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Probabilistic temporal logic with countably additive semantics ☆

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This work presents a proof-theoretical and model-theoretical approach to probabilistic temporal logic. We present two novel logics; each of them extends both the language of linear time logic (LTL) and the language of probabilistic logic with polynomial weight formulas. The first logic is designed for reasoning about probabilities of temporal events, allowing statements like "the probability that A will hold in next moment is at least the probability that B will always hold" and conditional probability statements like "probability that A will always hold, given that B holds, is at least one half", where A and B are arbitrary statements. We axiomatize this logic, provide corresponding sigma additive semantics and prove that the axiomatization is sound and strongly complete. We show that the satisfiability problem for our logic is decidable, by presenting a procedure which runs in polynomial space. We also present a logic with much richer language, in which probabilities are not attached only to temporal events, but the language allows arbitrary nesting of probability and temporal operators, allowing statements like "probability that tomorrow the chance of rain will be less than 80% is at least a half". For this logic we prove a decidability result.

1. Introduction

The study of temporal logics started with the seminal work of Arthur Prior [37]. Temporal logics are designed in order to analyze and reason about the way that systems change over time, and have been shown

☆ This paper is a revised and extended version of our conference paper [6] presented at the 31st Conference on Uncertainty in Artificial Intelligence (UAI 2015), in which we introduced a logic for reasoning about probabilities of temporal formulas, and where probability was expressed by linear weight formulas (linear weight formulas are introduced in the seminal paper "A logic for reasoning about probabilities" by [14]). In this work we develop an extension of the logic from [6] by allowing so called polynomial weight formulas (first introduced in the same paper by [14]) in order to allow one to express *conditional* probabilities of temporal events. In this paper we adopt the axiomatization and the proof of completeness from [6]. We also prove the new decidability and complexity results for satisfiability problem for the enriched logic. In addition, in this work we also present another, richer logic where both probabilistic and temporal modalities can be combined in an arbitrary way, and for which we prove decidability.

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to be useful tools in describing behavior of an agent's knowledge base, for specification and verification of programs, hardware, protocols in distributed systems etc. [10,11]. In many practical situations the temporal information is not known with certainty. A typical example is formal representation of information about tracking moving objects with GPS systems, in the case in which the locations or the identities of the objects are not certainly known [18].

In this work, our focus is on logical formalization of the above mentioned uncertainty about time. Many different tools have been developed for representing, and reasoning with, uncertain knowledge. One particular line of research concerns the formalization in terms of probabilistic logic. After Nilsson [32] presented a procedure for probabilistic entailment which, given probabilities of premises, calculates bounds on the probabilities of the derived sentences, researchers from the field started investigation about formal systems for probabilistic reasoning. Fagin et al. [14] proposed a logic with formulas which can express linear combinations of probabilities, called *linear weight formulas*, i.e., the formulas of the form $a_1w(\alpha_1)+...+a_kw(\alpha_k) \geq r$, where a_j's and r are rational numbers. They proposed a finitary axiomatization for the logic and proved weak completeness ("every consistent formula is satisfiable"), using a small model theorem. In the same paper, they also considered a richer probabilistic language, obtained by adding multiplication to the syntax (so called *polynomial weight formulas*), in order to allow representation of conditional probabilities. In order to axiomatize that logic, they needed to extend the language even further, allowing variables (intended to range over the reals) in the formulas, and first-order quantification over them (so called *first-order weight formulas*).

