
\cdot \arrow[r, "\mathrm{Id}"'] & \cdot \arrow[r, "s"'] & \cdot
\end{tikzcd}
\overset{\Gamma}{\Rightarrow}
\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "t"] \arrow[d, "f'"'] \arrow[dr, phantom, "\boxed{\gamma'}\ \swarrow\!\!\!\swarrow"] & \cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "g' \times h'"] \arrow[dr, phantom, "\boxed{\lambda \times \rho}\ \swarrow\!\!\!\swarrow"] & \cdot \arrow[d, "g \times h"] \\
\cdot \arrow[r, "s"'] & \cdot \arrow[r, "\mathrm{Id}"'] & \cdot
\end{tikzcd}$$

i.e., $\Gamma : \dfrac{[\mathrm{Id}\,|\gamma]}{[\xi\,|\,\mathrm{Id}_s]} \Rightarrow \dfrac{[\mathrm{Id}_t\,|\lambda \times \rho]}{[\gamma'|\,\mathrm{Id}]}$. *Moreover, if P_α, P_β are invertible, then so is Γ.*

As a consequence, we may formulate:

Corollary 3.13 *Given 1-cells $u, v : T \to A \times B$ and 2-cells $\alpha : p_1 u \Rightarrow p_1 v$ and $\beta : p_2 u \Rightarrow p_2 v$, which induce a 2-cell $\gamma : u \Rightarrow v$, and similarly assume that $\alpha' : p_1 u' \Rightarrow p_1 v'$ and $\beta : p_2 u' \Rightarrow p_2 v'$ induce $\gamma' : u \Rightarrow v'$, for $u', v' : T$*

$\to A \times B$. Suppose there are 2-cells $\xi : u \Rightarrow u'$ and $\zeta : v \Rightarrow v'$ and 3-cells $P_{\tilde{\alpha}} : \frac{\tilde{\alpha}}{[\zeta|\mathrm{Id}_{p_1}]} \Rrightarrow \frac{[\xi|\mathrm{Id}_{p_1}]}{\tilde{\alpha}'}$ and $P_{\tilde{\beta}} : \frac{\tilde{\beta}}{[\zeta|\mathrm{Id}_{p_2}]} \Rrightarrow \frac{[\xi|\mathrm{Id}_{p_2}]}{\tilde{\beta}'}$. Then, there is a unique 3-cell $\Gamma : \frac{\gamma}{\zeta} \Rrightarrow \frac{\xi}{\gamma'}$. If $P_{\tilde{\alpha}}$, $P_{\tilde{\beta}}$ are invertible, so is Γ.

Let us formulate the dualization of Proposition 3.12, whereas α and β are considered in the reversed order, as indicated in Remark 3.10, and in the setting of a 3-coproduct $\amalg_{i\in I} A_i$ for a set I (instead of a 3-coproduct $A \amalg B$). One has:

Proposition 3.14 *Let V be a 1-strict tricategory. Let $\alpha_i : t_{A'_i} g_i \Rightarrow f s_{A_i}$, for $i \in I$, induce $\gamma : t(\amalg_{i\in I} g_i) \Rightarrow f s$ in the diagram below:*

$$\begin{tikzcd}
A_i \arrow[r] \arrow[dr, "s_{A_i}"] \arrow[dd, "g_i"'] & \amalg_i A_i \arrow[d, "s"] \arrow[dd, "\amalg_i g_i"] \\
& S \arrow[dd, "t"] \\
A'_i \arrow[r] \arrow[dr, "t_{A'_i}"'] & \amalg_i A'_i \arrow[d, "t"] \\
& T
\end{tikzcd}$$

and similarly let $\alpha'_i : t_{A'_i} g'_i \Rightarrow f' s_{A_i}$, $i \in I$, induce $\gamma' : t(\amalg_{i\in I} g'_i) \Rightarrow f' s$. Suppose moreover that there are 2-cells $\xi : f \Rightarrow f'$ and $\sigma_i : g_i \Rightarrow g'_i$ and 3-cells

$$\begin{tikzcd}
\cdot \arrow[r, "g_i"] \arrow[d, "s_{A_i}"'] \arrow[dr, phantom, "\boxed{\alpha_i}\Swarrow"] & \cdot \arrow[d, "t_{A'_i}"] \\
\cdot \arrow[r, "f"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\boxed{\xi}\Swarrow"] & \cdot \arrow[d, "\mathrm{Id}"] \\
\cdot \arrow[r, "f'"'] & \cdot
\end{tikzcd}
\overset{P_{\alpha_i}}{\Rrightarrow}
\begin{tikzcd}
\cdot \arrow[r, "g_i"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\boxed{\sigma_i}\Swarrow"] & \cdot \arrow[d, "\mathrm{Id}"] \\
\cdot \arrow[r, "g'_i"] \arrow[d, "s_{A_i}"'] \arrow[dr, phantom, "\boxed{\alpha'_i}\Swarrow"] & \cdot \arrow[d, "t_{A'_i}"] \\
\cdot \arrow[r, "f'"'] & \cdot
\end{tikzcd}$$

for every $i \in I$. Then, there is a unique 3-cell

$$\begin{tikzcd}
\cdot \arrow[r, "\amalg_i g_i"] \arrow[d, "s"'] \arrow[dr, phantom, "\boxed{\gamma}\Swarrow"] & \cdot \arrow[d, "t"] \\
\cdot \arrow[r, "f"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\boxed{\xi}\Swarrow"] & \cdot \arrow[d, "\mathrm{Id}"] \\
\cdot \arrow[r, "f'"'] & \cdot
\end{tikzcd}
\overset{\Gamma}{\Rrightarrow}
\begin{tikzcd}
\cdot \arrow[r, "\amalg_i g_i"] \arrow[d, "="'] \arrow[dr, phantom, "\boxed{\amalg_i \sigma_i}\Swarrow"] & \cdot \arrow[d, "="] \\
\cdot \arrow[r, "\amalg_i g'_i"] \arrow[d, "s"'] \arrow[dr, phantom, "\boxed{\gamma'}\Swarrow"] & \cdot \arrow[d, "t"] \\
\cdot \arrow[r, "f'"'] & \cdot
\end{tikzcd}$$

If P_{α_i}, $i \in I$ are invertible, then so is Γ.

Observe here that g, h on the one hand and λ, ρ on the other, from Proposition 3.12, pass to g_i, respectively σ_i. Consequently, $\lambda \times \rho$ passes to $\amalg_i \sigma_i$.

The dual of Corollary 3.13 is:

Corollary 3.15 *Given 1-cells $u, v : \amalg_i A_i \to T$ and 2-cells $\alpha_i : u\iota_i \Rightarrow v\iota_i$ for every $i \in I$, which induce a 2-cell $\gamma : u \Rightarrow v$, and similarly assume that $\alpha'_i : u'\iota_i \Rightarrow v'\iota_i$ induce $\gamma' : u' \Rightarrow v'$, for $u', v' : \amalg_i A_i \to T$. Suppose there are 2-cells $\xi : u \Rightarrow u'$ and $\zeta : v \Rightarrow v'$ and 3-cells $P_{\alpha_i} : \frac{\alpha_i}{[\mathrm{Id}_{\iota_i} | \zeta]} \Rrightarrow \frac{[\mathrm{Id}_{\iota_i} | \xi]}{\alpha'_i}$, $i \in I$, then there is a unique 3-cell $\Gamma : \frac{\gamma}{\zeta} \Rrightarrow \frac{\xi}{\gamma'}$. If P_{α_i}, $i \in I$ are invertible, so is Γ.*

We now induce a 3-product 2-cell, *i.e.,* a 2-cell $\alpha \times \beta$ obtained as a 3-product of two 2-cells α, β.

Corollary 3.16 (Corollary of Lemma 3.9) *Let V be a 1-strict tricategory. Given four 3-products $M \times N$, $M' \times N'$, $P \times Q$ and $P' \times Q'$ of 0-cells and 2-cells:*

$$\begin{tikzcd} M \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\boxed{\alpha}"] & P \arrow[d, "p"] \\ M' \arrow[r, "a'"'] & P' \end{tikzcd} \qquad \begin{tikzcd} N \arrow[r, "b"] \arrow[d, "n"'] \arrow[dr, phantom, "\boxed{\beta}"] & Q \arrow[d, "q"] \\ N' \arrow[r, "b'"'] & Q' \end{tikzcd}$$

there is a 2-cell

$$\begin{tikzcd} M \times N \arrow[r, "a \times b"] \arrow[d, "m \times n"'] \arrow[dr, phantom, "\boxed{\alpha \times \beta}"] & P \times Q \arrow[d, "p \times q"] \\ M' \times N' \arrow[r, "a' \times b'"'] & P' \times Q' \end{tikzcd}$$

and two isomorphism 3-cells, which are the evident transversal prisms from back to front in the diagram:

$$\begin{tikzcd} M \arrow[d, "m"] \arrow[ddd, "a"'] & M \times N \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, "m \times n"] \arrow[ddd, "a \times b"] & N \arrow[d, "n"] \arrow[ddd, "b"] \\ M' \arrow[d] \arrow[ddd, "a'"] & M' \times N' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d] \arrow[ddd, "a' \times b'"] & N' \arrow[d] \arrow[ddd, "b'"] \\ P \arrow[d, "p"'] & P \times Q \arrow[l, "s_1"] \arrow[r, "s_2"'] \arrow[d, "p \times q"] & Q \arrow[d, "q"'] \\ P' & P' \times Q' \arrow[l, "p'_1"] \arrow[r, "p'_2"'] & Q' \end{tikzcd}$$

If α and β are equivalence 2-cells, then so is $\alpha \times \beta$.

Remark 3.17 If the 2-cells α and β were in the reversed order, that is $\overline{\alpha} : s_1 m \Rightarrow pa$, $\overline{\beta} : b'n \Rightarrow qb$, most importantly, if one considers the same squares as for α and β

above but directed in the opposite direction, whereas $\zeta_i, \theta_i, \omega_i$ and ω_i', $i = 1, 2$ remain the same, then one gets a 2-cell $\overline{\alpha} \times \overline{\beta}$ and 3-cells

$$\overline{C}^1_{\overline{\alpha}\times\overline{\beta}} : \frac{\mathrm{Id}_{p_1} \otimes (\overline{\alpha} \times \overline{\beta})}{\zeta_1^{-1} \otimes \mathrm{Id}_{a\times b}} \Rrightarrow \frac{\omega_1'^{-1} \otimes \mathrm{Id}_{p\times q}}{\mathrm{Id}_{a'} \otimes \theta_1^{-1}}, \qquad \overline{C}^2_{\overline{\alpha}\times\overline{\beta}} : \frac{\mathrm{Id}_{p_2} \otimes (\overline{\alpha} \times \overline{\beta})}{\zeta_2^{-1} \otimes \mathrm{Id}_{a\times b}} \Rrightarrow \frac{\omega_2'^{-1} \otimes \mathrm{Id}_{p\times q}}{\mathrm{Id}_{b'} \otimes \theta_2^{-1}}.$$

$$\frac{\mathrm{Id}_m \otimes \omega_1^{-1}}{\overline{\alpha} \otimes \mathrm{Id}_{s_1}} \qquad \frac{\mathrm{Id}_n \otimes \omega_2^{-1}}{\overline{\beta} \otimes \mathrm{Id}_{s_2}}$$

Remark 3.18 By our remarks on uniqueness, observe that the 1-cell $g \times h$ in Lemma 3.9 is unique up to an equivalence 2-cell, and that the equivalence 2-cells ω_1, ω_2 from there and $\alpha \times \beta$ from Corollary 3.16 are unique up to isomorphism.

We finally construct a 3-cell with an associated prism with basis $\alpha \times \beta$ out of two given 3-cells with associated prisms of basis α and β.

Proposition 3.19 *Given two 2-cells $\overline{\alpha} \times \overline{\beta}$ and $\overline{\alpha}' \times \overline{\beta}'$ with notations as in Corollary 3.16, but with the orientation as in Remark 3.17. Suppose that further 2-cells $\lambda_1 : m \Rightarrow m'$, $\lambda_2 : n \Rightarrow n'$, $\rho_1 : p \Rightarrow p'$, $\rho_2 : q \Rightarrow q'$ are given and two 3-cells*

$$\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\overline{\alpha}"] & \cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "p"] \arrow[dr, phantom, "\rho_1"] & \cdot \arrow[d, "p'"] \\
\cdot \arrow[r, "a'"'] \arrow[ur, phantom, "\Rrightarrow"] & \cdot \arrow[r, "\mathrm{Id}"'] \arrow[ur, phantom, "\Rrightarrow"] & \cdot
\end{tikzcd}
\overset{P_1}{\Rrightarrow}
\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "m"'] \arrow[dr, phantom, "\lambda_1"] & \cdot \arrow[r, "a"] \arrow[d, "m'"] \arrow[dr, phantom, "\overline{\alpha}'"] & \cdot \arrow[d, "p'"] \\
\cdot \arrow[r, "\mathrm{Id}"'] \arrow[ur, phantom, "\Rrightarrow"] & \cdot \arrow[r, "a'"'] \arrow[ur, phantom, "\Rrightarrow"] & \cdot
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "b"] \arrow[d, "n"'] \arrow[dr, phantom, "\overline{\beta}"] & \cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "q"] \arrow[dr, phantom, "\rho_2"] & \cdot \arrow[d, "q'"] \\
\cdot \arrow[r, "b'"'] \arrow[ur, phantom, "\Rrightarrow"] & \cdot \arrow[r, "\mathrm{Id}"'] \arrow[ur, phantom, "\Rrightarrow"] & \cdot
\end{tikzcd}
\overset{P_2}{\Rrightarrow}
\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "m"'] \arrow[dr, phantom, "\lambda_2"] & \cdot \arrow[r, "b"] \arrow[d, "n'"] \arrow[dr, phantom, "\overline{\beta}'"] & \cdot \arrow[d, "q'"] \\
\cdot \arrow[r, "\mathrm{Id}"'] \arrow[ur, phantom, "\Rrightarrow"] & \cdot \arrow[r, "b'"'] \arrow[ur, phantom, "\Rrightarrow"] & \cdot
\end{tikzcd}$$

Then, there is a unique 3-cell

$$\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "a\times b"] \arrow[d, "m\times n"'] \arrow[dr, phantom, "\overline{\alpha}\times\overline{\beta}"] & \cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "p\times q"] \arrow[dr, phantom, "\rho_1\times\rho_2"] & \cdot \arrow[d, "p'\times q'"] \\
\cdot \arrow[r, "a'\times b'"'] \arrow[ur, phantom, "\Rrightarrow"] & \cdot \arrow[r, "\mathrm{Id}"'] \arrow[ur, phantom, "\Rrightarrow"] & \cdot
\end{tikzcd}
\overset{\Gamma}{\Rrightarrow}
\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "m\times n"'] \arrow[dr, phantom, "\lambda_1\times\lambda_2"] & \cdot \arrow[r, "a\times b"] \arrow[d, "p\times q"] \arrow[dr, phantom, "\overline{\alpha}'\times\overline{\beta}'"] & \cdot \arrow[d, "p'\times q'"] \\
\cdot \arrow[r, "\mathrm{Id}"'] \arrow[ur, phantom, "\Rrightarrow"] & \cdot \arrow[r, "a'\times b'"'] \arrow[ur, phantom, "\Rrightarrow"] & \cdot
\end{tikzcd}.$$

Corollary 3.20 *Given equivalence 2-cells $\alpha, \beta, \alpha', \beta'$ and 3-cells*

$$\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "m'"'] \arrow[dr, phantom, "\lambda_1"] & \cdot \arrow[r, "a"] \arrow[d, "m"] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, "p"] \\
\cdot \arrow[r, "\mathrm{Id}"'] \arrow[ur, phantom, "\Leftrightarrow"] & \cdot \arrow[r, "a'"'] \arrow[ur, phantom, "\Leftrightarrow"] & \cdot
\end{tikzcd}
\overset{\tilde{P}_1}{\Rrightarrow}
\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "a"] \arrow[d, "m'"'] \arrow[dr, phantom, "\alpha'"] & \cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "p'"] \arrow[dr, phantom, "\rho_1"] & \cdot \arrow[d, "p"] \\
\cdot \arrow[r, "a'"'] \arrow[ur, phantom, "\Leftrightarrow"] & \cdot \arrow[r, "\mathrm{Id}"'] \arrow[ur, phantom, "\Leftrightarrow"] & \cdot
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "n'"'] \arrow[dr, phantom, "\lambda_2"] & \cdot \arrow[r, "b"] \arrow[d, "n"] \arrow[dr, phantom, "\beta"] & \cdot \arrow[d, "q"] \\
\cdot \arrow[r, "\mathrm{Id}"'] \arrow[ur, phantom, "\Leftrightarrow"] & \cdot \arrow[r, "b'"'] \arrow[ur, phantom, "\Leftrightarrow"] & \cdot
\end{tikzcd}
\overset{\tilde{P}_2}{\Rrightarrow}
\begin{tikzcd}[column sep=large, row sep=large]
\cdot \arrow[r, "b"] \arrow[d, "n'"'] \arrow[dr, phantom, "\beta'"] & \cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "q'"] \arrow[dr, phantom, "\rho_2"] & \cdot \arrow[d, "q"] \\
\cdot \arrow[r, "b'"'] \arrow[ur, phantom, "\Leftrightarrow"] & \cdot \arrow[r, "\mathrm{Id}"'] \arrow[ur, phantom, "\Leftrightarrow"] & \cdot
\end{tikzcd}$$

with 2-cells $\lambda_i, \rho_i, i = 1, 2$ as in Proposition 3.19. Then, there is a unique 3-cell

$$\begin{tikzcd} \cdot \arrow[r, "\mathrm{Id}"] \arrow[d, "m \times n"'] \arrow[dr, phantom, "\boxed{\lambda_1 \times \lambda_2}"] & \cdot \arrow[r, "a \times b"] \arrow[d] \arrow[dr, phantom, "\boxed{\alpha \times \beta}"] & \cdot \arrow[d, "p' \times q'"] \\ \cdot \arrow[r, "\mathrm{Id}"'] & \cdot \arrow[r, "a' \times b'"'] & \cdot \end{tikzcd} \overset{\Gamma'}{\Rrightarrow} \begin{tikzcd} \cdot \arrow[r, "a \times b"] \arrow[d, "m \times n"'] \arrow[dr, phantom, "\boxed{\alpha' \times \beta'}"] & \cdot \arrow[r, "\mathrm{Id}"] \arrow[d] \arrow[dr, phantom, "\boxed{\rho_1 \times \rho_2}"] & \cdot \arrow[d, "p' \times q'"] \\ \cdot \arrow[r, "a' \times b'"'] & \cdot \arrow[r, "\mathrm{Id}"'] & \cdot \end{tikzcd}$$

i.e., $\Gamma' : \dfrac{[\mathrm{Id}\,|\alpha \times \beta]}{[\lambda_1 \times \lambda_2 |\, \mathrm{Id}_{a' \times b'}]} \Rrightarrow \dfrac{[\mathrm{Id}_{a \times b}\,|\rho_1 \times \rho_2]}{[\alpha' \times \beta' |\, \mathrm{Id}]}$.