In this paper, we extend the approach from [14] with polynomial weight formulas (PWFs). We start with the propositional linear time logic (LTL) [16] with the "next" operator $\bigcirc$ and "until" operator U. The meaning of the formula $\bigcirc\alpha$ is "α holds in the next time instance", and $\alpha U\beta$ we read "α holds in every time instance until β holds". We present two logics that extend both PWF and LTL. In the first logic, we apply the probabilistic operator w to the formulas of LTL and define probabilistic formulas using the PWFs in order to allow conditional probabilities, like in [14]. In this logic, that we denote by PL_{LTL}, there are two types of formulas, LTL formulas and probabilistic formulas, with the requirement that if an LTL formula is true in a model, then its probability is equal to 1. The main technical challenge in axiomatizing such a logic lies in the fact that the set of models of the formula $\alpha U\beta$ can be represented as a countable union of models of temporal formulas which are pairwise disjoint. As a consequence, finitely additive semantics is obviously not appropriate for such a logic, and we propose σ-additive semantics for the logic. On the other hand, expressing σ-additivity with an axiom would require infinite disjunctions, and the resulting logic would be undecidable. We show in Section 3.1 that any finitary axiomatic system wouldn't be complete for the σ-additive semantics.

In order to overcome this problem, we axiomatize our language using infinitary rules of inference. Thus, in this work the term "infinitary" concerns the meta language only, i.e., the object language is countable and formulas are finite, while only proofs are allowed to be infinite. We prove that our axiomatization is sound and strongly complete ("every consistent set of formulas is satisfiable"). We prove strong completeness using an adaptation of Henkin's construction. There are several logics which combine time and probability in different ways [19,20,23,24,33,39], but, to the best of our knowledge, this is the first complete axiomatization for the σ-additive probabilistic semantics. We also prove that the satisfiability problem for our logic is decidable, using decidability of LTL and combining a variant of the method of filtration with a reduction to a system of inequalities. In addition, we present a decidability procedure which runs in polynomial space, thus showing that the problem is not more complex than LTL alone nor PWF alone.

Then we introduce our second logic MPL_{LTL}. It overcomes the limitation of the logic PL_{LTL}, in which we could apply probability operators on top of temporal operators, but not the other way around. The logic MPL_{LTL} uses non-restricted modal approach to probabilistic temporal logic, where all modalities can be nested in an arbitrary way. In this case, the semantics of LTL is naturally generalized in the way that all time instances of temporal models (*paths*) are equipped with probability spaces (over other paths). In addition to nesting of operators, we modify the language in the way that we allow that different agents place (possibly different) probabilities on events (following [13]), thus indexing the probability operators w_i with agents. Therefore, the formulas of MPL_{LTL} can express not only probability that expresses agents' uncertainty about temporal events (which was already expressible in PL_{LTL}), but they can also express uncertainty of one agent about uncertainty (possibly about time) of another agent. It seems a quite complex task to provide an axiomatization for σ-additive semantics of such a logic and we left that problem for future work, in which we hope we can combine the techniques presented here with those from [26]. On the other hand, we proved that the problem of satisfiability of formulas of our logic MPL_{LTL} is decidable.

The rest of the paper is organized as follows. In Section 2 we present the syntax and semantics of our logic PL_{LTL} in detail. In Section 3 we propose an axiomatization for PL_{LTL}, and we prove some the results about the axiomatization. In Section 4 we prove that the axiomatization is strongly complete with respect to the proposed class of measurable structures. In Section 5 we show that the satisfiability problem for PL_{LTL} is decidable in PSPACE. In Section 6, we present the syntax and semantics of our second logic MPL_{LTL}. In Section 7 we discuss decidability of MPL_{LTL}. We conclude in Section 8.

2. The logic PL_{LTL}: syntax and semantics

We present the syntax and semantics of the logic for probabilistic reasoning about linear time formulas, that we denote by PL_{LTL}. The logic contains two types of formulas: formulas of LTL without probabilities, and the polynomial weight formulas (PWFs) in the style of [14], with weights applied to temporal formulas.

In order to give semantics to the formulas of PL_{LTL}, we first briefly review some probability theory [1]. We denote by ω the set of all natural numbers, and we accept the convention that $0 \in \omega$. If $W \neq \emptyset$, then H is an algebra of subsets of W, if it is a set of subsets of W such that:

(a) $W \in H$,
(b) if $A, B \in H$, then $W \setminus A \in H$ and $A \cup B \in H$.

For an algebra H, a function $\mu : H \longrightarrow [0,1]$ is a (σ-additive) probability measure, if the following conditions hold:

(1) $\mu(W) = 1$,
(2) $\mu(\bigcup_{i\in\omega} A_i) = \sum_{i\in\omega} \mu(A_i)$, whenever each $A_i \in H$, $\bigcup_{i\in\omega} A_i \in H$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$.

For W, H and μ described above, the triple $\langle W, H, \mu\rangle$ is called a probability space. A function $\mu : H \longrightarrow [0,1]$ is a finitely additive probability measure, if the condition

(3) $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A \cap B = \emptyset$,

holds, instead of (2). We also say that an algebra H is a σ-algebra, if $\bigcup_{i\in\omega} A_i \in H$ whenever $A_i \in H$ for every $i \in \omega$.

For a finitely additive μ, the condition (2) is equivalent to the condition

(2') $\mu(\bigcup_{i\in\omega} A_i) = \lim_{n\to+\infty} \mu(\bigcup_{i=0}^{n} A_i)$.

We will actually use (2') in the axiomatization of our logic (see the inference rule R6).

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2.1. Syntax

First we introduce LTL formulas. Suppose that $\mathcal{P}$ is a nonempty finite set of propositional letters. We denote the elements of $\mathcal{P}$ with p and q, possibly with subscripts.

Definition 1 *(LTL formula).* An *LTL formula* is any formula built from propositional letters from $\mathcal{P}$, using the Boolean connectives $\neg$ and $\wedge$, and the temporal operators $\bigcirc$ and U.

We use For_{LTL} for the set of all LTL formulas and denote arbitrary LTL formulas by α, β and γ, possibly with subscripts.

Note that in this paper we use $\neg$ and $\wedge$ as the primitive connectives, while other Boolean connectives ($\rightarrow$, $\vee$, $\leftrightarrow$) can be introduced as usual. We also define other LTL operators F (sometime) and G (always) as abbreviations: $F\alpha$ is $\top U\alpha$, and $G\alpha$ is $\neg F\neg\alpha$. Note that we use the strong version of U, which means that if $\alpha U\beta$ holds in a path, then β must hold eventually.

Example 1. The expression

$$\bigcirc(p \wedge q) \rightarrow (pU\neg q)$$

is an LTL formula. Its intended meaning is "if both p and q hold in the next moment, then p will hold until q becomes false".

Semantics for LTL formulas consists of the set of *paths*, where a path is an ω-structure in $\mathcal{P}$, of the form $\sigma = s_0, s_1, s_2, \ldots$ Here s_i, called *the i-th time instance of* σ, is a subset of $\mathcal{P}$, and $p \in s_i$ means that the propositional letter p is true at time i in σ. We denote the set of all paths with $\overline{\Sigma}$. In the rest of the paper, we use the following abbreviations:

- $\sigma_{\geq i}$ is the path $s_i, s_{i+1}, s_{i+2}, \ldots$, and
- σ_i is the state s_i.

The evaluation function $v : \overline{\Sigma} \times For_{LTL} \longrightarrow \{0, 1\}$ is defined recursively as follows:

- if $p \in \mathcal{P}$, then $v(\sigma, p) = 1$ iff $p \in \sigma_0$,
- $v(\sigma, \neg\alpha) = 1$ iff $v(\sigma, \alpha) = 0$,
- $v(\sigma, \alpha \wedge \beta) = 1$ iff $v(\sigma, \alpha) = 1$ and $v(\sigma, \beta) = 1$,
- $v(\sigma, \bigcirc\alpha) = 1$ iff $v(\sigma_{\geq 1}, \alpha) = 1$,
- $v(\sigma, \alpha U\beta) = 1$ iff there is some $i \in \omega$ such that $v(\sigma_{\geq i}, \beta) = 1$, and for each $j \in \omega$, if $0 \leq j < i$ then $v(\sigma_{\geq j}, \alpha) = 1$.

We say that α is true in the path σ, if $v(\sigma, \alpha) = 1$.