3.4 On 3-pullbacks on higher cells

Analogously to Lemma 3.9, for 3-pullbacks we have:

Lemma 3.21 *Let V be a 1-strict tricategory. Given two 3-pullbacks $M \times_S N$ and $P \times_S Q$ of cospans $M \xrightarrow{m} S \xleftarrow{n} N$ and $P \xrightarrow{p} S \xleftarrow{q} Q$ in V, respectively, together with equivalence 2-cells $\zeta_1 : p_1 t \Rightarrow q_1, \zeta_2 : q_2 \Rightarrow p_2 t$ and $\theta_1 : s_1 \Rightarrow p_1' s, \theta_2 : s_2 \Rightarrow p_2' s$, as in the diagram below, and corresponding bijective 3-cells*

$$\Sigma_1 : \frac{\dfrac{\mathrm{Id}_n \otimes \zeta_2}{\omega \otimes \mathrm{Id}_t}}{\mathrm{Id}_m \otimes \zeta_1} \Rrightarrow \sigma_1 \quad \textit{and} \quad \Sigma_2 : \frac{\dfrac{\mathrm{Id}_q \otimes \theta_2}{\omega' \otimes \mathrm{Id}_s}}{\mathrm{Id}_p \otimes \theta_1} \Rrightarrow \sigma_2.$$

Then, one has

a) given 1-cells $g : M \to P$ and $h : N \to Q$ and equivalence 2-cells $\varphi_1 : m \Rightarrow pg$ and $\varphi_2 : qh \Rightarrow n$, then there exist a 1-cell $g \times_S h : M \times_S N \to P \times_S Q$, equivalence 2-cells $\omega_1 : gp_1 \Rightarrow p_1'(g \times_S h)$ and $\omega_2 : hp_2 \Rightarrow p_2'(g \times_S h)$, and an invertible 3-cell

$$\Sigma : \frac{\dfrac{\varphi_2 \otimes \mathrm{Id}_{p_2}}{\omega}}{\varphi_1 \otimes \mathrm{Id}_{p_1}} \Rrightarrow \frac{\dfrac{\mathrm{Id}_q \otimes \omega_2}{\omega' \otimes \mathrm{Id}_{g \times_S h}}}{\mathrm{Id}_p \otimes \omega_1^{-1}};$$

B) additionally to the data from a), given 1-cell $f : T \to R$, 2-cells $\alpha : gq_1 \Rightarrow s_1 f$ and $\beta : hq_2 \Rightarrow s_2 f$, and a 3-cell

$$\kappa_0 : \frac{\dfrac{\sigma_1}{\varphi_1 \otimes \mathrm{Id}_{q_1}}}{\mathrm{Id}_p \otimes \alpha} \Rrightarrow \frac{\dfrac{\varphi_2 \otimes \mathrm{Id}_{q_2}}{\mathrm{Id}_q \otimes \beta}}{\sigma_2 \otimes \mathrm{Id}_m},$$

then there is a 2-cell $\gamma : (g \times_S h)t \Rightarrow sf$ and invertible 3-cells

$$\Xi_\gamma^1 : \frac{\dfrac{\mathrm{Id}_g \otimes \zeta_1}{\omega_1 \otimes \mathrm{Id}_t}}{\mathrm{Id}_{p_1'} \otimes \gamma} \Rrightarrow \frac{\alpha}{\theta_1 \otimes \mathrm{Id}_f} \quad \textit{and} \quad \Xi_\gamma^2 : \frac{\dfrac{\mathrm{Id}_h \otimes \zeta_2}{\omega_2 \otimes \mathrm{Id}_t}}{\mathrm{Id}_{p_2'} \otimes \gamma} \Rrightarrow \frac{\beta}{\theta_2 \otimes \mathrm{Id}_f};$$

C) if α and β in b) are equivalence 2-cells, then so is γ.

$$\begin{tikzcd}
 & & T \arrow[dl, "q_1"'] \arrow[d, "t"] \arrow[dr, "q_2"] & & \\
 & M \arrow[dl, "g"'] \arrow[d, "f"] & M \times_S N \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "g \times_S h"] & N \arrow[dr, "h"] & \\
 & R \arrow[dl, "s_1"'] \arrow[d, "s"] \arrow[dr, "s_2"] & & & \\
P & P \times_S Q \arrow[l, "p'_1"] \arrow[r, "p'_2"'] & Q & &
\end{tikzcd}$$

4 The double categories of matrices and spans in a 1-category

In this section, we review the notion of extensivity used in [6] to characterize biequivalence of bicategories of spans and matrices in a 1-category $\mathcal{V}$, and we show this characterization. In the last two subsections, we then complete the characterization of equivalence of the discretely internal and enriched categories using 2-categorical tools, that was announced in [6], but was not carried out this way, as the authors chose to do the proof using 1-categories.

4.1 Review of extensivity

Let $\mathcal{V}$ be a category, I a set, and $(X_i)_{i\in I}$ an I-indexed family of objects of $\mathcal{V}$. If $\mathcal{V}$ admits all I-indexed coproducts, there is a functor

$$\Pi_{i\in I}(\mathcal{V}/X_i) \xrightarrow{\amalg} \mathcal{V}/(\amalg_{i\in I} X_i) \tag{12}$$

mapping a family $(f_i : A_i \to X_i)_{i\in I}$ to $\amalg_{i\in I} f_i : \amalg_{i\in I} A_i \to \amalg_{i\in I} X_i$.

Following [3, Chap 2, 6.3] and [5, 4.1], the definition below is introduced in [6, Definition 2.1].

Definition 4.1 A category $\mathcal{V}$ is *extensive* if $\mathcal{V}$ admits all small coproducts and, for each small family $(X_i)_{i\in I}$ of objects of $\mathcal{V}$, the functor in (12) is an equivalence of categories.

The right adjoint of $\amalg$, if it exists, has ith component $\sigma_i^* : \mathcal{V}/(\amalg_{i\in I} X_i) \to \mathcal{V}/X_i$ mapping $f : A \to \amalg_{i\in I} X_i$ to $\sigma_i^*(f) : \sigma_i^*(A) \to X_i$ defined by the pullback

$$\begin{tikzcd}
\sigma_i^*(A) \arrow[r] \arrow[d, "\sigma_i^*(f)"'] & A \arrow[d, "f"] \\
X_i \arrow[r, "\sigma_i"] & \amalg_{i\in I} X_i
\end{tikzcd}$$

in $\mathcal{V}$. Then, the adjunction has the form

$$\begin{tikzcd} \Pi_{i\in I}(\mathcal{V}/X_i) \arrow[r, shift left, "\sqcup"] \arrow[r, phantom, "\perp"] & \mathcal{V}/(\sqcup_{i\in I} X_i) \arrow[l, shift left, "\langle \sigma_i^* \rangle_{i\in I}"] \end{tikzcd}$$

For a set I and an object X of $\mathcal{V}$, let $I \bullet X$ denote the I-fold copower of X by I. If $\mathcal{V}$ has a terminal object 1, setting $X_i = 1$ for all $i \in I$ in (12), and observing that $\mathcal{V}/1 \cong \mathcal{V}^I$, one obtains the functor

$$\mathcal{V}^I \xrightarrow{\sqcup} \mathcal{V}/(I \bullet 1) \tag{13}$$

which is an adjunction

$$\begin{tikzcd} \mathcal{V}^I \arrow[r, shift left, "\sqcup"] \arrow[r, phantom, "\perp"] & \mathcal{V}/(I \bullet 1) \arrow[l, shift left, "\langle i^* \rangle_{i\in I}"] \end{tikzcd} \tag{14}$$

if for all $i \in I$ the category $\mathcal{V}$ admits all pullbacks along the coprojections $\bar{i} : 1 \to I \bullet 1$, due to [6, Proposition 2.2]. One sufficient condition for $\mathcal{V}$ to be extensive is:

Proposition 4.2 ([6, Proposition 2.5], [4, Proposition 4.1]) *Let $\mathcal{V}$ be a category with small coproducts and a terminal object. If for every small set I, the functor* (13) *is an equivalence, then $\mathcal{V}$ is extensive.*

4.2 The bicategories of matrices and spans

In [2] the bicategory $\mathcal{V}$-Mat of $\mathcal{V}$-matrices was introduced, and in [1] the bicategory $Span(\mathcal{V})$ of spans over $\mathcal{V}$. It is immediate to see that monads in the former bicategory are $\mathcal{V}$-categories (categories enriched over $\mathcal{V}$), while the monads in the latter are categories internal in $\mathcal{V}$. As a matter of fact, in [1, Section 5.4.3] categories internal in $\mathcal{V}$ are defined this way. For the purposes of examining the relation between enriched and internal categories in $\mathcal{V}$, the bicategory of spans over $\mathcal{V}$ is modified in [6, Section 3.2] so that the 0-cells are small sets, rather than objects of $\mathcal{V}$. This version of the bicategory is denoted by $Span_d(\mathcal{V})$, here "d" stands for *discrete*, as monads in $Span_d(\mathcal{V})$ are those internal categories whose object of objects is discrete. To shorten, these internal categories we will call throughout *discretely internal categories*, and spans in $Span_d(\mathcal{V})$ we will call *discrete spans*. For the biequivalence of the mentioned two bicategories the authors have proved the following.

Proposition 4.3 ([6, Proposition 3.2 and Theorem 3.3]) *Let $\mathcal{V}$ be a Cartesian closed category with finite limits and small coproducts. The following are equivalent:*

1. *for every set I the adjunction* (14) *is an adjoint equivalence;*
2. *$\mathcal{V}$ is extensive;*
3. *the oplax functor $Int : \mathcal{V}\text{-}Mat \to Span_d(\mathcal{V})$ is a biequivalence;*
4. *the lax functor $En : Span_d(\mathcal{V}) \to \mathcal{V}\text{-}Mat$ is a biequivalence.*

The functors Int and En are obtained from the adjunction (14) by substituting the set I from the latter by the set $I \times J$. Namely, 0-cells for both bicategories are sets I, J... and the hom-categories of the bicategories $\mathcal{V}\text{-}Mat$ and $Span_d(\mathcal{V})$ are given by $\mathcal{V}\text{-}Mat(I, J) = \mathcal{V}^{I \times J}$ (1-cells are matrices of dimension $|I| \times |J|$ whose entries are objects of $\mathcal{V}$, and 2-cells are families of morphisms in $\mathcal{V}$ between the corresponding objects in the matrices) and $Span_d(\mathcal{V})(I, J) = \mathcal{V}/((I \bullet 1) \times (J \bullet 1)) \cong \mathcal{V}/((I \times J) \bullet 1)$ (1-cells are spans of objects in $\mathcal{V}$ of the form: $I \bullet 1 \longleftarrow V \longrightarrow J \bullet 1$, and 2-cells are morphisms in $\mathcal{V}$ between such objects V making two evident triangle diagrams between two spans commute), respectively. The above isomorphism of slice categories is assured by Cartesian closedness of $\mathcal{V}$.

Concretely, we describe here the actions of Int and En on 1-cells, their actions on 2-cells can then be deduced easily, and they both are identities on 0-cells. Int maps a matrix $(M(i; j))_{i \in I, j \in J}$ to the span $I \bullet 1 \longleftarrow \coprod_{i \in I, j \in J} M(i, j) \longrightarrow J \bullet 1$. Here the arrows to $I \bullet 1$ and $J \bullet 1$ are induced by the following composite with the domain $M(i, j)$ for fixed $i \in I$, $j \in J$: the unique morphism to 1 followed by the coprojections to $I \bullet 1$ and $J \bullet 1$, respectively. En maps a span $I \bullet 1 \xleftarrow{v_I} V \xrightarrow{v_J} J \bullet 1$ to the matrix $En(V)$ whose (i, j)-th component is given by the pullback

$$\begin{tikzcd}
En(V)(i, j) \arrow[r, "!"] \arrow[d, "\iota_{i,j}"'] & 1 \arrow[d, "\langle \bar{i}, \bar{j} \rangle"] \\
V \arrow[r, "\langle v_I, v_J \rangle"] & (I \bullet 1) \times (J \bullet 1)
\end{tikzcd} \tag{15}$$

for each $i \in I$, $j \in J$, where $\bar{i}$ and $\bar{j}$ denote the i-th and j-th coprojections, respectively.

Although monads in $\mathcal{V}\text{-}Mat$ and $Span_d(\mathcal{V})$ are $\mathcal{V}$-categories and discretely internal categories in $\mathcal{V}$, respectively, the morphisms of monads in these bicategories are not morphisms in $\mathcal{V}$-Cat, the category of $\mathcal{V}$-categories, and $\mathrm{Cat}_d(\mathcal{V})$, the category of discretely internal categories in $\mathcal{V}$. Hence, as the authors comment, one cannot use the biequivalence of bicategories $\mathcal{V}\text{-}Mat$ and $Span_d(\mathcal{V})$ to conclude the equivalence of categories $\mathcal{V}$-Cat and $\mathrm{Cat}_d(\mathcal{V})$. They prove the latter equivalence in another way avoiding two-dimensional category theory, although they comment that one could proceed by using "additional two-dimensional structure, such as that of a pseudo-double category." For the purpose of our work, we will use the pseudo-double categories of $\mathcal{V}$-matrices and "discrete spans in $\mathcal{V}$".

4.3 The double categories of matrices and spans

In [11, Example 2.1] the authors introduced a pseudo-double category $\mathrm{Span}(\mathcal{V})$ of spans in $\mathcal{V}$ whose horizontal bicategory is precisely the bicategory $Span(\mathcal{V})$. In [11, Definition 2.4] a pseudo-double category $\mathrm{Mnd}(\mathbb{D})$ of monads in a pseudo-double category $\mathbb{D}$ is introduced, so that when $\mathbb{D} = \mathrm{Span}(\mathcal{V})$, the vertical 1-cells in $\mathrm{Mnd}(\mathrm{Span}(\mathcal{V}))$ are morphisms of internal categories in $\mathcal{V}$ (see [11, Example 2.6]). This inspires us to define the pseudo-double category $\mathrm{Span}_d(\mathcal{V})$ by modifying accordingly $\mathrm{Span}(\mathcal{V})$, and to introduce a pseudo-double category $\mathcal{V}$-Mat so to extend the biequivalence of bicategories from Proposition 4.3 to an equivalence of pseudo-double categories.

We define a pseudo-double category $\mathrm{Span}_d(\mathcal{V})$ as follows. Its 0-cells are small sets, for sets I, J 1h-cells $I \to J$ are spans $I \bullet 1 \xleftarrow{a_1} A \xrightarrow{a_2} J \bullet 1$, while 1v-cells $I \to J$ are set maps between I and J. For spans $I \bullet 1 \xleftarrow{a_1} A \xrightarrow{a_2} J \bullet 1$ and $K \bullet 1 \xleftarrow{b_1} B \xrightarrow{b_2} L \bullet 1$, and maps $u : I \to K$ and $v : J \to L$, 2-cells are given by the diagrams

$$\begin{tikzcd} I \bullet 1 \arrow[d, "u \bullet 1"'] & A \arrow[l, "a_1"'] \arrow[r, "a_2"] \arrow[d, "f"] & J \bullet 1 \arrow[d, "v \bullet 1"] \\ K \bullet 1 & B \arrow[l, "b_1"'] \arrow[r, "b_2"] & L \bullet 1 \end{tikzcd} \tag{16}$$

A pseudo-double category $\mathcal{V}$-Mat we define as follows. Its 0-cells are small sets, for sets I, J 1h-cells $I \to J$ are matrices of dimension $|I| \times |J|$ whose entries are objects of $\mathcal{V}$, 1v-cells $I \to K$ are maps of sets, and for matrices $(M(i, j))_{i \in I, j \in J}$ and $(N(k, l))_{k \in K, l \in L}$ and 1v-cells $u : I \to K$ and $v : J \to L$, 2-cells are given by the families f of morphisms in $\mathcal{V}$ determined so that the following diagram commutes:

$$\begin{tikzcd} \coprod_{i \in I, j \in J} M(i, j) \arrow[r] \arrow[d, "f"'] & I \bullet 1 \times J \bullet 1 \arrow[d, "u \bullet 1 \times v \bullet 1"] \\ \coprod_{k \in K, l \in L} N(k, l) \arrow[r] & K \bullet 1 \times L \bullet 1. \end{tikzcd} \tag{17}$$

(The horizontal arrows above are induced by the terminal morphism followed by the map to the product induced by the two corresponding coprojections.) This means that f is given by a family of morphisms

$$f_{i,j} : M(i, j) \to N(u(i), v(j))$$

in $\mathcal{V}$ for every $i \in I$, $j \in J$.

Remark 4.4 By the product property, the 2-cells (16) can equivalently be described by commutative squares (18). On the other hand, when $\mathcal{V}$ is a Cartesian closed category, the functors $X \times -$ and $- \times X$ for $X \in \mathcal{V}$ are left adjoint functors. As such they preserve colimits, implying that there is a natural isomorphism $\phi_{I,J} : (I \bullet 1) \times (J \bullet 1) \cong (I \times J) \bullet 1$ in $\mathcal{V}$. Then, this implies that the squares (18) can equivalently be described

by commutative squares (19).

$$\begin{tikzcd} A \arrow[r, "\langle a_1, a_2\rangle"] \arrow[d, "f"'] & (I \bullet 1) \times (J \bullet 1) \arrow[d, "u \bullet 1 \times v \bullet 1"] \\ B \arrow[r, "\langle b_1, b_2\rangle"] & (K \bullet 1) \times (L \bullet 1) \end{tikzcd} \tag{18}$$

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] & (I \times J) \bullet 1 \arrow[d, "(u \times v) \bullet 1"] \\ B \arrow[r, "b"] & (K \times L) \bullet 1. \end{tikzcd} \tag{19}$$

Remark 4.5 Adding natural isomorphism $\phi_{I,J}$ from the above remark to the square (17) yields that the 2-cells of $\mathcal{V}$-Mat can equivalently be described as the squares:

$$\begin{tikzcd} \amalg_{i \in I, j \in J} M(i, j) \arrow[r] \arrow[d, "f"'] & (I \times J) \bullet 1 \arrow[d, "(u \times v) \bullet 1"] \\ \amalg_{k \in K, l \in L} N(k, l) \arrow[r] & (K \times L) \bullet 1 \end{tikzcd} \tag{20}$$

In this case the horizontal arrows are induced by the unique morphism to 1 followed by the corresponding coprojections.

In the pseudo-double categories $\mathrm{Span}_d(\mathcal{V})$ and $\mathcal{V}$-Mat 0- and 1h-cells are the same as 0- and 1-cells in the bicategories $Span_d(\mathcal{V})$ and $\mathcal{V}$-Mat, respectively, and 1v-cells in both pseudo-double categories are the same. It is immediate to see that the lax functor En is compatible with 2-cells (a morphism between pullbacks $En(A)(i, j)$ to $En(B)(k, l)$ exists for $(k, l) = (u(i), v(j))$ precisely because (16) commutes). In the 1v-direction En is a strict functor; thus, we get a lax double functor $\mathrm{En} : \mathrm{Span}_d(\mathcal{V}) \to \mathcal{V}$-Mat.

Conversely, starting from a 2-cell (17), it clearly induces a 2-cell of the form (16). Hence, we have that if the oplax functor $Int : \mathcal{V}\text{-}Mat \to Span_d(\mathcal{V})$ is a biequivalence, then the oplax double functor $\mathrm{Int} : \mathcal{V}\text{-Mat} \to \mathrm{Span}_d(\mathcal{V})$ is a double equivalence. The converse is also true (by restriction to identity 1v-cells), so we have: Int is a biequivalence if and only if Int is a double equivalence. A similar statement holds for En and En. In view of Proposition 4.3 we have:

Proposition 4.6 *Let $\mathcal{V}$ be a Cartesian closed category with finite limits and small coproducts. The following are equivalent:*

1. *for every set I the adjunction* (14) *is an adjoint equivalence;*
2. *$\mathcal{V}$ is extensive;*
3. *the oplax double functor* $\mathrm{Int} : \mathcal{V}\text{-Mat} \to \mathrm{Span}_d(\mathcal{V})$ *is a double equivalence;*
4. *the lax double functor* $\mathrm{En} : \mathrm{Span}_d(\mathcal{V}) \to \mathcal{V}\text{-Mat}$ *is a double equivalence.*

4.4 Monads in the double categories of matrices and spans

Our pseudo-double category $\mathrm{Span}_d(\mathcal{V})$ is a pseudo-double subcategory of $\mathrm{Span}(\mathcal{V})$ (our 1h-cells are specific 1h-cells of the latter), and we have that the vertical 1-cells in $\mathrm{Mnd}(\mathrm{Span}_d(\mathcal{V}))$ are morphisms of discretely internal categories in $\mathcal{V}$ (see the beginning of Sect. 4.3). As for our pseudo-double category $\mathcal{V}$-Mat, we find the following. A monad is given by a matrix $M = (M(i,j))_{i,j\in I}$ (1-endocell over the 0-cell I) and 2-cells which are given by families of morphisms $\mu^M_{i,k} : \amalg_{j\in I} M(j,k) \times M(i,j) \to M(i,k)$ and $\eta^M_{i,k} : \mathbb{I} \to M(i,k)$, for every $i,k \in I$, both given by a commutative diagram (17) where the maps u and v are identities on I, satisfying associativity and unity laws. (Here $\mathbb{I}$ is the unit matrix, where 1 is the terminal object and 0 the initial object in $\mathcal{V}$.) Thus, a monad in $\mathcal{V}$-Mat is precisely a monad in the bicategory $\mathcal{V}$-Mat. Given another monad $(N = (N(k,l))_{k\in K, l\in L}, \mu^N, \eta^N)$, a 1v-cell in $\mathrm{Mnd}(\mathcal{V}\text{-Mat})$ between M and N is given by a set map $\omega : I \to I$ and a commutative square (17) in which u and v are equal to ω (this square comes down to a morphism $f_{i,j} : M(i,j) \to N(\omega(i),\omega(j))$ in $\mathcal{V}$ for every $i,j \in I$) and which satisfy:

$$\begin{tikzcd}
\amalg_{j\in I} M(j,k) \times M(i,j) \arrow[r, "\mu^M_{i,k}"] \arrow[d, "\amalg_{j\in I} f_{j,k} \times f_{i,j}"'] & M(i,k) \arrow[d, "f_{i,k}"] \\
\amalg_{j\in I} N(\omega(j),\omega(k)) \times N(\omega(i),\omega(j)) \arrow[r, "\mu^N_{i,k}"] & N(\omega(i),\omega(k))
\end{tikzcd}$$

and $f_{i,j} \circ \eta^M_{i,j} = \eta^N_{i,j}$. Thus, vertical 1-cells in $\mathcal{V}$-Mat clearly correspond to enriched functors in $\mathcal{V}$.