Remark 1. Note that in the literature, the evaluation of LTL formulas in paths is usually given in terms of satisfiability relation $\models$. We do not follow this notation, because in this paper we use $\models$ to denote satisfiability of formulas in PL_{LTL}-structures, that will be defined later. However, in Section 6, where we present the logic MPL_{LTL}, we will come back to the usual notation of satisfiability (since MPL_{LTL} does not contain two types of formulas).

Now we introduce the probabilistic formulas of PL_{LTL}. First we define the probabilistic terms.

Definition 2 *(Probabilistic term).* The set of probabilistic terms $Term$ is defined recursively as follows:

- $Term^0 = \{w_i(\alpha) \mid \alpha \in For_{LTL}\} \cup \{0,1\}$,
- $Term^{n+1} = Term^n(F) \cup \{(\mathbf{f}+\mathbf{g}), (\mathbf{f}\cdot\mathbf{g}), (-\mathbf{f}) \mid \mathbf{f},\mathbf{g} \in Term^n\}$, and
- $Term = \bigcup_{n=0}^{\infty} Term^n$.

We use $\mathbf{f}, \mathbf{g}$ and $\mathbf{h}$, possibly with indices, to denote probabilistic terms. We use a number of abbreviations throughout the paper for readability. For example, we introduce the usual abbreviations, like: $\mathbf{f}+\mathbf{g}$ is $(\mathtt{f}+\mathbf{g})$, $\mathtt{f}+\mathbf{g}+\mathbf{h}$ is $((\mathtt{f}+\mathbf{g})+\mathbf{h})$. Similarly, $-\mathtt{f}$ is $(-\mathtt{f})$, $\mathtt{f}-\mathbf{g}$ is $(\mathtt{f}+(-\mathbf{g}))$, and so on. In the same way, we can assume that the *integers are also probabilistic terms*, if we adopt the abbreviations 2 is $1+1$, 3 is $2+1$,. . . and, similarly, $2\mathtt{f}$ is $\mathtt{f}+\mathtt{f}$ and so on.

Definition 3 *(Probabilistic formula).* A basic probabilistic formula is any formula of the form $\mathbf{f} \geq 0$, where $\mathbf{f}$ is a probabilistic term. The set For_P of probabilistic formulas is the smallest set containing all basic probabilistic formulas, closed under Boolean connectives.

We denote by ϕ, ψ and θ (possibly with indices) the elements of For_P. To simplify notation, we define the following abbreviations: $\mathbf{f} \geq \mathbf{g}$ is $\mathbf{f}-\mathbf{g} \geq 0$, $\mathbf{f} \leq \mathbf{g}$ is $\mathbf{g} \geq \mathbf{f}$, $\mathbf{f} < \mathbf{g}$ is $\neg\mathbf{f} \geq \mathbf{g}$, $\mathbf{f} > \mathbf{g}$ is $\neg\mathbf{f} \leq \mathbf{g}$ and $\mathbf{f} = \mathbf{g}$ is $\mathbf{f} \geq \mathbf{g} \wedge \mathbf{f} \leq \mathbf{g}$.

More importantly, we may assume that *rational numbers are also terms*, since they can be eliminated from a formula by clearing the denominator. For example, the formula

$$\frac{2}{3}\mathbf{f} \geq \frac{5}{7}\mathbf{g}$$

is simply an abbreviation for $14\mathbf{f} - 15\mathbf{f} \geq 0$.

Example 2. The expression

$$w(p \vee q) = w(\bigcirc p) \rightarrow w(Gq) \leq \frac{1}{2}$$

is a probabilistic formula. Its meaning is “if the probability that either p or q hold in this moment is equal to the probability that p will hold in the next moment, then the probability that q will always hold is at most one half”.

Similarly, the expression

$$w(Gp) = \frac{1}{2} \wedge 5w(\bigcirc q \wedge Gp) \geq 3w(Gp)$$

is a probabilistic formula, which says that “the probability that p will always hold is one half, and the conditional probability that q will hold in the next moment if p will always hold is at least $\frac{3}{5}$”. Note that the second conjunct is a conditional probability statement about temporal events, simply rewritten as a probability formula of our language by clearing the denominators.

Definition 4 *(Formula).* The set For of all formulas of the logic PL_{LTL} is $For = For_{LTL} \cup For_P$.

We denote arbitrary formulas by Φ and Ψ (possibly with subscripts). We denote by $\bot$ both $\phi \wedge \neg\phi$ and $\alpha \wedge \neg\alpha$, letting the context determines the meaning. Similarly, we use $\top$ for both LTL and probabilistic formulas.

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Example 3. The expression

$$(p \vee \bigcirc q) \rightarrow w(p \vee \bigcirc q) = 1$$

is not a formula, since mixing LTL formulas and probabilistic formulas is not allowed, by Definition 4.

2.2. Semantics

The semantics of the logic PL_{LTL} is based on the possible-world approach.

Definition 5 *(PL_{LTL} structure).* A PL_{LTL} structure is a tuple $M = \langle W, H, \mu, \pi \rangle$ where:

- W is a nonempty set of worlds,
- $\langle W, H, \mu \rangle$ is a probability space, and
- $\pi : W \longrightarrow \overline{\Sigma}$ provides for each world $w \in W$ a path $\pi(w)$.

For a PL_{LTL} structure $M = \langle W, H, \mu, \pi \rangle$, we define

$$[\alpha]_M = \{w \in W \mid v(\pi(w), \alpha) = 1\}.$$

We say that M is *measurable*, if $[\alpha]_M \in H$ for every $\alpha \in For_{LTL}$. We denote the class of all measurable PL_{LTL} structures with PL_{LTL}^{Meas}. In this paper, we focus our attention on the class of measurable structures. We prove both a completeness theorem and decidability of PL_{LTL} with respect to the class PL_{LTL}^{Meas}.

Definition 6 *(Value of a probabilistic term).* Given a probabilistic term $\mathbf{f}$ and a measurable structure $M = \langle W, H, \mu, \pi \rangle \in PL_{LTL}^{Meas}$, we define the value of $\mathbf{f}$ in M, denoted by $\mathbf{f}^M$, recursively as follows:

- $0^M = 0$, $1^M = 1$.
- $w(\alpha)^M = \mu([\alpha]_M)$,
- $(\mathbf{f} + \mathbf{g})^M = \mathbf{f}^M + \mathbf{g}^M$.
- $(\mathbf{f} \cdot \mathbf{g})^M = \mathbf{f}^M \cdot \mathbf{g}^M$.
- $(-\mathbf{f})^M = -(\mathbf{f}^M)$.

Now we define the satisfiability of a formula from For in a structure from PL_{LTL}^{Meas}.

Definition 7 *(Satisfiability).* Let $M = \langle W, H, \mu, \pi \rangle$ be a PL_{LTL} structure. We define the satisfiability relation $\models \subseteq PL_{LTL}^{Meas} \times For$ recursively as follows:

- $M \models \alpha$ iff $v(\pi(w), \alpha) = 1$ for every $w \in W$,
- $M \models \mathbf{f} \geq 0$ iff $\mathbf{f}^M \geq 0$,
- $M \models \neg\phi$ iff $M \not\models \phi$,
- $M \models \phi \wedge \psi$ iff $M \models \phi$ and $M \models \psi$.

Now we define the notion of a model.

Definition 8 *(Model).* We say that $M \in PL_{LTL}^{Meas}$ is a model of Φ, if $M \models \Phi$. A formula Φ is valid, if $M \models \Phi$ holds for every $M \in PL_{LTL}^{Meas}$. We say that M is a model of a set of formulas T, and we write $M \models T$, iff $M \models \Phi$ for every $\Phi \in T$. A set of formulas T is satisfiable if there is M such that $M \models T$.

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The notion of entailment is defined in the usual way.

Definition 9 *(Entailment).* We say that the set of formulas T entails a formula Φ, and we write $T \models \Phi$, if for all $M \in PL_{LTL}^{Meas}$, $M \models T$ implies $M \models \Phi$.