It is immediately seen that a biequivalence of two 2-categories induces a biequivalence of the 2-categories of their monads. Passing from a strict to a weak context (from 2-categories to pseudo-double categories), one needs to be more careful with the technical details in the proof, but the analogous result is directly proved. Observe that the Definition 2.4 in [11] of the double category $\mathrm{Mnd}(\mathbb{D})$ of monads in a double category $\mathbb{D}$ is such that if $\mathbb{D}$ is a pseudo-double category, then $\mathrm{Mnd}(\mathbb{D})$ is a pseudo-double category, too.

Proposition 4.7 *If pseudo-double categories $\mathbb{D}$ and $\mathbb{E}$ are double equivalent, then their pseudo-double categories of monads* $\mathrm{Mnd}(\mathbb{D})$ *and* $\mathrm{Mnd}(\mathbb{E})$ *are also double equivalent.*

Proof The proof is straightforward, we only type the diagrams for the relevant structure 2-cells.

$$\begin{tikzcd}
F(X) \arrow[r, "F(P)"] \arrow[d, "="] \arrow[dr, phantom, "\boxed{F^{-1}_{P,P}}"] & F(X) \arrow[r, "F(P)"] \arrow[d, "="] & F(X) \arrow[d, "="] \\
F(X) \arrow[r, "="] \arrow[d, "="'] \arrow[dr, phantom, "\boxed{F^X}"] & F(X) \arrow[r, "F(PP)"] \arrow[d, "F(u)"] \arrow[dr, phantom, "\boxed{F(\mu_P)}"] & F(X) \arrow[r, "="] \arrow[d, "F(u)"'] \arrow[dr, phantom, "\boxed{(F^X)^{-1}}"] & F(X) \arrow[d, "="] \\
F(X) \arrow[r, "="] & F(X) \arrow[r, "F(P)"] & F(X) \arrow[r, "="] & F(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[r, "="] \arrow[d, "="'] \arrow[dr, phantom, "\boxed{F^{-1}_X}"] & F(X) \arrow[d, "="] \\
F(X) \arrow[r, "="] \arrow[d, "="'] \arrow[dr, phantom, "\boxed{F^X}"] & F(X) \arrow[r, "F(id_X)"] \arrow[d, "F(u)"] \arrow[dr, phantom, "\boxed{F(\eta_P)}"] & F(X) \arrow[r, "="] \arrow[d, "F(u)"'] \arrow[dr, phantom, "\boxed{(F^X)^{-1}}"] & F(X) \arrow[d, "="] \\
F(X) \arrow[r, "="] & F(X) \arrow[r, "F(P)"] & F(X) \arrow[r, "="] & F(X)
\end{tikzcd}$$

$$
\begin{tikzcd}
 & F(X) \arrow[r, "F(H)"] \arrow[d, "="'] & F(Y) \arrow[r, "F(Q)"] \arrow[d, phantom, "\boxed{F^{-1}_{P,P}}"] & F(Y) \arrow[d, "="] & \\
F(X) \arrow[r, "="] \arrow[d, "="'] \arrow[dr, phantom, "\boxed{F^X}"] & F(X) \arrow[rr, "F(QH)"] \arrow[d, "F(id)"] \arrow[dr, phantom, "\boxed{F(\Phi)}"] & & F(Y) \arrow[r, "="] \arrow[d, "F(id)"'] \arrow[dr, phantom, "\boxed{(F^Y)^{-1}}"] & F(Y) \arrow[d, "="] \\
F(X) \arrow[r, "="] & F(X) \arrow[rr, "F(HP)"] \arrow[d, "="] \arrow[dr, phantom, "\boxed{F_{P,P}}"] & & F(Y) \arrow[r, "="] \arrow[d, "="] & F(Y) \\
 & F(X) \arrow[r, "F(P)"] & F(X) \arrow[r, "F(H)"] & F(Y) &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[r, "F(P)"] \arrow[d, "F(u)"'] \arrow[dr, phantom, "\boxed{F(\overline{u})}"] & F(X) \arrow[d, "F(u)"] \\
F(X) \arrow[r, "F(P)"] & F(X)
\end{tikzcd}
$$

□

From the above said, we obtain

Corollary 4.8 *In the conditions of Proposition 4.6 the categories* $\mathcal{V}$-Cat *of* $\mathcal{V}$*-enriched categories and* $\mathrm{Cat}_d(\mathcal{V})$ *of discretely internal categories in* $\mathcal{V}$ *are equivalent.*

5 (1 × 2)-category of spans in a tricategory

The term (1 × 2)-*category* is due to [16] (see the top of page 2). It is a 1-category internal to a tricategory.

In this section, we define a structure that would be a tricategorical version of the pseudo-double category $\mathrm{Span}_d(\mathcal{V})$ from Sect. 4.3 for a category $\mathcal{V}$. We will build a category internal in a certain 1-strict tricategory T, which to shorten we will call a (1 × 2)-category $\mathcal{S}$. Then, we will also have the horizontal tricategory $\mathcal{H}(\mathcal{S})$. We will do this gradually, by introducing first two bicategories C_0 (the bicategory of objects) and C_1 (the bicategory of morphisms). Thus, the 0-cells of the tricategory T will be bicategories, then 1-cells will be some kind of two-dimensional functors between them, and we have that T is 1-strict. The precise notion of a category internal in a 1-strict tricategory was introduced in [10]. We remark that a category internal in a Gray category was introduced in [8].

5.1 Bicategories C_0 and C_1 in $\mathcal{S}$

Assume that V is a 1-strict tricategory with a terminal object 1, small 3-coproducts and 3-pullbacks. Let $C_0 = \mathrm{Cat}_2$ be the 2-category of *small* categories. Before we define C_1 we observe the following.

In Remark 4.4, we saw that morphisms between spans in a 1-category $\mathcal{V}$ could be described in two equivalent ways. Already turning to dimension 2 (if $\mathcal{V}$ were a 2-category) requires the involvement of two 2-cells, in the case of (16), respectively, of one 2-cell, in the case of the squares (18) and (19). These two approaches yield two ways of defining 1-cells in the bicategory of spans in V. However, the former is more

suitable for defining their composition. At the end of the subsection we will show that the two ways of defining these 1-cells are equivalent.

Let $\mathcal{C}$ and $\mathcal{D}$ be small categories (0-cells in C_0). The 0-cells of C_1 are *spans in* V, given by 1-cells $\underline{\mathcal{C}} \bullet 1 \longleftarrow A \longrightarrow \underline{\mathcal{D}} \bullet 1$ in V. Here, $\underline{\mathcal{C}} := \mathcal{O}\lfloor\mathcal{C}$ denotes the set of objects of $\mathcal{C}$ and $\underline{\mathcal{C}} \bullet 1$ the $\underline{\mathcal{C}}$-fold copower of 1.

Given two spans $\underline{\mathcal{C}} \bullet 1 \xleftarrow{a_1} A \xrightarrow{a_2} \underline{\mathcal{D}} \bullet 1$ and $\underline{\mathcal{E}} \bullet 1 \xleftarrow{b_1} B \xrightarrow{b_2} \underline{\mathcal{H}} \bullet 1$ and two functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{H}$ (1-cells in C_0) a 1-cell in C_1 is given by a 1-cell $f : A \to B$ and two equivalence 2-cells $\alpha : (F \bullet 1)a_1 \Rightarrow b_1 f$ and $\beta : (G \bullet 1)a_2 \to b_2 f$ in V:

$$\begin{tikzcd}
\underline{\mathcal{C}} \bullet 1 \arrow[d, "F \bullet 1"'] \arrow[dr, phantom, "\boxed{\alpha}"] & A \arrow[l, "a_1"'] \arrow[r, "a_2"] \arrow[d, "f"] \arrow[dr, phantom, "\boxed{\beta}"] & \underline{\mathcal{D}} \bullet 1 \arrow[d, "G \bullet 1"] \\
\underline{\mathcal{E}} \bullet 1 & B \arrow[l, "b_1"'] \arrow[r, "b_2"] & \underline{\mathcal{H}} \bullet 1.
\end{tikzcd} \tag{21}$$

Given another such 1-cell in C_1 with the same 0-cells:

$$\begin{tikzcd}
\underline{\mathcal{C}} \bullet 1 \arrow[d, "F' \bullet 1"'] \arrow[dr, phantom, "\boxed{\gamma}"] & A \arrow[l, "a_1"'] \arrow[r, "a_2"] \arrow[d, "g"] \arrow[dr, phantom, "\boxed{\delta}"] & \underline{\mathcal{D}} \bullet 1 \arrow[d, "G' \bullet 1"] \\
\underline{\mathcal{E}} \bullet 1 & B \arrow[l, "b_1"'] \arrow[r, "b_2"] & \underline{\mathcal{H}} \bullet 1
\end{tikzcd}$$

and natural transformations $\lambda : F \Rightarrow F' : \mathcal{C} \to \mathcal{E}$ and $\rho : G \Rightarrow G' : \mathcal{D} \to \mathcal{H}$ (2-cells in C_0), a 2-cell in C_1 between them is given by a 2-cell $\xi : f \Rightarrow g$ and two 3-cells

$$\Sigma : \frac{\alpha}{[\xi \,|\, \mathrm{Id}_{b_1}]} \Rrightarrow \frac{[\mathrm{Id}_{a_1} \,|\lambda \bullet 1]}{\gamma} \qquad \text{and} \qquad \Omega : \frac{\beta}{[\xi \,|\, \mathrm{Id}_{b_2}]} \Rrightarrow \frac{[\mathrm{Id}_{a_1} \,|\rho \bullet 1]}{\delta} \tag{22}$$

in V, which are to be considered in the transversal direction, perpendicular to the parallel planes of the 1-cells (α, f, β) and (δ, g, γ). The 2-cells in C_1 we think as transversal prisms whose bases are 1-cells in C_1.

Composition of 1-cells in C_1, which contains the horizontal composition of 2-cells in $\mathcal{H}(\mathcal{S})$, the underlying horizontal tricategory of $\mathcal{S}$, is defined by the 3-pullback. Given 1-cells (α, f, β) and (γ, g, δ) as below:

$$\begin{tikzcd}
 & A \arrow[dl, "a_1"'] \arrow[dr, "a_2"] \arrow[d, "f"] & & A' \arrow[dl, "a'_1"'] \arrow[dr, "a'_2"] \arrow[d, "g"] & \\
\underline{\mathcal{C}} \bullet 1 \arrow[d, "F \bullet 1"'] \arrow[r, phantom, "\boxed{\alpha}"] & B \arrow[dl, "b_1"] \arrow[dr, "b_2"] \arrow[r, phantom, "\boxed{\beta}"] & \underline{\mathcal{D}} \bullet 1 \arrow[d, "G \bullet 1"] \arrow[r, phantom, "\boxed{\gamma}"] & B' \arrow[dl, "b'_1"] \arrow[dr, "b'_2"] \arrow[r, phantom, "\boxed{\delta}"] & \underline{\mathcal{J}} \bullet 1 \arrow[d, "H \bullet 1"] \\
\underline{\mathcal{E}} \bullet 1 & & \underline{\mathcal{H}} \bullet 1 & & \underline{\mathcal{K}} \bullet 1
\end{tikzcd} \tag{23}$$

Taking for σ in Definition 3.4 the composite equivalence 2-cell on the right below, there are: a 1-cell h, equivalence 2-cells ζ_1, ζ_2 and an isomorphism 3-cell Σ in V:

$$\begin{tikzcd}
 & A \times_{\underline{\mathcal{D}}\bullet 1} A' \arrow[dl, "p_1"'] \arrow[d, "h"] \arrow[dr, "p_2"] & \\
A \arrow[d, "f"'] \arrow[r, phantom, "\boxed{\zeta_1}"] & B \times_{\underline{\mathcal{H}}\bullet 1} B' \arrow[dl, "\overline{p}_1"'] \arrow[dr, "\overline{p}_2"] \arrow[r, phantom, "\boxed{\zeta_2}"] & A' \arrow[d, "g"] \\
B \arrow[dr, "b_2"'] & \boxed{\zeta} & B' \arrow[dl, "b'_1"] \\
 & \underline{\mathcal{H}}\bullet 1 &
\end{tikzcd}
\quad \overset{\Sigma}{\Rrightarrow} \quad
\begin{tikzcd}
 & A \times_{\underline{\mathcal{D}}\bullet 1} A' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A \arrow[d, "f"'] \arrow[dr, "a_2"] & \boxed{\omega} & A' \arrow[d, "g"] \arrow[dl, "a'_1"'] \\
B \arrow[dr, "b_2"'] & \boxed{\beta}\ \underline{\mathcal{D}}\bullet 1 \arrow[d, "G\bullet 1"] \ \boxed{\gamma} & B' \arrow[dl, "b'_1"] \\
 & \underline{\mathcal{H}}\bullet 1 &
\end{tikzcd}$$

For the desired 1-cell $A \times_{\mathcal{D}\bullet 1} A' \to B \times_{\mathcal{H}\bullet 1} B'$ in V we take this h, and for the desired pair of equivalence 2-cells (α', β') in V we set the horizontal juxtapositions: $\alpha' = (\alpha|\zeta_1)$ and $\beta' = (\zeta_2|\delta)$ of 2-cells in V. Properly speaking, $\alpha' = \frac{[\zeta_1|\,\mathrm{Id}_{b_1}]}{[\mathrm{Id}_{p_1}\,|\alpha]}$ and $\beta' = \frac{[\zeta_2|\,\mathrm{Id}_{b_2}]}{[\mathrm{Id}_{p_2}\,|\delta]}$, where p_1, p_2 are the projections of the 3-pullback $A \times_{\mathcal{D}\bullet 1} A'$.

Since the composition of spans of the underlying 1-category of V (and consequently of 1-cells in $\mathcal{H}(\mathcal{S})$) is not strictly associative, the same holds for the 2-cells of $\mathcal{H}(\mathcal{S})$ and 1-cells of C_1. This is why C_1 is a bicategory, and not a 2-category.

Vertical composition of horizontal 2-cells in $\mathcal{S}$ (and 2-cells in $\mathcal{H}(\mathcal{S})$)—as in (21) and below it—is given in the obvious way: $(\frac{[\alpha|\,\mathrm{Id}_{F'\bullet 1}]}{[\mathrm{Id}_f\,|\gamma]}, gf, \frac{[\beta|\,\mathrm{Id}_{G'\bullet 1}]}{[\mathrm{Id}_f\,|\delta]})$. It is not strictly associative: interchange law of V must be used, as well as the following isomorphisms between 2-cells in V: $\mathrm{Id}_{X\otimes Y} \cong [\mathrm{Id}_X\,|\,\mathrm{Id}_Y]$ and the one for the associativity of the horizontal composition in V. These isomorphisms can be expressed in terms of 3-cells in $\mathcal{H}(\mathcal{S})$.

The interchange law in C_1 is expressed in terms of 3-cells of V and it holds strictly, as one hoped.

The vertical composition of 2-cells in C_1 (and of 3-cells in $\mathcal{H}(\mathcal{S})$ and horizontal 3-cells in $\mathcal{S}$) is given by obvious vertical juxtaposition of prisms. In the case of their horizontal composition we proceed as follows. Suppose we are given two composable horizontal 1-cells in $\mathcal{S}$ as in (23) (horizontally composable 2-cells in $\mathcal{H}(\mathcal{S})$) and another such a pair with the same 1-cells in V, so that only the vertically denoted 1-cells and 2-cells in V are different: $F'\bullet 1, \alpha', f', \beta', G'\bullet 1, \gamma', g', \delta', H'\bullet 1$. Suppose that we are given two horizontally composable 2-cells in C_1, each of which we present by a pair of prisms. Concretely, for simplicity reasons, the pair of 3-cells as in (22) we will identify with prisms which we will write in the present case as: $\mathcal{A} : \alpha \Rrightarrow \alpha'$, $\mathcal{B} : \beta \Rrightarrow \beta'$, $\Gamma : \gamma \Rrightarrow \gamma'$ and $\Delta : \delta \Rrightarrow \delta'$. We think $\mathcal{A}$, $\mathcal{B}$, Γ and Δ as transversal prisms going from the base squares $\alpha, \beta, \gamma, \delta$ in the back toward the base squares $\alpha', \beta', \gamma', \delta'$, in the front. On the vertically transversal planes of these four prisms are the following 2-cells in V: $\lambda\bullet 1 : F\bullet 1 \Rrightarrow F'\bullet 1$, $\xi : f \Rrightarrow f'$, $\rho : G\bullet 1 \Rrightarrow G'\bullet 1$, $\xi' : g \Rrightarrow g'$, $\rho' : H\bullet 1 \Rrightarrow H'\bullet 1$, with the obvious meanings.