For every $\alpha, \beta \in For_{LTL}$, let us denote by $\alpha \overline{U}_n \beta$ the formula

$$(\bigwedge_{k=0}^{n-1} \bigcirc^k \alpha) \wedge \bigcirc^n \beta,$$

and by $\alpha U_n \beta$ the formula

$$\bigvee_{k=0}^{n} \alpha \overline{U}_k \beta.$$

Those formulas will play the important role in our axiomatization. Obviously, $v(\sigma, \alpha U \beta) = 1$ iff there is some $n \in \omega$ such that $v(\sigma, \alpha \overline{U}_n \beta) = 1$, and

$$[\alpha U \beta]_M = \bigcup_{n \in \omega} [\alpha \overline{U}_n \beta]_M. \tag{1}$$

Similarly,

$$[\alpha U \beta]_M = \bigcup_{n \in \omega} [\alpha U_n \beta]_M. \tag{2}$$

Since (1) follows directly from the definition of the evaluation function v, we will use it to properly axiomatize LTL part of our logic. On the other hand, (2) is more convenient for capturing σ-additivity.

3. The axiomatization of PL_{LTL}

In this section we provide an axiomatization for PL_{LTL}, which we denote by $AX_{PL_{LTL}}$. Let us first discuss some axiomatization issues. By (2) and σ-additivity, we obtain

$$\mu([\alpha U \beta]_M) = \mu(\bigcup_{n \in \omega} [\alpha U_n \beta]_M) = \lim_{n \to +\infty} \mu(\bigcup_{k=1}^{n} [\alpha U_k \beta]_M).$$

We can see that the set

$$T = \{w(\alpha U \beta) > r\} \cup \{w(\alpha U_n \beta) \leq r \mid n \in \omega\}$$

is an unsatisfiable set of formulas. On the other hand, it is easy to check that every finite subset of T is satisfiable. In other words, the logic is not compact. It is known that, in this case, any finitary axiomatization would fail to be strongly complete [25]. Here we use an infinitary rule (R6) to obtain completeness, and, in particular, to make the set T inconsistent. It turns out that it is necessary (see the proof of Theorem 4) to introduce another infinitary rule (R4) to properly axiomatize LTL part of the logic, since the set of LTL formulas $\{\alpha U \beta\} \cap \{\neg(\alpha \overline{U}_n \beta) \mid n \in \omega\}$ is also an example of non-compactness.

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3.1. The axiomatic system $AX_{PL_{LTL}}$

The axiomatization $AX_{PL_{LTL}}$ contains 8 axioms and 6 rules of inference. We divide the axioms into 3 groups as given below.

Tautologies

A1. All instances of classical propositional tautologies for both LTL and probabilistic formulas.

Temporal axioms

A2. $\bigcirc(\alpha \to \beta) \to (\bigcirc\alpha \to \bigcirc\beta)$.
A3. $\neg \bigcirc \alpha \leftrightarrow \bigcirc\neg\alpha$.
A4. $\alpha U\beta \leftrightarrow \beta \vee (\alpha \wedge \bigcirc(\alpha U\beta))$.

Axioms about commutative ordered rings

A5. All For_P-instances of axioms about commutative ordered rings.

Probabilistic axioms

A6. $w(\alpha) \geq 0$.
A7. $w(\alpha \wedge \beta) + w(\alpha \wedge \neg\beta) = w(\alpha)$.
A8. $w(\alpha \to \beta) = 1 \to w(\alpha) \leq w(\beta)$.

Inference rules

R1. From Φ and $\Phi \to \Psi$ infer Ψ (where either $\Phi, \Psi \in For_{LTL}$ or $\Phi, \Psi \in For_P$).
R2. From α infer $\bigcirc\alpha$.
R3. From α infer $w(\alpha) = 1$.
R4. From the set of premises

$$\{\gamma \to \neg(\alpha\overline{U}_n\beta) \mid n \in \omega\}$$

infer $\gamma \to \neg(\alpha U\beta)$.
R5. From the set of premises

$$\{\phi \to \mathbf{f} \geq r - \frac{1}{n} \mid n \in \omega \setminus \{0\}\}$$

infer $\phi \to \mathbf{f} \geq r$.
R6. From the set of premises

$$\{\phi \to w(\alpha U_n\beta) \leq r \mid n \in \omega\}$$

infer $\phi \to w(\alpha U\beta) \leq r$.

Let us briefly discuss the axiomatic system. A1 and R1 allow propositional reasoning with all formulas from For. The axioms A2–A4 are some standard axioms in various axiomatizations of LTL. Although all the axiomatizations contain some additional axioms, we show in Lemma 1(1) that all the valid temporal formulas can be deduced in $AX_{PL_{LTL}}$. Moreover, by Lemma 2, A1–A4 together with R1,R2 and R4 make a strongly complete system for LTL. Note that we use the temporal necessitation R2 with the next operator, while the standard generalization can be derived, as it is shown in the proof of Lemma 1(1). The rule R4 is an infinitary rule that characterizes the until operator. It is similar to a rule introduced by [31]. The axiom A5 includes all axioms about commutative ordered rings. They formally provide the usual manipulations with terms (commutativity, associativity etc) on the syntactical level. For example, $\mathbf{f}\cdot(\mathbf{g}+\mathbf{h}) = (\mathbf{f}\cdot\mathbf{g})+(\mathbf{f}\cdot\mathbf{h})$ and $(\mathbf{f} \geqslant \mathbf{g} \ \wedge \ \mathbf{h} < 0) \ \rightarrow \mathbf{f}\cdot\mathbf{h} \leqslant \mathbf{g}\cdot\mathbf{h}$ are instances of A5. Of course, any particular complete set of axioms about commutative ordered rings can be used for A5. The probabilistic axioms A6 and A7 correspond to nonnegativity and finite additivity, respectively. They are two of the four axioms presented by [14]. The other two axioms are theorems of $AX_{PL_{LTL}}$ (see Lemma 1). The rule R3 states that if we know that α holds, then we believe that it is true with probability 1. The rules R5 and R6 are two infinitary rules of inference. R6 is crucial for the proof of σ-additivity, while R5 ensures that the values of probability measures belong to the set of reals. R5 is a variant of a rule introduced by [34].

Definition 10 *(Proof).* A formula Φ is a *theorem* of the logic PL_{LTL}, $(\vdash \Phi)$, if there is an at most countable sequence of formulas $\Phi_0, \Phi_1, \dots, \Phi$, such that every Φ_i is an axiom, or it is derived from the preceding formulas by an inference rule.

A formula Φ is *deducible* from a set of formulas T $(T \vdash \Phi)$ if there is an at most countable sequence of formulas $\Phi_0, \Phi_1, \dots, \Phi$, such that every Φ_i is a theorem or a formula from T, or it is derived from the preceding formulas by one of the inference rules, excluding R2. The corresponding sequence $\Phi_0, \Phi_1, \dots, \Phi$ is the *proof* of Φ from T.

By the previous definition, application of the rule R2 is restricted to theorems only. Otherwise, any change over the course of time would be impossible. Note that the length of a proof (the number of formulas in the corresponding sequence) can be any countable successor ordinal.

Remark 2. We can see that the only way to infer a probabilistic formula from a temporal formula is by an application of R3. On the other hand, there is no way to derive a temporal formula from probabilistic formulas. In other words, every temporal formula can only be derived from other temporal formulas and the axioms A1–A4 using the inference rules R1, R2 and R4. Actually, we will show in Lemma 2 that this restricted proof system is a strongly complete axiomatization for LTL. We denote by $Cn_{LTL}(T)$ the set of all LTL formulas derivable from T.

Definition 11 *(Consistency).* A set of formulas T is consistent if there is no $\alpha \in For_{LTL}$ such that $T \vdash \alpha \wedge \neg\alpha$ and no $\phi \in For_P$ such that $T \vdash \phi \wedge \neg\phi$; otherwise it is inconsistent.