We know that the horizontal composition of the bases (back) 2-cells (α, f, β) and (γ, g, δ) is given by $((\alpha|\zeta_1), h, (\zeta_2|\delta))$ and we have the isomorphism 3-cell Σ (as we explained below (23)). Analogously, at the front we have $((\alpha'|\zeta'_1), h', (\zeta'_2|\delta'))$ and we

have an analogous isomorphism 3-cell Σ':

$$\begin{tikzcd}
 & A \times_{\underline{\mathcal{D}}\bullet 1} A' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "h'"] & \\
A \arrow[d, "f'"'] \arrow[rr, phantom, "\boxed{\zeta_1'}"] & & A' \arrow[d, "g'"] \arrow[ll, phantom, "\boxed{\zeta_2'}"] \\
B \arrow[dr, "b_2"'] & B \times_{\underline{\mathcal{H}}\bullet 1} B' \arrow[l, "\overline{p}_1"] \arrow[r, "\overline{p}_2"'] \arrow[d, phantom, "\boxed{\zeta}"] & B' \arrow[dl, "b_1'"] \\
 & \underline{\mathcal{H}} \bullet 1 &
\end{tikzcd}
\quad \overset{\Sigma'}{\Rightarrow} \quad
\begin{tikzcd}
 & A \times_{\underline{\mathcal{D}}\bullet 1} A' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[d, phantom, "\boxed{\omega}"] & \\
A \arrow[d, "f'"'] \arrow[dr, "a_2"] & \underline{\mathcal{D}} \bullet 1 \arrow[d, "G' \bullet 1"] \arrow[l, phantom, "\boxed{\beta'}"] \arrow[r, phantom, "\boxed{\gamma'}"] & A' \arrow[d, "g'"] \arrow[dl, "a_1'"'] \\
B \arrow[dr, "b_2"'] & \underline{\mathcal{H}} \bullet 1 & B'. \arrow[l, "b_1'"]
\end{tikzcd}$$

Observe the following (transversal) composition of 3-cells:

$$\begin{tikzcd}
 & A \times_{\underline{\mathcal{D}}\bullet 1} A' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "h"] & \\
\underline{\mathcal{C}} \bullet 1 \arrow[d, "F \bullet 1"'] & A \arrow[l, "a_1"'] \arrow[d, "f"] \arrow[rr, phantom, "\boxed{\zeta_1}"] & & A' \arrow[r, "a_2'"] \arrow[d, "g"] \arrow[ll, phantom, "\boxed{\zeta_2}"] & \underline{\mathcal{J}} \bullet 1 \arrow[d, "H \bullet 1"] \arrow[ll, phantom, "\boxed{\delta}"] \\
\underline{\mathcal{E}} \bullet 1 \arrow[rr, phantom, "\boxed{\alpha}"] & B \arrow[l, "b_1"] \arrow[dr, "b_2"'] & B \times_{\underline{\mathcal{H}}\bullet 1} B' \arrow[l, "\overline{p}_1"] \arrow[r, "\overline{p}_2"'] \arrow[d, phantom, "\boxed{\zeta}"] & B' \arrow[r, "b_2'"] & \underline{\mathcal{K}} \bullet 1 \\
 & & \underline{\mathcal{H}} \bullet 1 \arrow[ur, "b_1'"'] & &
\end{tikzcd}
\quad \overset{(\mathrm{Id}_\alpha\,|\Sigma|\,\mathrm{Id}_\delta)}{\Rightarrow} \quad
\begin{tikzcd}
 & A \times_{\underline{\mathcal{D}}\bullet 1} A' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[d, phantom, "\boxed{\omega}"] & \\
\underline{\mathcal{C}} \bullet 1 \arrow[d, "F \bullet 1"'] & A \arrow[l, "a_1"'] \arrow[d, "f"] \arrow[dr, "a_2"] & A' \arrow[r, "a_2'"] \arrow[d, "g"] \arrow[dl, "a_1'"'] & \underline{\mathcal{J}} \bullet 1 \arrow[d, "H \bullet 1"] \arrow[ll, phantom, "\boxed{\delta}"] \\
\underline{\mathcal{E}} \bullet 1 \arrow[rr, phantom, "\boxed{\alpha}"] & B \arrow[l, "b_1"] \arrow[dr, "b_2"'] & \underline{\mathcal{D}} \bullet 1 \arrow[d, "G \bullet 1"] \arrow[l, phantom, "\boxed{\beta}"] \arrow[r, phantom, "\boxed{\gamma}"] & B' \arrow[r, "b_2'"] & \underline{\mathcal{K}} \bullet 1 \\
 & & \underline{\mathcal{H}} \bullet 1 \arrow[ur, "b_1'"'] & &
\end{tikzcd}$$

$$\Downarrow \left(\mathcal{A}|\frac{\mathrm{Id}_\omega}{(\mathcal{B}|\Gamma)}|\Delta\right)$$

$$\begin{tikzcd}
 & A \times_{\underline{\mathcal{D}}\bullet 1} A' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "h'"] & \\
\underline{\mathcal{C}} \bullet 1 \arrow[d, "F' \bullet 1"'] & A \arrow[l, "a_1"'] \arrow[d, "f'"] \arrow[rr, phantom, "\boxed{\zeta_1'}"] & & A' \arrow[r, "a_2'"] \arrow[d, "g'"] \arrow[ll, phantom, "\boxed{\zeta_2'}"] & \underline{\mathcal{J}} \bullet 1 \arrow[d, "H' \bullet 1"] \arrow[ll, phantom, "\boxed{\delta'}"] \\
\underline{\mathcal{E}} \bullet 1 \arrow[rr, phantom, "\boxed{\alpha'}"] & B \arrow[l, "b_1"] \arrow[dr, "b_2"'] & B \times_{\underline{\mathcal{H}}\bullet 1} B' \arrow[l, "\overline{p}_1"] \arrow[r, "\overline{p}_2"'] \arrow[d, phantom, "\boxed{\zeta}"] & B' \arrow[r, "b_2'"] & \underline{\mathcal{K}} \bullet 1 \\
 & & \underline{\mathcal{H}} \bullet 1 \arrow[ur, "b_1'"'] & &
\end{tikzcd}
\quad \overset{(\mathrm{Id}_{\alpha'}\,|\overline{\Sigma'}|\,\mathrm{Id}_{\delta'})}{\Leftarrow} \quad
\begin{tikzcd}
 & A \times_{\underline{\mathcal{D}}\bullet 1} A' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[d, phantom, "\boxed{\omega}"] & \\
\underline{\mathcal{C}} \bullet 1 \arrow[d, "F' \bullet 1"'] & A \arrow[l, "a_1"'] \arrow[d, "f'"] \arrow[dr, "a_2"] & A' \arrow[r, "a_2'"] \arrow[d, "g'"] \arrow[dl, "a_1'"'] & \underline{\mathcal{J}} \bullet 1 \arrow[d, "H' \bullet 1"] \arrow[ll, phantom, "\boxed{\delta'}"] \\
\underline{\mathcal{E}} \bullet 1 \arrow[rr, phantom, "\boxed{\alpha'}"] & B \arrow[l, "b_1"] \arrow[dr, "b_2"'] & \underline{\mathcal{D}} \bullet 1 \arrow[d, "G' \bullet 1"] \arrow[l, phantom, "\boxed{\beta'}"] \arrow[r, phantom, "\boxed{\gamma'}"] & B' \arrow[r, "b_2'"] & \underline{\mathcal{K}} \bullet 1 \\
 & & \underline{\mathcal{H}} \bullet 1 \arrow[ur, "b_1'"'] & &
\end{tikzcd}$$

Here, $\overline{\Sigma'}$ denotes the inverse of Σ'. Compose the domain and codomain 2-cells above vertically with $(\mathrm{Id}_{b_1}\,|\overline{\zeta}|\,\mathrm{Id}_{b_2'})$ from below, where $\overline{\zeta}$ is a quasi-inverse of ζ, and compose the above composition 3-cell with the according 3-cell induced by the identity on $\overline{\zeta}$. From the result one obtains the wanted 3-cell $((\alpha|\zeta_1), h, (\zeta_2|\delta)) \Rrightarrow ((\alpha'|\zeta_1'), h', (\zeta_2'|\delta'))$.

The transversal composition of 3-cells (Σ, ξ, Ω) and (Σ', ξ', Ω') in $\mathcal{H}(\mathcal{S})$ (as in (22)) is given by the transversal composition of the 3-cell components and the vertical composition of the 2-cells: $\frac{\xi}{\xi'}$.

This finishes the construction of 0-cells C_0 and C_1 of T, the first step to define a (1×2)-category $\mathcal{S}$ of spans in V. We finish this subsection with the promised result.

Proposition 5.1 *Provided the existence of 3-products, a 1-cell* (21) *in the bicategory C_1 of spans in V can equivalently be described by an equivalence 2-cell γ in V:*

$$\begin{tikzcd}
A \arrow[r, "\langle a_1, a_2\rangle"] \arrow[d, "f"'] \arrow[dr, phantom, "\boxed{\gamma}"] & \underline{\mathcal{C}} \bullet 1 \times \underline{\mathcal{D}} \bullet 1 \arrow[d, "F \bullet 1 \times G \bullet 1"] \\
B \arrow[r, "\langle b_1, b_2\rangle"] & \underline{\mathcal{E}} \bullet 1 \times \underline{\mathcal{H}} \bullet 1.
\end{tikzcd}$$

A 2-cell (22) *in* C_1 *can equivalently be described by a 3-cell* $\Gamma : \dfrac{[\mathrm{Id}\,|\gamma]}{[\xi\,|\,\mathrm{Id}_{\langle b_1,b_2\rangle}]} \Rrightarrow \dfrac{[\mathrm{Id}_{\langle a_1,a_2\rangle}\,|\lambda \bullet 1 \times \rho \bullet 1]}{[\gamma'|\,\mathrm{Id}]}$.

Proof This is Lemma 3.9 c) and d). The converse holds by Remark 3.11. The claim for 2-cells in C_1 follows analogously by Proposition 3.12. □

5.2 The 1-cells *u*, *s*, *t*, *c* in *T*

For the sake of saving space, we just state that $c : C_1 \times_{C_0} C_1 \to C_1$ and $u : C_0 \to C_1$ are pseudofunctors and the source and target are strict 2-functors. On 0-cells $c : C_1 \times_{C_0} C_1 \to C_1$ is given by the 3-pullback and on 1- and 2-cells we defined it in the previous section. The pseudofunctor $u : C_0 \to C_1$ we define using initial object and cells (see Sect. 3.2). We leave the construction of pseudonatural transformations a^*, l^* and r^* and modifications π^*, μ^*, λ^*, ρ^* and ϵ^* from [10, Definition 6.2] to the reader. Accordingly, we take for T, the tricategory from the beginning of the section, to be the tricategory Bicat_3 of bicategories, pseudofunctors, pseudonatural transformations and modification. For a recent reference on the construction of Bicat_3 [15]. Mind that Bicat_3 is a 1-strict tricategory weaker than a Gray category.

6 (1 × 2)-category of matrices in a tricategory

In this section, we are going to construct another category internal in the tricategory $T = \mathrm{Bicat}_3$, which is a tricategorification of the pseudo-double category $\mathcal{V}$-Mat of matrices from Sect. 4.3. We will denote this (1×2)-category by $\mathcal{M}$.

Suppose that V is a 1-strict tricategory with a terminal object 1, 3-products and small 3-coproducts. Let $D_0 = \mathrm{Cat}_2$ be the 2-category of *small* categories, as in the case of spans in V. Before defining the bicategory D_1, we first set up some notational conventions.

For a small category $\mathcal{C}$ and $A \in \mathcal{C}$, 1-cells $1 \xrightarrow{\sigma_A} \underline{\mathcal{C}} \bullet 1$ will stand for coprojections. Given two small categories $\mathcal{C}$ and $\mathcal{D}$, we will write for short $\overline{\mathcal{C}} \times \overline{\mathcal{D}} := \underline{\mathcal{C}} \bullet 1 \times \underline{\mathcal{D}} \bullet 1$. A 1-cell to the 3-product $\overline{\mathcal{C}} \times \overline{\mathcal{D}}$ induced by coprojections σ_A and σ_B with $B \in \mathcal{D}$ we will denote by $1 \xrightarrow{\langle A,B\rangle} \overline{\mathcal{C}} \times \overline{\mathcal{D}}$. Accordingly, for small categories $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{H}$ and functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{H}$ we will denote $\overline{F} \times \overline{G} := F \bullet 1 \times G \bullet 1$, and $\overline{\lambda} \times \overline{\rho} := \lambda \bullet 1 \times \rho \bullet 1$ for natural transformations λ and ρ.

6.1 Defining the bicategory D_1

We now define the bicategory D_1 of matrices in V as follows.

Given small categories $\mathcal{C}$ and $\mathcal{D}$, a *matrix in* V is a 1-cell $\coprod_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A, B) \xrightarrow{m} \overline{\mathcal{C}} \times \overline{\mathcal{D}}$ in V, where the domain 0-cell we think as a matrix of objects $M(A, B)$ in V indexed by

the objects of the categories $\mathcal{C}$ and $\mathcal{D}$. The 1-cell m is the unique 1-cell to the terminal object followed by $1 \xrightarrow{\langle A,B\rangle} \overline{\mathcal{C}} \times \overline{\mathcal{D}}$. Matrices in V are 0-cells of D_1.

Given two matrices $\coprod_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B) \xrightarrow{m} \overline{\mathcal{C}} \times \overline{\mathcal{D}}$ and $\coprod_{\substack{A'\in\underline{\mathcal{E}}\\ B'\in\underline{\mathcal{H}}}} N(A',B') \xrightarrow{n} \overline{\mathcal{E}} \times \overline{\mathcal{H}}$, for small categories $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{H}$, and given two functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{H}$ (1-cells in D_0) a *morphism of matrices* and a 1-cell of D_1 is a square:

$$\begin{tikzcd}
\coprod_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B) \arrow[r, "m"] \arrow[d, "f"] \arrow[dr, phantom, "\boxed{\nu}"] & \overline{\mathcal{C}} \times \overline{\mathcal{D}} \arrow[d, "\overline{F} \times \overline{G}"] \\
\coprod_{\substack{A'\in\underline{\mathcal{E}}\\ B'\in\underline{\mathcal{H}}}} N(A',B') \arrow[r, "n"] & \overline{\mathcal{E}} \times \overline{\mathcal{H}}
\end{tikzcd}$$

which consists of a family of pairs of equivalence 2-cells in V:

$$\begin{tikzcd}
M(A,B) \arrow[r, "{[A,B]}"] \arrow[d, "f_{A,B}"'] \arrow[dr, phantom, "\boxed{\nu_{A,B}}"] & \overline{\mathcal{C}} \times \overline{\mathcal{D}} \arrow[d, "\overline{F} \times \overline{G}"] \\
N(A',B') \arrow[r, "{[A',B']}"] & \overline{\mathcal{E}} \times \overline{\mathcal{H}}
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "{\langle A,B\rangle}"] \arrow[dr, "{\langle A',B'\rangle}"'] & \overline{\mathcal{C}} \times \overline{\mathcal{D}} \arrow[d, "\overline{F} \times \overline{G}"] \arrow[dl, phantom, "\boxed{\chi^{F,G}_{A,B}}"] \\
& \overline{\mathcal{E}} \times \overline{\mathcal{H}}
\end{tikzcd} \tag{24}$$

where $\chi^{F,G}_{A,B}$ is induced, according to Lemma 3.9 d), by equivalence 2-cells

$$\begin{tikzcd}
1 \arrow[r, "\sigma_A"] \arrow[dr, "\sigma_{A'}"'] & \underline{\mathcal{C}} \bullet 1 \arrow[d, "F \bullet 1"] \arrow[dl, phantom, "\boxed{\chi^{F}_{A}}"] \\
& \underline{\mathcal{E}} \bullet 1
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "\sigma_B"] \arrow[dr, "\sigma_{B'}"'] & \underline{\mathcal{D}} \bullet 1 \arrow[d, "G \bullet 1"] \arrow[dl, phantom, "\boxed{\chi^{G}_{B}}"] \\
& \underline{\mathcal{H}} \bullet 1,
\end{tikzcd} \tag{25}$$

indexed by pairs $(A, B) \in \underline{\mathcal{C}} \times \underline{\mathcal{D}}$, whenever there exist isomorphisms $F(A) \cong A'$ and $G(B) \cong B'$. (These isomorphisms condition also the existence of the 1-cells $f_{A,B}$.) In the case that F and G are identities we consider $A = A'$, $B = B'$. The 1-cells $[A, B]$ are the unique morphism to 1 followed by $1 \xrightarrow{\langle A,B\rangle} \overline{\mathcal{C}} \times \overline{\mathcal{D}}$.

If $\chi^{F^{-1},G^{-1}}_{A,B}$ is an equivalence 2-cell corresponding to $\overline{F^{-1}} \times \overline{G^{-1}}$, then for quasi-inverses of $\chi^{F,G}_{A,B}$ one has: $(\chi^{F,G}_{A,B})^{-1} \cong [\chi^{F^{-1},G^{-1}}_{A',B'} \mid \mathrm{Id}_{\overline{F}\times\overline{G}}]$.

Finally, given another morphism of matrices among the same matrices:

$$\begin{tikzcd}
\coprod_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B) \arrow[r, "m"] \arrow[d, "f'"] \arrow[dr, phantom, "\boxed{\nu'}"] & \overline{\mathcal{C}} \times \overline{\mathcal{D}} \arrow[d, "\overline{F'} \times \overline{G'}"] \\
\coprod_{\substack{A'\in\underline{\mathcal{E}}\\ B'\in\underline{\mathcal{H}}}} N(A',B') \arrow[r, "n"] & \overline{\mathcal{E}} \times \overline{\mathcal{H}}
\end{tikzcd}$$

and natural transformations $\lambda : F \Rightarrow F' : \mathcal{C} \to \mathcal{E}$ and $\rho : G \Rightarrow G' : \mathcal{D} \to \mathcal{H}$ (2-cells in D_0), a 2-cell in D_1 between ν and ν' is given by a 2-cell $\xi : f \Rightarrow f'$ and a

transversal prism whose bases are vertical squares of the two 1-cells, where this prism consists of a family of prisms, *i.e.,* 3-cells in V:

$$\Sigma_{A,B} : \frac{\nu_{A,B}}{[\xi_{A,B} \,|\, \mathrm{Id}_{[A',B']}]} \Rrightarrow \frac{[\mathrm{Id}_{[A,B]} \,|\overline{\lambda} \times \overline{\rho}]}{\nu'_{A,B}} \tag{26}$$

for every $(A, B) \in \underline{\mathcal{C}} \times \underline{\mathcal{D}}$. Vertical composition of 2-cells in D_1 is clear: it is induced by vertical concatenation of the corresponding 2-cells $\nu_{A,B}$.

6.2 Composition of 1-cells in D_1

The composition of matrices in V is analogous to that of matrices in a 1-category, namely in the bicategory $\mathcal{V}$-Mat from Sect. 4.2. Given matrices $\big(M(A, B)\big)_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}}$ and $\big(N(B, C)\big)_{\substack{B\in\underline{\mathcal{D}}\\ C\in\underline{\mathcal{J}}}}$ their composition is given by the matrix $\big(\sqcup_{B\in\underline{\mathcal{D}}} M(A, B) \times N(B, C)\big)_{\substack{A\in\underline{\mathcal{C}}\\ C\in\underline{\mathcal{J}}}}$ and the corresponding 1-cell to $\overline{\mathcal{C}} \times \overline{\mathcal{J}}$. This defines the composition of 1h-cells in the (1×2)-category $\mathcal{M}$ of matrices. We now define the composition of 1-cells in the bicategory D_1.

Given 1-cells ν and ν' with their respective families of 2-cells, we consider the following diagrams:

$$\begin{tikzcd}
M(A,B) \arrow[d, "="'] \arrow[drr, "{[A]}"] & & \\
M(A,B) \arrow[r, "{[A,B]}"] \arrow[d, "f_{A,B}"'] \arrow[u, phantom, "\boxed{\tilde{\theta}_1}"] \arrow[dr, phantom, "\boxed{\nu_{A,B}}"] & \overline{\mathcal{C}} \times \overline{\mathcal{D}} \arrow[r, "p_1"] \arrow[d, "\overline{F} \times \overline{G}"'] \arrow[dr, phantom, "\boxed{\omega_1}"] & \underline{\mathcal{C}} \bullet 1 \arrow[d, "F \bullet 1"] \\
M'(A',B') \arrow[r, "{[A',B']}"] \arrow[d, "="'] \arrow[u, phantom, "\boxed{\tilde{\theta}'_1}"] & \overline{\mathcal{E}} \times \overline{\mathcal{H}} \arrow[r, "p'_1"] & \underline{\mathcal{E}} \bullet 1 \\
M'(A',B') \arrow[urr, "{[A']}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \underline{\mathcal{J}} \bullet 1 \arrow[d, "="] \\
N(B,C) \arrow[r, "{[B,C]}"] \arrow[d, "g_{B,C}"'] \arrow[urr, "{[C]}"] \arrow[dr, phantom, "\boxed{\nu'_{B,C}}"] & \overline{\mathcal{D}} \times \overline{\mathcal{J}} \arrow[r, "p_2"] \arrow[d, "\overline{G} \times \overline{H}"'] \arrow[u, phantom, "\boxed{\tilde{\theta}_2}"] \arrow[dr, phantom, "\boxed{\omega_2}"] & \underline{\mathcal{J}} \bullet 1 \arrow[d, "H \bullet 1"] \\
N'(B',C') \arrow[r, "{[B',C']}"] \arrow[drr, "{[C']}"'] & \overline{\mathcal{H}} \times \overline{\mathcal{K}} \arrow[r, "p'_2"] & \underline{\mathcal{K}} \bullet 1 \arrow[d, "="] \\
& & \underline{\mathcal{K}} \bullet 1 \arrow[u, phantom, "\boxed{\tilde{\theta}'_2}"]
\end{tikzcd}$$

where the 2-cells ω_1, ω_2 are the ones from Lemma 3.9 b) and $[A] = \sigma_A!$. From Lemma 3.9 a), we have an equivalence 2-cell $\theta_1 : \sigma_A \Rightarrow p_1\sigma_{A,B}$, then let $\tilde{\theta}_1 := [\mathrm{Id}_! \,|\theta_1]$. Similarly, we define $\tilde{\theta}'_1, \tilde{\theta}_2, \tilde{\theta}'_2$. To simplify the notations, let us denote the above composite equivalence 2-cells by: $\tilde{\nu}_{B,C} : (F \bullet 1)[A] \Rightarrow [A']f_{A,B}$ and $\tilde{\nu}'_{A,B} : (H \bullet 1)[C] \Rightarrow [C']g_{B,C}$. They induce an equivalence 2-cell $\tilde{\nu}_{A,B} \times \tilde{\nu}'_{B,C}$ in the middle of the diagram:

$$
\begin{tikzcd}
 & 1 \arrow[dr, "\langle A, C\rangle"] \arrow[d, phantom, "\boxed{\gamma}"] & \\
M(A,B)\times N(B,C) \arrow[ur, "!"] \arrow[rr, "{[A]\times[C]}"] \arrow[d, "f_{A,B}\times g_{B,C}"'] \arrow[rr, phantom, shift right=6ex, "\boxed{\tilde{\nu}_{A,B}\times\tilde{\nu}'_{B,C}}"] & & \overline{\mathcal{C}}\times\overline{\mathcal{J}} \arrow[d, "\overline{F}\times\overline{H}"] \\
M'(A',B')\times N'(B',C') \arrow[rr, "{[A']\times[C']}"'] \arrow[dr, "!"'] & \arrow[d, phantom, "\boxed{\gamma'}"] & \overline{\mathcal{E}}\times\overline{\mathcal{K}} \\
 & 1 \arrow[ur, "\langle A', C'\rangle"'] &
\end{tikzcd}
\tag{27}
$$

It is easily seen how the 2-cells γ and γ' are induced.

We next observe the following diagram:

$$
\begin{tikzcd}
 & 1 \arrow[dr, "\langle A, C\rangle"] \arrow[d, phantom, "\boxed{\theta^B}"] & \\
M(A,B)\times N(B,C) \arrow[ur, "!"] \arrow[r, "\iota^B"] \arrow[d, "f_{A,B}\times g_{B,C}"] \arrow[r, phantom, shift right=6ex, "\boxed{\zeta^B}"] & \amalg_{B\in\underline{\mathcal{D}}} M(A,B)\times N(B,C) \arrow[r, "{[A,C]}"] \arrow[d, "h_{A,C}"'] & \overline{\mathcal{C}}\times\overline{\mathcal{J}} \arrow[d, "\overline{F}\times\overline{H}"] \\
M'(A',B')\times N'(B',C') \arrow[r, "\iota^{B'}"] \arrow[dr, "!"'] & \amalg_{B'\in\underline{\mathcal{H}}} M'(A',B')\times N'(B',C') \arrow[r, "{[A',C']}"] \arrow[d, phantom, "\boxed{\theta^{B'}}"] & \overline{\mathcal{E}}\times\overline{\mathcal{K}} \\
 & 1 \arrow[ur, "\langle A', C'\rangle"'] &
\end{tikzcd}
\tag{28}
$$

Taking into account every B for which there exist the 1-cells $f_{A,B}$ and $g_{B,C}$, the composite 1-cells $\iota^{B'}(f_{A,B}\times g_{B,C})$ on the left induce a 1-cell $h_{A,C}$ and an equivalence 2-cell ζ^B. We set for the total 2-cell $(\overline{F}\times\overline{H})\langle A,C\rangle! \Rightarrow \langle A',C'\rangle!(f_{A,B}\times g_{B,C})$ to be the above composite equivalence 2-cell (27). We now apply the dual of Lemma 3.9 d) by making the following dual correspondence:

$$
\begin{aligned}
&p_2 \mapsto \iota^{B'},\ p_2' \mapsto \iota^B,\ f \mapsto \overline{F}\times\overline{H},\ g\times h \mapsto h_{A,C},\\
&s_2 \mapsto \langle A,C\rangle!,\ q_2 \mapsto \langle A',C'\rangle!,\\
&s \mapsto [A,C],\ t \mapsto [A',C'],\\
&\omega_2 \mapsto \zeta^B,\ \zeta_2 \mapsto \theta^{B'},\ \theta_2 \mapsto \theta^B\\
&\beta \mapsto \text{2-cell}\quad (27).
\end{aligned}
$$

Then, there exists an equivalence 2-cell $(\overline{F}\times\overline{H})[A,C] \Rightarrow [A',C']h_{A,C}$ corresponding to the right rectangular in the diagram above (the dual of γ from Lemma 3.9). We take this 2-cell for the desired equivalence 2-cell $\nu_{A,C}$ as in (24) for the composition of matrices $\big(M(A,B)\big)_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}}$ and $\big(N(B,C)\big)_{\substack{B\in\underline{\mathcal{D}}\\ C\in\underline{\mathcal{J}}}}$.

To get an equivalence 2-cell $\chi^{F,H}_{A,C}$ from (24) is easy: it is induced by the given equivalence 2-cells χ^F_A and χ^H_C.

6.3 Composition of 2-cells in D_1

Let two 2-cells in D_1

$$\Sigma^1_{A,B} : \frac{\nu^1_{A,B}}{[\xi \,|\, \mathrm{Id}_{[A',B']}]} \Rrightarrow \frac{[\mathrm{Id}_{[A,B]} \,|\overline{\lambda} \times \overline{\rho}]}{\nu'^1_{A,B}} \quad \text{and} \quad \Sigma^2_{B,C} : \frac{\nu^2_{B,C}}{[\xi \,|\, \mathrm{Id}_{[A',B']}]} \Rrightarrow \frac{[\mathrm{Id}_{[A,B]} \,|\overline{\lambda} \times \overline{\rho}]}{\nu'^2_{B,C}}$$

be given. Recall how the 2-cell $\tilde{\nu}_{A,B}$ in V above (27) is induced by $\nu_{A,B}$. Let us denote the 2-cells induced analogously by $\nu^1_{A,B}, \nu'^1_{A,B}, \nu^2_{B,C}, \nu'^2_{B,C}$ as follows: $\tilde{\nu}^1_{A,B}, \tilde{\nu}'^1_{A,B}, \tilde{\nu}^2_{B,C}, \tilde{\nu}'^2_{B,C}$, respectively. Consider the prism $P_{\tilde{\nu}^1_{A,B}}$ with basis $\tilde{\nu}^1_{A,B}$ (and analogously the prism $P_{\tilde{\nu}^2_{B,C}}$ with basis $\tilde{\nu}^2_{B,C}$) obtained by concatenation of the following prisms: $\Sigma^1_{A,B}, C^1_{F\times G}$ from Corollary 3.16, identity 3-cells on $\tilde{\theta}_1$ and $\tilde{\theta}^i_1$. Analogously, the prism $P_{\tilde{\nu}^2_{B,C}}$ is obtained by concatenation of the prisms $\Sigma^2_{B,C}, C^2_{F\times G}$, identity 3-cells on $\tilde{\theta}_2$ and $\tilde{\theta}^i_2$. Seeing $P_{\tilde{\nu}^1_{A,B}}$ and $P_{\tilde{\nu}^2_{B,C}}$ as $\tilde{P}^1$ and $\tilde{P}^2$ in Corollary 3.20, we obtain a unique 3-cell

$$\Gamma' : \frac{[\mathrm{Id} \,|\tilde{\nu}^1_{A,B} \times \tilde{\nu}^2_{B,C}]}{[\xi \times \xi' \,|\, \mathrm{Id}_{[A',B']\times[B',C']}]} \Rrightarrow \frac{[\mathrm{Id}_{[A,B]\times[B,C]} \,|\overline{\lambda} \times \overline{\sigma}]}{[\tilde{\nu}'^1_{A,B} \times \tilde{\nu}'^2_{B,C} \,|\, \mathrm{Id}]}.$$

This is a prism with basis $\tilde{\nu}^1_{A,B} \times \tilde{\nu}^2_{B,C}$, as in the middle of (27). Concatenate to it the identity 3-cells on γ and γ' from (27) and consider the obtained prism P_{total} as the prism whose basis is the total 2-cell in (28). This is total 2-cell we treat as the 2-cell α_i in Proposition 3.14 (in the reversed direction) and it induces the equivalence 2-cell $\gamma : (\overline{F} \times \overline{H})[A, C] \Rightarrow [A', C']h_{A,C}$, which is taken for $\nu_{A,C}$. Observe that the 1-cell $h_{A,C}$ can be written as $\amalg_{B\in\underline{\mathcal{D}}} f_{A,B} \times g_{B,C}$. This 2-cell $\gamma = \nu_{A,C}$ corresponds to "γ with reversed order" in Proposition 3.14. The prism P_{total} corresponds to the 3-cell P_{α_i} in there (in the corresponding mapping direction), and we finally obtain a unique 3-cell Γ with basis γ, that is, a 3-cell $\Gamma : \nu_{A,C} \Rrightarrow \nu'_{A,C}$. This Γ is the horizontal composition 2-cell of the 2-cells $\Sigma^1_{A,B}$ and $\Sigma^2_{B,C}$ in D_1.

7 Relating matrices and spans in a tricategory

In this section, we are going to construct functors between the (1×2)-categories of matrices $\mathcal{M}$ and spans $\mathcal{S}$ in a 1-strict tricategory V with a terminal object, small 3-coproducts, 3-products and 3-pullbacks. Such a functor is internal in Bicat_3, so for that purpose we will define pseudofunctors between the bicategories C_1 of spans from Sect. 5 and D_1 of matrices from Sect. 6, and additionally check their compatibility with the (horizontal) composition on the 3-pullback. Recall that $C_0 = D_0$ is the 2-category

of small categories. We will obtain a lax internal functor $\mathcal{S} \to \mathcal{M}$ and a colax internal functor $\mathcal{M} \to \mathcal{S}$. For completeness and the sake of the next section, we introduce two formal definitions.

7.1 Internal and enriched functors in 1-strict tricategories

In this subsection, we only give the two definitions. Referring to the notion and notation from [10, Definition 6.2], we introduce:

Definition 7.1 Let $\mathcal{C}, \mathcal{D}$ be categories internal in a 1-strict tricategory V. We say that $F : \mathcal{C} \to \mathcal{D}$ is a (pseudo-/lax/colax) functor internal in V if it consists of:

1. pseudofunctors $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ such that $s \circ F_1 = F_0 \circ s$, $t \circ F_1 = F_0 \circ t$;
2. pseudonatural transformations

   $$F_\times(f, g) : F_1(g) \times_{D_0} F_1(f) \Rightarrow F_1(g \times_{C_0} f) \quad \text{and} \quad F_u(A) : u_{F_0(A)} \Rightarrow F_1(u_A)$$

   in the lax functor case ($F_\times(f, g) : F_1(g \times_{C_0} f) \Rightarrow F_1(g) \times_{D_0} F_1(f)$ and $F_u(A) : F_1(u_A) \Rightarrow u_{F_0(A)}$ in the colax functor case, and for a pseudofunctor require $F_\times(f, g)$ and $F_u(A)$ to be equivalence 2-cells) for objects $A \in C_0$ and 1h-cells $f, g \in C_1$, whose components are globular equivalences, and
3. modifications $\Omega_{a^*}, \Omega_{l^*}, \Omega_{r^*}$:

$$\begin{tikzcd}
(F_1(R) \times F_1(S)) \times F_1(T) \arrow[r, Rightarrow, "a^*_{F_1(R), F_1(S), F_1(T)}"] \arrow[d, Rightarrow, "F_\times \times \mathrm{Id}_{F_1(T)}"'] & F_1(R) \times (F_1(S) \times F_1(T)) \arrow[r, Rightarrow, "\mathrm{Id}_{F_1(R)} \times F_\times"] \arrow[dl, Rightarrow, "\Omega_{a^*}"] & F_1(R) \times F_1(S \times T) \arrow[d, Rightarrow, "F_\times"] \\
F_1(R \times S) \times F_1(T) \arrow[r, Rightarrow, "F_\times"] & F_1((R \times S) \times T) \arrow[r, Rightarrow, "F_1(\alpha_{R,S,T})"] & F_1(R \times (S \times T))
\end{tikzcd}$$

$$\begin{tikzcd}
F(R) \times F(u) \arrow[d, Rightarrow, "F_\times"'] \arrow[rr, Rightarrow, "\Omega_{r^*}"] & & F(R) \arrow[ll, Rightarrow, "\mathrm{Id}_{F_1(R)} \times F_u"'] \arrow[rr, Rightarrow, "F_u \times \mathrm{Id}_{F_1(R)}"] \arrow[d, equal] \arrow[dl, phantom, "\Omega_{l^*}"] & & F(u) \times F(R) \arrow[ll, Rightarrow, "\Omega_{l^*}"'] \arrow[d, Rightarrow, "F_\times"] \\
F(R \times u) \arrow[rr, Rightarrow, "F(r^*)"] & & F(R) & & F(u \times R) \arrow[ll, Rightarrow, "F(l^*)"']
\end{tikzcd}$$

which satisfy the two diagrammatic equations (A1) and (A2) in the "Appendix."

Referring to the notion and notation from [10, Definition 8.1] we introduce:

Definition 7.2 Let $\mathcal{T}, \mathcal{T}'$ be categories enriched in a 1-strict tricategory V. We say that $F : \mathcal{T} \to \mathcal{T}'$ is a functor enriched in V if it consists of:

1. an assignment $F_0 : \mathcal{O}\lfloor\mathcal{T} \to \mathcal{O}\lfloor\mathcal{T}'$;
2. a 1-cell $F_1 : \mathcal{T}(A, B) \to \mathcal{T}'(F(A), F(B))$ in V for all $A, B \in \mathcal{O}\lfloor\mathcal{T}$;
3. equivalence 2-cells $F_c : F_1 \cdot \circ \Rightarrow \circ' \cdot (F_1 \times F_1)$ and $F_{I_A} : F_1 \cdot I_A \Rightarrow I'_{F(A)}$, and

4. bijective 3-cells $\Omega_{a^\dagger}, \Omega_{l^\dagger}, \Omega_{r^\dagger}$:

$$\begin{tikzcd}
F_1(-\circ(-\circ-)) \arrow[rr, Rightarrow, "F_1(a^\dagger)"] \arrow[d, Rightarrow, "F_c^L"'] \arrow[drr, phantom, "\Omega_{a^\dagger}"] & & F_1((-\circ-)\circ-) \arrow[d, Rightarrow, "F_c^R"] \\
F_1(-)\circ(F_1(-)\circ F_1(-)) \arrow[rr, Rightarrow, "a'^\dagger"] & & (F(-)\circ F(-))\circ F(-)
\end{tikzcd}$$

where F_c^L and F_c^R are the obvious 2-cells induced by F_c and $\mathrm{Id}\times F_C$, and by F_c and $F_c\times\mathrm{Id}$, respectively,

$$\begin{tikzcd}
F_1(I_B\circ-) \arrow[rr, Rightarrow, "F_1(l^\dagger)"] \arrow[d, Rightarrow, "F_c(\mathrm{Id})"'] \arrow[drr, phantom, "\Omega_{l^\dagger}"] & & F_1(-) \arrow[d, Rightarrow, "l'^\dagger"] \\
(F_1\cdot I_B)\circ' F_1(-) \arrow[rr, Rightarrow, "F_{I_B}\circ'\mathrm{Id}"] & & I'_{F(B)}\circ' F_1(-)
\end{tikzcd}$$

$$\begin{tikzcd}
F_1(-\circ I_A) \arrow[rr, Rightarrow, "F_1(r^\dagger)"] \arrow[d, Rightarrow, "F_c(\mathrm{Id})"'] \arrow[drr, phantom, "\Omega_{r^\dagger}"] & & F_1(-) \arrow[d, Rightarrow, "r'^\dagger"] \\
F_1(-)\circ'(F_1\cdot I_A) \arrow[rr, Rightarrow, "\mathrm{Id}\circ' F_{I_A}"] & & F_1(-)\circ' I'_{F(A)}
\end{tikzcd}$$

which satisfy axioms analogous to those from the "Appendix" (substitute the horizontal composition on pullbacks, there denoted by $\times$ for short, with $\circ$, the pseudonatural transformations $F_\times$ and F_u by the 2-cells F_c^{-1} and F_I^{-1}, respectively, and the modifications $\Omega_{a^*}, \Omega_{l^*}, \Omega_{r^*}$ by 3-cells $\Omega_{a^\dagger}, \Omega_{l^\dagger}, \Omega_{r^\dagger}$, respectively.

7.2 From spans to matrices

We start by defining a pseudo functor $En : C_1 \to D_1$. It maps a span $\underline{\mathcal{C}}\bullet 1 \xleftarrow{r_1} R \xrightarrow{r_2} \underline{\mathcal{D}}\bullet 1$ to the matrix $En(R)$ whose (A, B)-component is given by the 3-pullback

$$\begin{tikzcd}
En(R)(A,B) \arrow[r, "!"] \arrow[d, "\iota^R_{A,B}"'] \arrow[dr, phantom, "\boxed{\omega^R_{A,B}}"] & 1 \arrow[d, "\langle A,B\rangle"] \\
R \arrow[r, "\langle r_1, r_2\rangle"] & (\underline{\mathcal{C}}\bullet 1)\times(\underline{\mathcal{D}}\bullet 1)
\end{tikzcd} \tag{29}$$

for each $A\in\mathcal{C}$, $B\in\mathcal{D}$.

We next compose the above equivalence 2-cell with the equivalence 2-cells inducing morphisms into 3-products (so far they are well known and we do not label them) and set for the composite equivalence 2-cells ω_A^R and ω_B^R:

$$\omega_A^R := \begin{tikzcd} En(R)(A,B) \arrow[r, "!"] \arrow[d, "\iota^R_{A,B}"'] \arrow[dr, phantom, "\boxed{\omega^R_{A,B}}"] & 1 \arrow[d, "\langle A,B\rangle"] \arrow[ddr, "\sigma_A"] & \\ R \arrow[r, "\langle r_1,r_2\rangle"] \arrow[drr, "r_1"'] & (\underline{\mathcal{C}} \bullet 1) \times (\underline{\mathcal{D}} \bullet 1) \arrow[dr, "p_1"] & \\ & & \underline{\mathcal{C}} \bullet 1 \end{tikzcd} \qquad \omega_B^R := \begin{tikzcd} En(R)(A,B) \arrow[r, "!"] \arrow[d, "\iota^R_{A,B}"'] \arrow[dr, phantom, "\boxed{\omega^R_{A,B}}"] & 1 \arrow[d, "\langle A,B\rangle"] \arrow[ddr, "\sigma_B"] & \\ R \arrow[r, "\langle r_1,r_2\rangle"] \arrow[drr, "r_2"'] & (\underline{\mathcal{C}} \bullet 1) \times (\underline{\mathcal{D}} \bullet 1) \arrow[dr, "p_2"] & \\ & & \underline{\mathcal{D}} \bullet 1 \end{tikzcd}$$

Lemma 7.3 *Given a morphism of spans* (21) *(we write R and S instead of A and B there), together with the above equivalence 2-cells $\omega^R_{A,B}$ and $\omega^S_{A,B}$ it induces equivalence 2-cells χ^F_A, χ^G_B as in* (25)*, which in turn induce an equivalence 2-cell $\chi^{F,G}_{A,B}$ as in* (24)*.*

Proof Consider the composite equivalence 2-cell

$$\alpha_A := \begin{tikzcd} En(R)(A,B) \arrow[r, "\iota^R_{A,B}"] \arrow[d, "="'] \arrow[dr, phantom, "\boxed{\kappa}"] & R \arrow[r, "!"] \arrow[d, "="] & 1 \arrow[r, "\sigma_A"] \arrow[d, "="] & \underline{\mathcal{C}} \bullet 1 \arrow[d, "="] \\ = & & & \\ & & & \\ = & & & \\ & & & \end{tikzcd} \qquad \begin{tikzcd} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{tikzcd} \qquad \begin{tikzcd} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{tikzcd} \qquad \begin{tikzcd} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{tikzcd}$$

$$\begin{tikzcd} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{tikzcd} \qquad \begin{tikzcd} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{tikzcd} \qquad \begin{tikzcd} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{tikzcd}$$

$$\begin{tikzcd} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{tikzcd} \qquad \begin{tikzcd} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{tikzcd} \qquad \begin{tikzcd} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{tikzcd}$$

and similarly α_B (changing to σ_B, ω^R_B and r_2). By the 3-coproduct property there are equivalence 2-ells $\gamma_A : \sigma_A! \Rightarrow r_1$ and $\gamma_B : \sigma_B! \Rightarrow r_2$. Then, set

$$\chi^F_A := F \bullet 1 \qquad \begin{tikzcd} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{tikzcd} \qquad \begin{tikzcd} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{tikzcd} \qquad \begin{tikzcd} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{tikzcd} \qquad \begin{tikzcd} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{tikzcd}$$

and similarly for χ^G_B. Finally apply Lemma 3.9 d) to get an equivalence 2-cell $\chi^{F,G}_{A,B}$. □

Given a 1-cell (21) in C_1. To define its image by En we consider the following cube:

$$\begin{tikzcd}
En(R)(A,B) \arrow[rr, "!"] \arrow[dd, "En(f)(A,B)"'] \arrow[dr, "\iota^R_{A,B}"] & \boxed{\omega^R_{A,B}} & 1 \arrow[dr, "\langle A,B\rangle"] & \\
& R \arrow[rr, "\langle r_1,r_2\rangle"] \arrow[dd, "f"] \arrow[ur, phantom, "\boxed{\kappa}"] \arrow[ur, phantom, "="] & & \overline{\mathcal{C}}\times\overline{\mathcal{D}} \arrow[dd, "\overline{F}\times\overline{G}"] \arrow[ul, phantom, "\boxed{\chi^{F,G}_{A,B}}"] \\
En(S)(A',B') \arrow[rr, "!"] \arrow[dr, "\iota^S_{A',B'}"'] & \boxed{\omega^S_{A',B'}} & 1 \arrow[dr, "\langle A',B'\rangle"] & \\
& S \arrow[rr, "\langle s_1,s_2\rangle"] & & \overline{\mathcal{E}}\times\overline{\mathcal{H}}
\end{tikzcd} \tag{30}$$

The 1-cell $En(f)(A,B)$ and the left 2-cell are to be defined. The top and bottom are the equivalence 2-cells (29), the right 2-cell is the equivalence 2-cell from Lemma 7.3, the back is a terminal (equivalence) 2-cell, and the front one is the equivalence 2-cell from Proposition 5.1, let us denote it by γ^f. By the 3-pullback property of $En(S)(A',B')$ the composite equivalence 2-cell

$$\begin{tikzcd}
\cdot \arrow[r, "!"] \arrow[d, "="] & \cdot \arrow[r, "\langle A',B'\rangle"] \arrow[d, "="] \arrow[dr, phantom, "\boxed{\chi^{-1}_{A,B}}"] & \cdot \arrow[d, "="] \\
\cdot \arrow[r, "!"] \arrow[d, "="'] & \cdot \arrow[r, "\langle A,B\rangle"] \arrow[d, "="] \arrow[dr, phantom, "\boxed{\omega^R_{A,B}}"] & \cdot \arrow[r, "\overline{F}\times\overline{G}"] \arrow[d, "="] & \cdot \arrow[d, "="] \\
\cdot \arrow[r, "\iota^R"] \arrow[d, "="'] & \cdot \arrow[r, "\langle r_1,r_2\rangle"] \arrow[d, "="] \arrow[dr, phantom, "\boxed{\gamma^f}"] & \cdot \arrow[r, "\overline{F}\times\overline{G}"] & \cdot \arrow[d, "="] \\
\cdot \arrow[r, "\iota^R"] & \cdot \arrow[r, "f"] & \cdot \arrow[r, "\langle s_1,s_2\rangle"] & \cdot
\end{tikzcd}$$

induces a 1-cell $En(f)(A,B) : En(R)(A,B) \to En(S)(A,B)$, an equivalence 2-cell $\zeta^f : f\iota^R_{A,B} \Rightarrow \iota^S_{A',B'}En(f)(A,B)$, the left face of the cube, and a bijective 3-cell $\Sigma : \dfrac{\dfrac{[\kappa\,|\,\mathrm{Id}_{\langle A',B'\rangle}]}{[\mathrm{Id}_{En(f)}\,|\omega^S_{A',B'}]}}{[\zeta^f\,|\,\mathrm{Id}_{\langle s_1,s_2\rangle}]} \Rrightarrow \dfrac{\dfrac{[\mathrm{Id}_!\,|\chi^{-1}_{A,B}]}{[\omega^R_{A,B}\,|\,\mathrm{Id}_{\overline{F}\times\overline{G}}]}}{[\mathrm{Id}_{\iota^R}\,|\gamma^f]}$. Then, we set for the equivalence 2-cell (24) determining a 1-cell in D_1 the concatenation of the four faces of the cube:

$$\nu_{A,B} := \frac{\dfrac{[\omega^R_{A,B}\,|\,\mathrm{Id}_{\overline{F}\times\overline{G}}]}{(\zeta^f|\gamma^f)}}{[\mathrm{Id}_{En(f)(A,B)}\,|(\omega^S_{A',B'})^{-1}]}. \tag{31}$$

(We recall that by $(\zeta^f|\gamma^f)$ we mean the horizontal concatenation of 2-cells, simplifying the writing of the horizontal composition of 2-cells.) Thus, $En(\alpha|f|\beta)$ is the family of these $\nu_{A,B}$ and $\chi^{F,G}_{A,B}$ from Lemma 7.3, for $(A,B) \in \underline{\mathcal{C}} \times \underline{\mathcal{D}}$.

For the image of 2-cells in D_1, we proceed as follows. Given a 2-cell (22) in C_1. Observe that for the domain and codomain 1-cells in the corresponding cubes (30) three faces remain the same (ω's and κ) and the resting three faces differ. Let

$$\nu_{A,B} = \frac{[\omega^R_{A,B}|\,\mathrm{Id}_{\overline{F}\times\overline{G}}]}{\dfrac{(\zeta^f|\gamma^f)}{[\mathrm{Id}_{En(f)(A,B)}\,|(\omega^S_{A',B'})^{-1}]}} \text{ and } \nu'_{A,B} := \frac{[\omega^R_{A,B}|\,\mathrm{Id}_{\overline{F'}\times\overline{G'}}]}{\dfrac{(\zeta^g|\gamma^g)}{[\mathrm{Id}_{En(g)(A,B)}\,|(\omega^S_{A',B'})^{-1}]}} \text{ denote}$$

the images of the 1-cells $(\alpha|f|\beta)$ and $(\gamma|g|\delta)$ by En.

The 3-cells Σ and Ω from (22) induce, on the one hand, 3-cells (prisms) over χ^F_A and χ^G_B in Lemma 7.3 and then by Proposition 5.1 a prism between $\chi^{F,G}_{\alpha,\beta}$ and $\chi^{F',G'}_{\gamma,\delta}$, and on the other hand, a 3-cell (prism) between γ^f and γ^g also by Proposition 5.1. Taking identity 3-cell over ω^R and concatenating the obtained prisms, we obtain a 3-cell κ_0 as in Lemma 3.21, and thus also a 2-cell $\gamma : En(f)(A,B) \Rightarrow En(g)(A,B)$ and a prism with basis ζ^f and opposite face ζ^g, so that all the 1-cells joining their corresponding vertexes are identities, with $\alpha = \mathrm{Id}_!$ and $\beta = [\mathrm{Id}_{\iota^R}|\xi]$. Next, apart from the prism over γ^f, between the resting 2-cells comprising $\nu_{A,B}$ and $\nu'_{A,B}$ take identity 3-cells and their corresponding prisms. Then, for the obvious concatenation of these prisms we set to be the image by En of the 2-cell (22).

For the sake of saving space, we skip the proof of compatibility of En with the composition of 1- and 2-cells. We only record that we constructed a 1-cell $w : \amalg_{B\in\underline{\mathcal{D}}} En(R)(A,B) \times En(S)(B,C) \to En(R \times_{\underline{\mathcal{D}}\bullet 1} S)(A,C)$ for spans $\underline{\mathcal{C}}\bullet 1 \xleftarrow{r_1} R \xrightarrow{r_2} \underline{\mathcal{D}}\bullet 1$ and $\underline{\mathcal{D}}\bullet 1 \xleftarrow{s_1} S \xrightarrow{s_2} \underline{\mathcal{E}}\bullet 1$, proving the laxity of an internal functor $\mathcal{S} \to \mathcal{M}$.

7.3 From matrices to spans

We define here the pseudofunctor $Int : D_1 \to C_1$. Given a matrix $\big(M(A,B)\big)_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}}$ in V (with the corresponding 1-cell to $\overline{\mathcal{C}} \times \overline{\mathcal{D}}$), the pseudofunctor Int maps it into the span $\underline{\mathcal{C}}\bullet 1 \xleftarrow{m_1} \amalg_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B) \xrightarrow{m_2} \underline{\mathcal{D}}\bullet 1$. The 1-cells m_1, m_2 are induced by the following 1-cells on $M(A,B)$, for fixed $A \in \underline{\mathcal{C}}$, $B \in \underline{\mathcal{D}}$: the unique morphism to 1 followed by the coprojections to $\underline{\mathcal{C}}\bullet 1$ and $\underline{\mathcal{D}}\bullet 1$, respectively.

Given a 1-cell in D_1, with a family of 2-cells $\nu_{A,B}$ as in the most left rectangular diagram below, we are going to define a 1-cell $\amalg f_{A,B}$ and 2-cells α and β as on the right in:

$$\begin{tikzcd}
 & \amalg_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B) \arrow[dr, "m_1"] \arrow[d, phantom, "\boxed{\zeta^M_1}"] & \\
M(A,B) \arrow[ur, "\iota^M_{A,B}"] \arrow[r, "{[A,B]}"] \arrow[d, "f_{A,B}"'] \arrow[dr, phantom, "\boxed{\nu_{A,B}}"] & \overline{\mathcal{C}} \times \overline{\mathcal{D}} \arrow[r, "p_1"] \arrow[d, "\overline{F}\times\overline{G}"] \arrow[dr, phantom, "\boxed{\omega_1}"] & \underline{\mathcal{C}}\bullet 1 \arrow[d, "F\bullet 1"] \\
N(A',B') \arrow[r, "{[A',B']}"] \arrow[dr, "\iota^N_{A',B'}"'] & \overline{\mathcal{E}} \times \overline{\mathcal{H}} \arrow[r, "p'_1"] \arrow[d, phantom, "\boxed{\zeta^N_1}"] & \underline{\mathcal{E}}\bullet 1 \\
 & \amalg_{\substack{A'\in\underline{\mathcal{E}}\\ B'\in\underline{\mathcal{H}}}} N(A',B') \arrow[ur, "n_1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathcal{C}}\bullet 1 \arrow[d, "F\bullet 1"'] \arrow[dr, phantom, "\boxed{\alpha}"] & \amalg_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B) \arrow[l, "m_1"'] \arrow[r, "m_2"] \arrow[d, "\amalg f_{A,B}"] \arrow[dr, phantom, "\boxed{\beta}"] & \underline{\mathcal{D}}\bullet 1 \arrow[d, "G\bullet 1"] \\
\underline{\mathcal{E}}\bullet 1 & \amalg_{\substack{A'\in\underline{\mathcal{E}}\\ B'\in\underline{\mathcal{H}}}} N(A',B') \arrow[l, "n_1"'] \arrow[r, "n_2"] & \underline{\mathcal{H}}\bullet 1.
\end{tikzcd}
\tag{32}$$

The 2-cells ω_1, ζ_1^M and ζ_1^N are from (the dual of) Lemma 3.9. Observe now the next diagram, where the upper 2-cell is the composite 2-cell from the left diagram above:

$$\begin{tikzcd}
M(A,B) \arrow[r, "\iota^M_{A,B}"] \arrow[d, "="] & \coprod_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B) \arrow[r, "m_1"] \arrow[d, phantom, "\boxed{\text{left composite 2-cell from above}}"] & \underline{\mathcal{C}}\bullet 1 \arrow[r, "F\bullet 1"] & \underline{\mathcal{E}}\bullet 1 \arrow[d, "="] \\
N(A',B') \arrow[r, "\iota^N_{A',B'}"] \arrow[d, "="] \arrow[r, phantom, shift right=6ex, "\boxed{\zeta^{-1}_{A,B}}"] & \coprod_{\substack{A'\in\underline{\mathcal{E}}\\ B'\in\underline{\mathcal{H}}}} N(A',B') \arrow[r, "n_1"] \arrow[d, "="] & {} \arrow[d, "="] \\
\coprod_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B) \arrow[r, "\coprod f_{A,B}"] & \coprod_{\substack{A'\in\underline{\mathcal{E}}\\ B'\in\underline{\mathcal{H}}}} N(A',B') \arrow[r, "n_1"] & {}
\end{tikzcd}$$

and $\zeta^{-1}_{A,B}$ is a quasi-inverse of a 2-cell $\zeta_{A,B}$, which together with a 1-cell $\coprod f_{A,B}$, is induced by the 1-cell $\iota^N_{A',B'} f_{A,B}$ and the 3-coproduct property. Since this new composite 2-cell is an equivalence 2-cell, by the dual of Lemma 3.8 we obtain an equivalence 2-cell α, as desired. Doing the same as above, but projecting to the second coordinate, one obtains a desired equivalence 2-cell β.

Let a 2-cell in D_1 be given, which in turn is given by a family of 3-cells (prisms)

$$\Sigma_{A,B} : \frac{\nu_{A,B}}{[\xi_{A,B}|\,\mathrm{Id}_{[A',B']}]} \Rightarrow \frac{[\mathrm{Id}_{[A,B]}\,|\overline{\lambda}\times\overline{\rho}]}{\nu'_{A,B}}$$

as in (26). Let α, β and α', β' be the 2-cells from the image by Int of the 1-cells given by $\nu_{A,B}$ and $\nu'_{A,B}$, respectively. Denote the total 2-cell in the left-hand side of (32) by α_0 and the analogous opposite 2-cell related to $\nu'_{A,B}$ by α'_0. Concatenate the equivalence 2-cell $\zeta^{-1}_{A,B}$ (the one defining $\coprod f_{A,B}$) to α_0, and analogously $(\zeta'_{A,B})^{-1}$ to α'_0. Observe that $\xi_{A,B} : f_{A,B} \Rightarrow f'_{A,B}$ induces an equivalence 2-cell $\coprod\xi_{A,B} : \coprod f_{A,B} \Rightarrow \coprod f'_{A,B}$ and that moreover there is an invertible 3-cell $P_\zeta : \frac{\zeta^{-1}_{A,B}}{[\mathrm{Id}_{\iota^M}\,|\coprod\xi_{A,B}]} \Rightarrow \frac{[\xi_{A,B}|\,\mathrm{Id}_{\iota^N}]}{(\zeta'_{A,B})^{-1}}$—both come from the dual of Corollary 3.16. In Corollary 3.15 set for ξ to be $\overline{\lambda}\otimes\mathrm{Id}_{m_1} : \overline{F}m_1 \Rightarrow \overline{F}'m_1$ and for ζ in there to be $\mathrm{Id}_{n_1}\otimes(\coprod\xi_{A,B}) : n_1(\coprod f_{A,B}) \Rightarrow n_1(\coprod f'_{A,B})$. Apart from the 3-cell P_ζ, we also have all the prisms whose bases are the constituting 2-cells in α_0 and whose opposite faces make α'_0. Now by Corollary 3.15 there is a unique 3-cell $\frac{\alpha}{[\coprod\xi_{A,B}|\,\mathrm{Id}_{n_1}]} \Rightarrow \frac{[\mathrm{Id}_{m_1}\,|\overline{\lambda}]}{\alpha'}$. Similarly one obtains a unique 3-cell $\frac{\beta}{[\coprod\xi_{A,B}|\,\mathrm{Id}_{n_2}]} \Rightarrow \frac{[\mathrm{Id}_{m_2}\,|\overline{\rho}]}{\beta'}$.

7.4 Compatibility of *Int* with 0-cells

Before checking the compatibility of Int with the composition of 1- and 2-cells, that we will not type here for the sake of saving space, one first needs to prove that there is a 1-cell $v : \coprod_{\substack{A\in\underline{\mathcal{C}}\\ C\in\underline{\mathcal{E}}}}\big(\coprod_{B\in\underline{\mathcal{D}}} M(A,B)\times N(B,C)\big) \to \big(\coprod_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B)\big)\times_{\underline{\mathcal{D}}\bullet 1}\big(\coprod_{\substack{B\in\underline{\mathcal{D}}\\ C\in\underline{\mathcal{E}}}} N(B,C)\big)$ in V. We show this as we will use it in the last section. This also leads to a colax internal functor $\mathcal{M}\to\mathcal{S}$.

To construct v, we need to find an equivalence 2-cell

$$\sigma : n_1 q_N \Rightarrow m_2 q_M \tag{33}$$

(the total 2-cell in (34)), then apart from v we will get also equivalence 2-cells λ and ρ as in (34) and an invertible 3-cell $\Sigma : \dfrac{\dfrac{\mathrm{Id}_{n_1}\otimes\rho}{\omega\otimes\mathrm{Id}_v}}{\mathrm{Id}_{m_2}\otimes\lambda} \Rrightarrow \sigma$. In the diagram we set for short $(\coprod M)\times_{\overline{\mathcal{D}}}(\coprod N) = \big(\coprod_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B)\big)\times_{\underline{\mathcal{D}}\bullet 1}\big(\coprod_{\substack{B\in\underline{\mathcal{D}}\\ C\in\underline{\mathcal{E}}}} N(B,C)\big)$:

$$\begin{tikzcd}
\coprod_{\substack{A\in\underline{\mathcal{C}}\\ C\in\underline{\mathcal{E}}}}\big(\coprod_{B\in\underline{\mathcal{D}}} M(A,B)\times N(B,C)\big) \arrow[dr, "v"] \arrow[drr, "q_N"] \arrow[ddr, "q_M"'] & \boxed{\rho} & \\
& (\coprod M)\times_{\overline{\mathcal{D}}}(\coprod N) \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dl, phantom, "\boxed{\lambda}"] \arrow[dr, phantom, "\boxed{\omega}"] & \coprod_{\substack{B\in\underline{\mathcal{D}}\\ C\in\underline{\mathcal{E}}}} N(B,C) \arrow[d, "n_1"] \\
& \coprod_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B) \arrow[r, "m_2"'] & \underline{\mathcal{D}}\bullet 1.
\end{tikzcd} \tag{34}$$

We first explain how to get q_M and q_N. In the next diagram the 1-cell $\iota^M_B p_1$ induces a 1-cell h_M and an equivalence 2-cell ζ^M_1. In turn, $\iota^M_A h_M$ similarly induces q_M and ζ^M_2. Let us denote the composite equivalence 2-cell $(\zeta^M_1|\zeta^M_2)$ by ζ_{q_M}. Analogous 1-cell q_N and equivalence 2-cell ζ_{q_N} are obtained similarly, by projecting to the second coordinate on the most left.

$$\begin{tikzcd}
M(A,B)\times N(B,C) \arrow[r, "\iota^B"] \arrow[d, "p_1"] \arrow[dr, phantom, "\boxed{\zeta^M_1}"] & \coprod_{B\in\underline{\mathcal{D}}} M(A,B)\times N(B,C) \arrow[r, "\iota^{A,C}"] \arrow[d, "h_M"] \arrow[dr, phantom, "\boxed{\zeta^M_2}"] & \coprod_{\substack{A\in\underline{\mathcal{C}}\\ C\in\underline{\mathcal{E}}}}\big(\coprod_{B\in\underline{\mathcal{D}}} M(A,B)\times N(B,C)\big) \arrow[d, "q_M"] \\
M(A,B) \arrow[r, "\iota^M_B"'] & \coprod_{B\in\underline{\mathcal{D}}} M(A,B) \arrow[r, "\iota^M_A"'] & \coprod_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B).
\end{tikzcd}$$

We set for short $f = \iota^M_A\iota^M_B p_1$, $g = \iota^N_C\iota^N_B p_2$ and $\iota = \iota^{A,C}\iota^B$, then so far we have equivalence 2-cells ζ_{q_M} and ζ_{q_N} in the diagram:

$$\begin{tikzcd}
\coprod_{\substack{A\in\underline{\mathcal{C}}\\ C\in\underline{\mathcal{E}}}}\big(\coprod_{B\in\underline{\mathcal{D}}} M(A,B)\times N(B,C)\big) \arrow[dr, "q_N"] \arrow[ddr, "q_M"'] & & \\
& M(A,B)\times N(B,C) \arrow[ul, "\iota"'] \arrow[r, "g"] \arrow[d, "f"] \arrow[dl, phantom, "\boxed{\zeta_{q_M}}"] \arrow[ur, phantom, "\boxed{\zeta_{q_N}}"] & \coprod_{\substack{B\in\underline{\mathcal{D}}\\ C\in\underline{\mathcal{E}}}} N(B,C) \arrow[d, "n_1"] \\
& \coprod_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B) \arrow[r, "m_2"'] & \underline{\mathcal{D}}\bullet 1
\end{tikzcd} \tag{35}$$

and we define a 2-cell $n_1 g \Rightarrow m_2 f$ to be the following composite equivalence 2-cell:

$$\begin{tikzcd}
M(A,B)\times N(B,C) \arrow[r, "p_2"] \arrow[d, "="] & N(B,C) \arrow[r, "\iota^N_C\iota^N_B"] \arrow[d, "="] & \coprod_{\substack{B\in\underline{\mathcal{D}}\\ C\in\underline{\mathcal{E}}}} N(B,C) \arrow[r, "n_1"] \arrow[d, phantom, "\boxed{\zeta_{n_1}}"] & \underline{\mathcal{D}}\bullet 1 \arrow[d, "="] \\
\arrow[rr, "p_2"] \arrow[rr, phantom, shift right=4ex, "\boxed{\kappa}"] \arrow[d, "="'] & \arrow[rr, "!"] \arrow[d, "="] & \arrow[rr, "\sigma_B"] & \arrow[d, "="] \\
\arrow[rr, "!"] \arrow[rr, phantom, shift right=4ex, "\boxed{\kappa^{-1}}"] \arrow[d, "="'] & M(A,B) \arrow[rr, "!"] \arrow[d, "="] & \arrow[rr, "\sigma_B"] \arrow[d, "="] & \arrow[d, "="] \\
\arrow[rr, "p_1"] & \arrow[rr, "\iota^M_A\iota^M_B"] & \coprod_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B) \arrow[rr, "m_2"] \arrow[u, phantom, "\boxed{\zeta_{m_2}}"] & \arrow[d, "="] \\
\arrow[rr, "p_1"] & & & \arrow[rr, "m_2"] & & \arrow[d, "="]
\end{tikzcd}$$

where ζ_{n_1} and ζ_{m_2} are the obvious equivalence 2-cells. Now we set the composite equivalence 2-cell $n_1 q_N \Rightarrow m_2 q_M$ in (35) to be the desired 2-cell σ in (33). Finally, by the 3-pullback property of $\big(\coprod_{\substack{A\in\underline{\mathcal{C}}\\ B\in\underline{\mathcal{D}}}} M(A,B)\big) \times_{\underline{\mathcal{D}}\bullet 1} \big(\coprod_{\substack{B\in\underline{\mathcal{D}}\\ C\in\underline{\mathcal{E}}}} N(B,C)\big)$, we get a 1-cell v and equivalence 2-cells λ and ρ in (34) and an isomorphism 3-cell Σ, as announced.

8 Equivalence of matrices and spans in a tricategory

In this final section, we examine equivalence conditions for the (1×2)-categories of matrices $\mathcal{M}$ and spans $\mathcal{S}$ in a 1-strict tricategory V to be equivalent, and also for the 1-categories of discretely internal and enriched categories in V to be equivalent. Inspired by the ideas that we exposed in Sect. 4 we start by introducing monads in (1×2)-categories and then summarize our findings in the tricategorical setting.

8.1 Monads in (1 × 2)-categories

In this subsection, we are going to introduce monads and vertical monad morphisms in a (1×2)-category $\mathbb{V}$. In the analogy with the definition of a monad in a double category [11, Definition 2.4], we introduce:

Definition 8.1 A monad in a (1×2)-category $\mathbb{V}$ is a monad in the horizontal tricategory $\mathcal{H}(\mathbb{V})$ of $\mathbb{V}$ (see Definition 8.2).

Whereas:

Definition 8.2 A monad in a tricategory V is given by a 1-endocell $T : \mathcal{A} \to \mathcal{A}$ with two 2-cells $\mu : TT \Rightarrow T$ and $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$ and 3-cells $\alpha : \frac{\mathrm{Id}_T \otimes \mu}{\mu} \Rrightarrow \frac{\mu \otimes \mathrm{Id}_T}{\mu}$, $\lambda : \frac{\mathrm{Id}_T \otimes \eta}{\mu} \Rrightarrow \mathrm{Id}_T$, $\rho : \frac{\mathrm{Id}_T \otimes \eta}{\mu} \Rrightarrow \mathrm{Id}_T$ which satisfy the usual five axioms that expressed in terms of equations of the transversal compositions of 3-cells have the form:

$$\begin{aligned}
&\frac{\alpha \otimes \mathrm{Id}}{\mathrm{Id}} \cdot \frac{\mathrm{Id}}{\alpha} \cdot \frac{\mathrm{Id} \otimes \alpha}{\mathrm{Id}} = \frac{\mathrm{Id}}{\alpha} \cdot \frac{inter}{\mathrm{Id}} \cdot \frac{\mathrm{Id}}{\alpha} \\
&\frac{\mathrm{Id} \otimes \lambda}{\mathrm{Id}} \cdot \frac{\mathrm{Id}}{\alpha} = \frac{\rho}{\mathrm{Id}} \\
&\frac{\mathrm{Id}}{\lambda} \cdot inter \cdot \frac{\mathrm{Id}}{\alpha} = \frac{\lambda}{\mathrm{Id}} \\
&\frac{\mathrm{Id}}{\rho} \cdot inter = \frac{\rho \otimes \mathrm{Id}}{\mathrm{Id}} \cdot \frac{\mathrm{Id}}{\alpha} \\
&\frac{\mathrm{Id}}{\rho} \cdot inter = \frac{\mathrm{Id}}{\lambda}.
\end{aligned}$$

We are interested in monads in the (1×2)-categories of matrices $\mathcal{M}$ and spans $\mathcal{S}$ in a 1-strict tricategory V. Being monads in their respective horizontal tricategories $\mathcal{H}(\mathcal{M})$ and $\mathcal{H}(\mathcal{S})$, observe that their 3-cells α, λ, ρ for associativity and unitality are given through both 2-cells and 3-cells in V. We explain now how the five axioms for α, λ, ρ come down to 3-cells analogous to those in [10, Definition 6.2, 4)]. We will restrict to a particular kind of monads. Namely, we consider those monads in $\mathcal{H}(\mathcal{S})$ whose 2-cells μ are given by two identity 2-cells and identity vertical 1-cells (*i.e.,* identity functors), see (21). We will refer to such monads *strict monads*. Let a strict monad be given by a cospan $\overline{\mathcal{C}} \xleftarrow{s} T \xrightarrow{t} \overline{\mathcal{C}}$ and a 1-cell $c : T \times_{\underline{\mathcal{C}}} T \to T$ in V determining the 2-cell μ for the monad. Then, we have $sc = sp_1$ and $tc = tp_2$ (we restrict to strict monads precisely in order to have the latter identities hold strictly, for the purpose of Proposition 8.3). The associativity 3-cell α is then given by (a pair of prisms determined by) a 2-cell $a^* : c(\mathrm{Id}_T \times_{\underline{\mathcal{C}}} c) \Rightarrow c(c \times_{\underline{\mathcal{C}}} \mathrm{Id}_T)$ and a pair of 3-cells of the form $\frac{\mathrm{Id}_{(sp_1)p_1 = sc(c \times \mathrm{Id})}}{\mathrm{Id}_s \otimes a*} \Rrightarrow \mathrm{Id}_{\mathrm{Id}_{(sp_1)p_1 = sc(\mathrm{Id} \times c)}}$ and an analogous one for t. To study the first of the five axioms above, observe that each of the 3-cells that are being composed in the equation comes down to the horizontal composition of a^* and the relating identity 2-cells ($\otimes$ becomes $\times_{\underline{\mathcal{C}}}$ and $\frac{\bullet}{\bullet}$ becomes horizontal composition of 2-cells in V), while the transversal composition of those 3-cells comes down to the vertical composition of the obtained 2-cells. Thus, the first of the five axioms means

that two pairs of 3-cells (prisms) $(P_s^{\Lambda}, P_t^{\Lambda})$ and (P_s^{P}, P_t^{P}) are equal so that $P_s^{\Lambda} = P_s^{P}$ yields that two 3-cells of the form $\Omega_{\Lambda} : \mathrm{Id}_s \otimes \Lambda \Rrightarrow \mathrm{Id}$ and $\Omega_P : \mathrm{Id}_s \otimes P \Rrightarrow \mathrm{Id}$ are equal, where

$$\Lambda = \frac{\dfrac{\mathrm{Id}_c \otimes (\mathrm{Id}_{id_T} \times_{\underline{\mathcal{C}}} a^*)}{a^* \otimes \mathrm{Id}_{1 \times_{\underline{\mathcal{C}}} c \times_{\underline{\mathcal{C}}} 1}}}{\mathrm{Id}_c \otimes (a^* \times_{\underline{\mathcal{C}}} \mathrm{Id}_{id_T})} \quad \text{and} \quad P = \frac{\dfrac{a^* \otimes \mathrm{Id}_{1 \times_{\underline{\mathcal{C}}} 1 \times_{\underline{\mathcal{C}}} c}}{\mathrm{Id}_c \otimes \mathrm{Nat}}}{a^* \otimes \mathrm{Id}_{c \times_{\underline{\mathcal{C}}} 1 \times_{\underline{\mathcal{C}}} 1}}.$$

This implies that the 2-cells $\mathrm{Id}_s \otimes \Lambda$ and $\mathrm{Id}_s \otimes P$ are equal, yielding a bijective 3-cell $\pi^* : \Lambda \Rrightarrow P$. (Proposition 5.1 gives a hint that it is sufficient to consider the equality of components related to s.) The similar reasoning is applied to the other four axioms, and one finds that they induce bijective 3-cells $\mu^*, \lambda^*, \rho^*, \varepsilon^*$, respectively, which satisfy the axioms from [10, Definition 6.2, 4)].

We do the same for matrices in V: we consider strict monads in matrices in V and come to analogous conclusions. A *strict monad* in $\mathcal{H}(\mathcal{M})$ is a monad in $\mathcal{H}(\mathcal{M})$ for which F and G are identities and for every $A, B \in \mathcal{C}$ it is $\nu_{A,B} \cong \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}\times\mathcal{C}}} \otimes \kappa_{f_{A,B}}$ and $\chi_{A,B}^{F,G}$ are identities, see (24).

Analogously to the well-known fact that monads in the bicategories of matrices and spans in a 1-category $\mathcal{C}$ which has pullbacks, products and coproducts are categories enriched, respectively internal, in $\mathcal{C}$, we have:

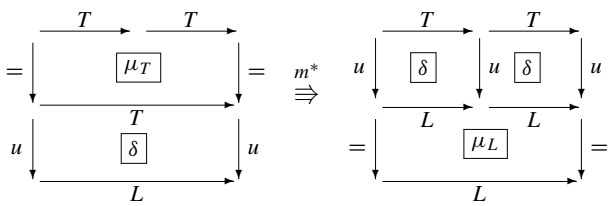
Proposition 8.3 *A strict monad in $\mathcal{S}$, the (1×2)-category of spans in a 1-strict tricategory V, is a category discretely internal in V in the sense of* [10, Definition 6.2].

A strict monad in $\mathcal{M}$, the (1×2)-category of matrices in a 1-strict tricategory V, is a category enriched in V in the sense of [10, Definition 8.1].

Discretely internal here means that the object of objects is a 3-coproduct of copies of the terminal object.

Rather than defining a (1×2)-category of monads in a (1×2)-category $\mathbb{V}$, in analogy to the double category of monads in a double category from [11], for simplicity reasons we restrict ourselves to defining only the *vertical morphisms* of monads in $\mathbb{V}$.

Definition 8.4 A vertical morphism between monads $T : \mathcal{A} \to \mathcal{A}$ and $L : \mathcal{A}' \to \mathcal{A}'$ in a (1×2)-category $\mathbb{V}$ is a horizontal 2-cell δ as below together with 3-cells:



and

$$\begin{tikzcd} \cdot \arrow[r, "="] \arrow[d, "="'] \arrow[dr, phantom, "\boxed{\eta_T}"] & \cdot \arrow[d, "="] \\ \cdot \arrow[r, "T"'] \arrow[d, "u"'] \arrow[dr, phantom, "\boxed{\delta}"] & \cdot \arrow[d, "u"] \\ \cdot \arrow[r, "L"'] & \cdot \end{tikzcd} \overset{i^*}{\Rrightarrow} \begin{tikzcd} \cdot \arrow[r, "="] \arrow[d, "u"'] \arrow[dr, phantom, "\boxed{\mathrm{Id}}"] & \cdot \arrow[d, "u"] \\ \cdot \arrow[r, "="] \arrow[d, "="'] \arrow[dr, phantom, "\boxed{\eta_L}"] & \cdot \arrow[d, "="] \\ \cdot \arrow[r, "L"'] & \cdot \end{tikzcd}$$

satisfying the following axioms which we express in terms of equations of the transversal compositions of 3-cells:

$$\frac{\mathrm{Id}}{\alpha} \cdot \frac{\xi^{-1}(\mathrm{Id} \otimes m^*)\xi}{\mathrm{Id}} \cdot \frac{\mathrm{Id}}{m^*} = \frac{\xi^{-1}(m^* \otimes \mathrm{Id})\xi}{\mathrm{Id}} \cdot \frac{\mathrm{Id}}{m^*} \cdot \frac{\alpha}{\mathrm{Id}}$$
$$l \cdot \frac{\mathrm{Id}}{\lambda} \cdot \frac{\xi^{-1}(\mathrm{Id} \otimes i^*)\xi}{\mathrm{Id}} = \frac{\lambda}{\mathrm{Id}} \cdot \frac{\mathrm{Id}}{(m^*)^{-1}}$$
$$r \cdot \frac{\mathrm{Id}}{\rho} \cdot \frac{\xi^{-1}(i^* \otimes \mathrm{Id})\xi}{\mathrm{Id}} = \frac{\rho}{\mathrm{Id}} \cdot \frac{\mathrm{Id}}{(m^*)^{-1}}.$$

Here, ξ stands for the interchange 3-cell, and l, r are left and right unity constraints for the horizontal composition of 2-cells in V.

We denote the category of monads and their vertical morphisms in a (1×2)-category $\mathbb{V}$, with the vertical composition of horizontal 2-cells in $\mathbb{V}$ (that is, of vertical monad morphisms δ), by $\mathrm{Mnd}(\mathbb{V})$.

Analogously to Proposition 4.7, the following is straightforward to prove:

Proposition 8.5 *For two equivalent* (1×2)*-categories* $\mathbb{V}_1$ *and* $\mathbb{V}_2$*, their respective categories of monads* $\mathrm{Mnd}(\mathbb{V}_1)$ *and* $\mathrm{Mnd}(\mathbb{V}_2)$ *are equivalent.*

8.2 Equivalence of (1×2)-categories of matrices and spans and of discretely internal and enriched categories in a tricategory

In analogy to bicategorical biequivalence functors, a trifunctor is a triequivalence if and only if it is pseudo by nature (*i.e.,* it is compatible with the composition of 1-cells up to an equivalence 2-cell), it is essentially surjective on the class of objects and its every component bicategorical functor is an equivalence pseudofunctor (a biequivalence).

By their construction, the internal categories $\mathcal{M}$ and $\mathcal{S}$ are equivalent in any of the two following cases:

- the bicategories C_1 and D_1 from Sects. 5 and 6 are biequivalent;
- the horizontal tricategories $\mathcal{H}(\mathcal{M})$ and $\mathcal{H}(\mathcal{S})$ are triequivalent.

For the second case we may consider the trifunctor $\mathcal{I} : \mathcal{H}(\mathcal{M}) \to \mathcal{H}(\mathcal{S})$ which is identity on 0-cells and on hom-bicategories for fixed 0-cells, which are small categories $\mathcal{C}$ and $\mathcal{D}$, consider the clear restriction $Int_{\mathcal{C},\mathcal{D}} : V\text{-}\mathrm{Mat}(\mathcal{C}, \mathcal{D}) \to \mathrm{Span}_d(V)(\mathcal{C}, \mathcal{D})$ of the pseudofunctor $Int : D_1 \to C_1$ to the hom-bicategories of $\mathcal{H}(\mathcal{M})$ and $\mathcal{H}(\mathcal{S})$,

i.e., the obvious sub-bicategories V-Mat$(\mathcal{C}, \mathcal{D})$ of D_1 and $\mathrm{Span}_d(V)(\mathcal{C}, \mathcal{D})$ of C_1. By the above observation $\mathcal{I}$ is a triequivalence if and only if for all small categories $\mathcal{D}$ the 1-cell v in (34) is a biequivalence 1-cell in V and the pseudofunctors $Int_{\mathcal{C},\mathcal{D}}$ are biequivalences for all small categories $\mathcal{C}$ and $\mathcal{D}$.

We next study when v is a biequivalence 1-cell. Let us consider the following sub-tricategories of $\mathcal{H}(\mathcal{M})$ and $\mathcal{H}(\mathcal{S})$. First consider the sub-tricategory of $\mathcal{H}(\mathcal{M})$ in which at all cell levels matrices indexed over pairs of small categories $(\mathcal{C}, \mathcal{D})$ are replaced by matrices indexed over pairs $(*, \mathcal{D})$, that is, lists indexed over small categories $\mathcal{D}$. Correspondingly, all higher cells on pairs $(\mathcal{C}, \mathcal{D})$ are replaced by identity higher cells over categories $\mathcal{D}$. Next, take a sub-tricategory of the latter sub-tricategory, where we fix a single 0-cell $\mathcal{D}$. We denote this tricategory by $V^{\mathcal{D}}$. Importantly, observe that its only hom-bicategory is the sub-bicategory V-Mat$(*, \mathcal{D})$ of D_1.

Fully analogously to $V^{\mathcal{D}}$, let $V/(\underline{\mathcal{D}} \bullet 1)$ be the sub-tricategory of $\mathcal{H}(\mathcal{S})$ with a fixed 0-cell $\mathcal{D}$ and whose only hom-bicategory is the sub-bicategory $\mathrm{Span}_d(V)(*, \mathcal{D})$ of C_1.

Let us now consider the trifunctor

$$\amalg : V^{\underline{\mathcal{D}}} \to V/(\underline{\mathcal{D}} \bullet 1) \tag{36}$$

that is identity on the unique 0-cell $\mathcal{D}$ and on hom-bicategories set

$$\amalg_{\mathcal{D},\mathcal{D}} = Int_{*,\mathcal{D}} : V\text{-Mat}(*, \mathcal{D}) \to \mathrm{Span}_d(V)(*, \mathcal{D}),$$

the restriction of the pseudofunctor $Int : D_1 \to C_1$.

Remark 8.6 Saying that the trifunctor $\coprod : V^{\underline{\mathcal{D}}} \to V/(\underline{\mathcal{D}} \bullet 1)$ has a property P for every small category $\mathcal{D}$ is the same as saying that the trifunctor $\coprod : V^{\underline{\mathcal{C}} \times \underline{\mathcal{D}}} \to V/((\underline{\mathcal{C}} \times \underline{\mathcal{D}}) \bullet 1)$ has it for all small categories $\mathcal{C}$ and $\mathcal{D}$ (replace $\mathcal{D}$ by $\mathcal{C} \times \mathcal{D}$ in one direction, and $\mathcal{C}$ by the trivial category *, in the other).

If V is 3-Cartesian closed the trifunctors $X \times -$, $- \times X$ preserve 3-coproducts, for any object X of V. Then, there is a natural biequivalence 1-cell $\phi : \underline{\mathcal{C}} \bullet 1 \times \underline{\mathcal{D}} \bullet 1 \to (\underline{\mathcal{C}} \times \underline{\mathcal{D}}) \bullet 1$ in V with (naturality) equivalence 2-cells Φ below for all functors F, G:

$$\begin{tikzcd}
\underline{\mathcal{C}} \bullet 1 \times \underline{\mathcal{D}} \bullet 1 \arrow[r, "\phi"] \arrow[d, "F \bullet 1 \times G \bullet 1"'] \arrow[dr, phantom, "\boxed{\Phi}"] & (\underline{\mathcal{C}} \times \underline{\mathcal{D}}) \bullet 1 \arrow[d, "(F \times G) \bullet 1"] \\
\underline{\mathcal{E}} \bullet 1 \times \underline{\mathcal{H}} \bullet 1 \arrow[r, "\phi'"] & (\underline{\mathcal{E}} \times \underline{\mathcal{H}}) \bullet 1.
\end{tikzcd}$$

In this case by Proposition 5.1, we obtain that there is a biequivalence of bicategories $\mathrm{Span}_d(V)(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Span}_d(V)(*, \mathcal{C} \times \mathcal{D})$, being the latter the hom-bicategory of the (sub-)tricategory $V/((\underline{\mathcal{C}} \times \underline{\mathcal{D}}) \bullet 1)$ (concatenate the above equivalence 2-cells Φ to the 2-cell γ in Proposition 5.1). On the other hand, it is clear that V-Mat$(*, \mathcal{C} \times \mathcal{D}) \simeq V$-Mat$(\mathcal{C}, \mathcal{D})$. Then, we may observe that the trifunctor $\coprod : V^{\underline{\mathcal{C}} \times \underline{\mathcal{D}}} \to V/((\underline{\mathcal{C}} \times \underline{\mathcal{D}}) \bullet 1)$ on hom-bicategories is given indeed by $Int_{1,\mathcal{C}\times\mathcal{D}} = Int_{\mathcal{C},\mathcal{D}}$.

Due to the remark, we may state:

Proposition 8.7 *If V is 3-Cartesian closed the trifunctor $\coprod : V^{\underline{\mathcal{D}}} \to V/(\underline{\mathcal{D}} \bullet 1)$ is a triequivalence for all $\mathcal{D}$ if and only if it is "pseudo" and the pseudofunctors $Int_{\mathcal{C},\mathcal{D}}$ are biequivalences of bicategories for all small categories $\mathcal{C}$ and $\mathcal{D}$.*

The following result is a tricategorification of [6, Proposition 3.1].

Proposition 8.8 *Let V be 3-Cartesian closed and assume that for every small category $\mathcal{D}$, the trifunctor $\coprod : V^{\underline{\mathcal{D}}} \to V/(\underline{\mathcal{D}} \bullet 1)$ preserves binary 3-products. Then, the 1-cell v in (34) is a biequivalence in V (and consequently, the trifunctor $\mathcal{I} : \mathcal{H}(\mathcal{M}) \to \mathcal{H}(\mathcal{S})$ is "pseudo").*

Proof Binary 3-products in $V^{\underline{\mathcal{D}}}$ are given pointwise, while binary 3-products in $V/(\underline{\mathcal{D}} \bullet 1)$ are given by 3-pullback. By the Cartesian closedness of V the 3-product trifunctors $X \times -$ and $- \times X$ commute with 3-coproducts. Thus, the 3-coproduct $\coprod_{\substack{A \in \underline{\mathcal{C}} \\ C \in \underline{\mathcal{E}}}} \big(M(A, B) \times N(B, C)\big)$ can be seen as a 3-product $\big(\coprod_{A \in \underline{\mathcal{C}}} M(A, B)\big) \times \big(\coprod_{C \in \underline{\mathcal{E}}} N(B, C)\big)$, i.e., a 3-pullback over 1. When the trifunctor $\coprod_{B \in \underline{\mathcal{D}}}$ acts on it, by assumption it sands it to a 3-product, which in $V/(\underline{\mathcal{D}} \bullet 1)$ is a 3-pullback over $\mathcal{D}$. This means that the outer arrows in (34) denote a 3-pullback, while obviously the inner square diagram in there denotes a 3-pullback for the same cospan in V. Since 3-limits are unique up to biequivalence 1-cells (recall Remark 3.6), the comparison 1-cell v is a biequivalence in V, making $\mathcal{I}$ into a pseudo trifunctor. □

For the converse, we may assume less:

Proposition 8.9 *Let V be 3-Cartesian closed and assume that for every small category $\mathcal{D}$ the 1-cell $v : \coprod_{B \in \underline{\mathcal{D}}} M(*, B) \times N(B, *) \to \big(\coprod_{B \in \underline{\mathcal{D}}} M(*, B)\big) \times_{\underline{\mathcal{D}} \bullet 1} \big(\coprod_{B \in \underline{\mathcal{D}}} N(B, *)\big)$ (a special case of (34)) is a biequivalence in V (and consequently, the trifunctor $\coprod : V^{\underline{\mathcal{D}}} \to V/(\underline{\mathcal{D}} \bullet 1)$ is "pseudo"). Then, the trifunctor $\coprod : V^{\underline{\mathcal{D}}} \to V/(\underline{\mathcal{D}} \bullet 1)$ preserves binary 3-products.*

Proof The proof is fully analogous to the direct direction of [6, Proposition 3.1]. □

Corollary 8.10 *In a 3-Cartesian closed 1-strict tricategory V with terminal object, 3-(co)products and 3-pullbacks, the following are equivalent:*

1. *the trifunctor $\coprod : V^{\underline{\mathcal{D}}} \to V/(\underline{\mathcal{D}} \bullet 1)$ is "pseudo";*
2. *the trifunctor $\mathcal{I} : \mathcal{H}(\mathcal{M}) \to \mathcal{H}(\mathcal{S})$ is "pseudo";*
3. *the trifunctor $\coprod : V^{\underline{\mathcal{D}}} \to V/(\underline{\mathcal{D}} \bullet 1)$ preserves binary 3-products.*

As a by-product of the above corollary, one obtains that the 1-cell

$$v : \coprod_{\substack{A \in \underline{\mathcal{C}} \\ C \in \underline{\mathcal{E}}}} \big(\coprod_{B \in \underline{\mathcal{D}}} M(A, B) \times N(B, C)\big) \to \big(\coprod_{\substack{A \in \underline{\mathcal{C}} \\ B \in \underline{\mathcal{D}}}} M(A, B)\big) \times_{\underline{\mathcal{D}} \bullet 1} \big(\coprod_{\substack{B \in \underline{\mathcal{D}} \\ C \in \underline{\mathcal{E}}}} N(B, C)\big)$$

from (34) is a biequivalence for all $\mathcal{C}, \mathcal{D}, \mathcal{E}$ if and only if so is its special case 1-cell $v : \coprod_{B \in \underline{\mathcal{D}}} M(*, B) \times N(B, *) \to \big(\coprod_{B \in \underline{\mathcal{D}}} M(*, B)\big) \times_{\underline{\mathcal{D}} \bullet 1} \big(\coprod_{B \in \underline{\mathcal{D}}} N(B, *)\big)$ for every $\mathcal{D}$.

From all the above said, we obtain:

Corollary 8.11 *In a 3-Cartesian closed 1-strict tricategory V with terminal object, 3-(co)products and 3-pullbacks, the following are equivalent:*

1. *the trifunctor* $\coprod : V^{\underline{\mathcal{D}}} \to V/(\underline{\mathcal{D}} \bullet 1)$ *is a triequivalence for all* $\mathcal{D}$*;*
2. *the trifunctor* $\mathcal{I} : \mathcal{H}(\mathcal{M}) \to \mathcal{H}(\mathcal{S})$ *is a triequivalence;*
3. *the colax internal functor* $\mathcal{M} \to \mathcal{S}$ *(constructed in Sects. 7.3 and 7.4) is an equivalence of* (1×2)*-categories.*

Remark 8.12 Analogously to $\mathcal{I} : \mathcal{H}(\mathcal{M}) \to \mathcal{H}(\mathcal{S})$, we may consider the trifunctor $\mathcal{E} : \mathcal{H}(\mathcal{S}) \to \mathcal{H}(\mathcal{M})$ which is identity on 0-cells and on hom-bicategories the clear restriction $En_{\mathcal{C},\mathcal{D}} : \mathrm{Span}_d(V)(\mathcal{C}, \mathcal{D}) \to V\text{-Mat}(\mathcal{C}, \mathcal{D})$ of the pseudofunctor $En : C_1 \to D_1$. By analogy to [6, Proposition 2.2] one has that the trifunctors $\mathcal{I} : \mathcal{H}(\mathcal{M}) \to \mathcal{H}(\mathcal{S})$ and $\mathcal{E} : \mathcal{H}(\mathcal{S}) \to \mathcal{H}(\mathcal{M})$ are 3-adjoint. Then, the following two equivalent statements can be added as two additional equivalent conditions in the above corollary: $\mathcal{E} : \mathcal{H}(\mathcal{S}) \to \mathcal{H}(\mathcal{M})$ is a triequivalence if and only if the lax internal functor $\mathcal{S} \to \mathcal{M}$ (constructed in Sect. 7.2) is an equivalence of (1×2)-categories.

In any of the equivalent conditions of the above corollary, by Proposition 8.5 the categories of monads of $\mathcal{M}$ and $\mathcal{S}$ are equivalent. As a matter of fact, analogously as lax functors of monoidal categories preserve monoids, any lax trifunctor preserves monads in tricategories. Hence, $\mathcal{E}$ preserves monads, and $\mathcal{I}$ does so if it is "pseudo" (for example in the conditions of Proposition 8.8). Consequently, in exactly the same conditions the internal functors $\mathcal{M} \to \mathcal{S}$ and $\mathcal{S} \to \mathcal{M}$ preserve monads. It is easily and directly proved that $\mathcal{E}$ and $\mathcal{I}$ (the latter being "pseudo") preserve strict monads. Thus, we have:

Proposition 8.13 *Under conditions of Proposition 8.8, the trifunctor $\mathcal{I}$ preserves strict monads,* i.e., *due to Proposition 8.3 it maps categories enriched in V into categories discretely internal in V.*

Moreover, let us restrict to *strict vertical morphisms of monads in $\mathcal{S}$*—those for which the equivalence 2-cells α and β in (21) are identities, and to *strict vertical morphisms of monads in $\mathcal{M}$*—those for which $\nu_{A,B} \cong \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}\times\mathcal{C}}} \otimes \kappa_{f_{A,B}}$ and $\chi^{F,G}_{A,B}$ are identities for all $A, B \in \underline{\mathcal{C}}$ in (24). Then, it is not difficult to see that strict vertical morphisms of monads in $\mathcal{S}$ yield internal functors in V and strict vertical morphisms of monads in $\mathcal{M}$ yield enriched functors in V, recall Sect. 7.1.

Now by Proposition 8.3 and the latter, we finally obtain:

Corollary 8.14 *Let V be 3-Cartesian closed with terminal object, 3-(co)products and 3-pullbacks and assume that for every small category $\mathcal{D}$, the trifunctor* $\coprod : V^{\underline{\mathcal{D}}} \to V/(\underline{\mathcal{D}} \bullet 1)$ *is a triequivalence. Then, the subcategories of strict monads and strict vertical morphisms of monads* $\mathrm{Mnd}_s(\mathcal{M})$ *and* $\mathrm{Mnd}_s(\mathcal{S})$ *of* $\mathrm{Mnd}(\mathcal{M})$ *and* $\mathrm{Mnd}(\mathcal{S})$*, respectively, are equivalent. Equivalently, the categories of discretely internal and enriched categories in V are equivalent.*

We finish the paper with a comparison of our results with those from [10, Section 8]. We assume Cartesian closedness in Proposition 8.8 in order to have that the trifunctors $X \times -$ and $- \times X$ commute with 3-coproducts, which is the second preservation

assumption in [10, Proposition 8.4]. That the trifunctor $\coprod : V^{\underline{\mathcal{D}}} \to V/(\underline{\mathcal{D}}\bullet 1)$ preserves binary 3-products can be restated so that the 3-coproduct functor maps binary 3-products, which are special kind of 3-pullbacks, into 3-pullbacks in V. Thus requiring that the 3-coproduct functor preserves 3-pullbacks, which is the first preservation assumption in [10, Proposition 8.4], is a little bit stronger than the latter. The result of [10, Proposition 8.4] concerns an enriched category $\mathcal{T}$ in V and n-ary 3-products of its endo-hom 0-cells in V, so that for $n = 2$ it claims the existence of biequivalence 1-cells

$$\begin{aligned}&\textstyle\coprod_{B\in Ob\mathcal{T}}\big((\coprod_{A\in Ob\mathcal{T}}\mathcal{T}(A,B)) \times (\coprod_{C\in Ob\mathcal{T}}\mathcal{T}(B,C))\big)\\ &\quad\to (\textstyle\coprod_{A,B\in Ob\mathcal{T}}\mathcal{T}(A,B)) \times_{(\coprod_{B\in Ob\mathcal{T}}1_B)} (\coprod_{B,C\in Ob\mathcal{T}}\mathcal{T}(B,C)).\end{aligned}$$

Thus, it is a special case of our Proposition 8.8. Finally, under the above assumptions in [10, Proposition 8.5] it is proved that an enriched category in V is an internal category in V, which is our Proposition 8.13. It is clear that our present constructions give a much broader picture than the approach that we employed in [10, Section 8], for which we followed [9].

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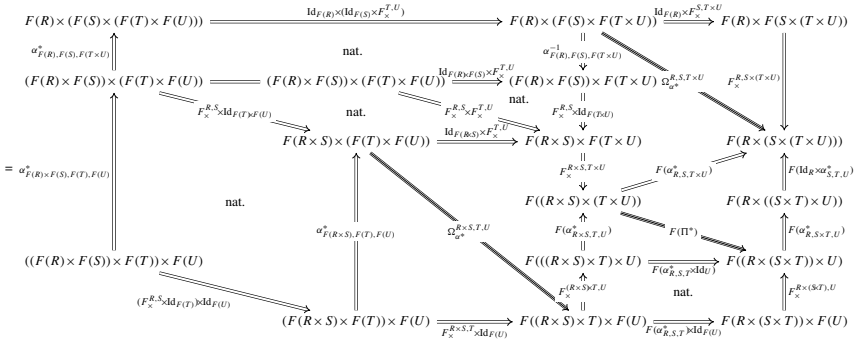
Data availability This is an original research manuscript in higher category theory. All data generated or analyzed during this study are included in this published article and in the scientific articles it refers to.

Appendix: Axioms for 3-cells of an internal functor

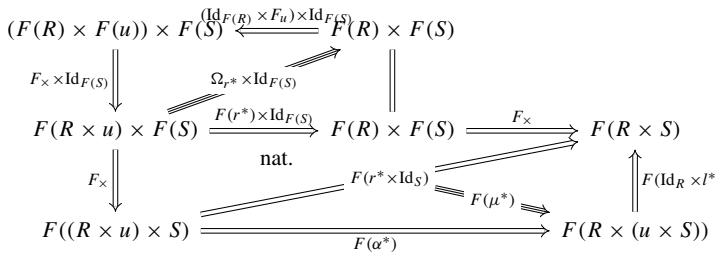
In these diagrams, the symbol $\times$ will stand for short for the pullback $- \times_{C_0} -$ or $- \times_{D_0} -$.

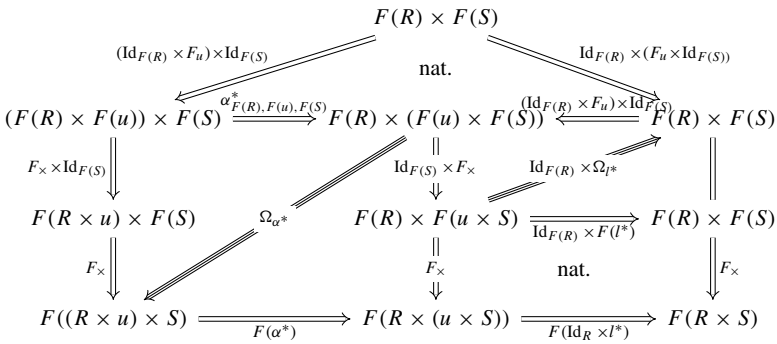
(A1)

$$\begin{tikzcd}
 & F(R)\times(F(S)\times F(T\times U)) \arrow[r, "\mathrm{Id}_{F(R)}\times F_{\times}^{S,T\times U}"] \arrow[dd, "\mathrm{Id}_{F(R)}\times(\mathrm{Id}_{F(S)}\times F_{\times}^{T,U})"'] \arrow[ddr, Rightarrow] & F(R)\times F(S\times(T\times U)) \arrow[r, "F_{\times}^{R,S\times(T\times U)}"] \arrow[d, "\mathrm{Id}_{F(R)}\times F(\alpha^{*}_{S,T,U})"] \arrow[dr, phantom, "\text{nat.}"] & F(R\times(S\times(T\times U))) \\
(F(R)\times F(S))\times(F(T)\times F(U)) \arrow[r, "\alpha^{*}_{F(R),F(S),F(T\times U)}"] \arrow[ur, Rightarrow, "\pi^{*}"] \arrow[uur, Rightarrow] \arrow[dr, Rightarrow] & F(R)\times(F(S)\times(F(T)\times F(U)) \arrow[d, equal] \arrow[dr, Rightarrow, "\mathrm{Id}_{F(R)}\times\Omega_{\alpha}^{S,T,U}"] & F(R)\times F((S\times T)\times U) \arrow[r, "F_{\times}^{R,(S\times T)\times U}"] \arrow[d, equal] & F(R\times((S\times T)\times U)) \arrow[u, "F(\mathrm{Id}_{R}\times\alpha^{*}_{S,T,U})"'] \\
 & F(R)\times((F(S)\times F(T))\times F(U)) \arrow[r, "(\mathrm{Id}_{F(R)}\times F_{\times}^{S,T})\times\mathrm{Id}_{F(U)}"] \arrow[u, "\mathrm{Id}_{F(R)}\times\alpha^{*}_{F(S),F(T),F(U)}"] \arrow[d, equal] & F(R)\times(F(S\times T)\times F(U)) \arrow[u, "\mathrm{Id}_{F(R)}\times F_{\times}^{S\times T,U}"'] \arrow[d, equal] \arrow[dr, Rightarrow] & \\
 & (F(R)\times(F(S)\times F(T)))\times F(U) \arrow[r, "\mathrm{Id}_{F(R)}\times(F_{\times}^{S,T}\times\mathrm{Id}_{F(U)})"] \arrow[u, "\alpha^{*}_{F(R),F(S),F(S\times T),F(U)}"] \arrow[dr, Rightarrow] & (F(R)\times F(S\times T))\times F(U) \arrow[u, "\alpha^{*}_{F(R),F(S\times T),F(U)}"'] \arrow[d, equal] \arrow[dr, Rightarrow, "\Omega_{\alpha}^{R,S\times T,U}"] \arrow[dr, phantom, "\text{nat.}"] & \\
((F(R)\times F(S))\times F(T))\times F(U) \arrow[uu, "\alpha^{*}_{F(R)\times F(S),F(T),F(U)}"] \arrow[ur, Rightarrow, "\alpha^{*}_{F(R),F(S),F(T)}\times\mathrm{Id}_{F(U)}"] \arrow[dr, Rightarrow, "(F_{\times}^{R,S}\times\mathrm{Id}_{F(T)})\times\mathrm{Id}_{F(U)}"'] & & F(R\times(S\times T))\times F(U) \arrow[r, "F_{\times}^{R,S\times T}\times\mathrm{Id}_{F(U)}"] \arrow[d, equal] & F((R\times(S\times T))\times U) \arrow[uu, "F(\alpha^{*}_{R,S\times T,U})"'] \\
 & (F(R\times S)\times F(T))\times F(U) \arrow[r, "F_{\times}^{R\times S,T}\times\mathrm{Id}_{F(U)}"] & F((R\times S)\times T)\times F(U) \arrow[u, "F(\alpha^{*}_{R,S,T})\times\mathrm{Id}_{F(U)}"'] &
\end{tikzcd}$$



(A2)





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