T is maximal consistent if it is consistent and the following conditions hold:

- for every $\alpha \in For_{LTL}$, if $T \vdash \alpha$, then $\alpha \in T$ and $w(\alpha) = 1 \in T$,
- for every $\phi \in For_P$, either $\phi \in T$ or $\neg\phi \in T$.

Next we make several observations about the notions of consistency and maximal consistency:

- In the definition of consistency, the condition that there is no $\alpha \in For_{LTL}$ such that $T \vdash \alpha \wedge \neg\alpha$ is redundant, i.e., it follows from the condition that there is no $\phi \in For_P$ such that $T \vdash \phi \wedge \neg\phi$. Indeed, from $T \vdash \alpha \wedge \neg\alpha$ we can obtain $T \vdash w(\alpha) = 1 \wedge w(\neg\alpha) = 1$ by R3, and $T \vdash w(\alpha) = 1 \wedge \neg w(\alpha) = 1$ by probabilistic axioms.

- Maximal consistency of T doesn't imply that for every $\alpha \in For_{LTL}$ either $T \vdash \alpha$ or $T \vdash \neg\alpha$. Such a requirement would trivialize probabilities, i.e., it would force probability of every temporal formula to be either 0 or 1. Indeed, suppose that $w(\alpha) = r \in T$ for some α and some $r \in (0, 1)$. If $T \vdash \alpha$ or $T \vdash \neg\alpha$, then by R3 (and some probabilistic reasoning) we have $T \vdash w(\alpha) = 1$ or $T \vdash w(\alpha) = 0$, which would make T inconsistent. On the other hand, for a $\phi \in For_P$ we have either $T \vdash \phi$ or $T \vdash \neg\phi$.

- If T is a maximal consistent set of formulas, then T is *deductively closed*, i.e., if $T \vdash \Phi$ then $\Phi \in T$.

3.2. Some theorems about $AX_{PL_{LTL}}$

It is straightforward to check that all the axioms of $AX_{PL_{LTL}}$ are valid, and that the rules of inference maintain the validity of formulas. Thus, we omit the proof of the following result.

Theorem 1 *(Soundness). The axiomatization $AX_{PL_{LTL}}$ is sound with respect to the class of models PL_{LTL}^{Meas}.*

Theorem 2 *(Deduction theorem). Let T be a set of formulas and let Φ and Ψ be two formulas such that either $\Phi, \Psi \in For_{LTL}$ or $\Phi, \Psi \in For_P$. Then $T \cup \{\Phi\} \vdash \Psi$ iff $T \vdash \Phi \to \Psi$.*

Proof. We will prove the direction from left to right, because the other direction follows immediately from R1. We will use induction on the length of the inference. The cases when we consider application of the inference rules R1–R3 are standard. Let us consider the case when R6 is applied. Suppose that $T \cup \{\phi\} \vdash \psi \to w(\alpha U \beta) \leq r$ is obtained by R6. Then $T \cup \{\phi\} \vdash \psi \to w(\alpha U_n \beta) \leq r$ holds, by assumption, for every $n \in \omega$. Using the induction hypothesis, we have:
$T \vdash \phi \to (\psi \to w(\alpha U_n \beta) \leq r)$, for every $n \in \omega$;
$T \vdash (\phi \wedge \psi) \to w(\alpha U_n \beta) \leq r$, for every $n \in \omega$;
$T \vdash (\phi \wedge \psi) \to w(\alpha U \beta) \leq r$, by R6;
$T \vdash \phi \to (\psi \to w(\alpha U \beta) \leq r)$.

The cases when we apply R4 and R5 are similar. $\square$

Lemma 1.

1. *If $v(\sigma, \alpha) = 1$ for all $\sigma \in \overline{\Sigma}$, then $\vdash \alpha$.*
2. $\vdash w(\top) = 1$
3. *If $T \vdash \alpha \leftrightarrow \beta$, then $T \vdash w(\alpha) = w(\beta)$*
4. *If T is maximal consistent then either $\phi \in T$ or $\neg\phi \in T$, for every $\phi \in For_P$.*

Proof. (1) It is sufficient to prove that all the axioms of any complete axiomatization of LTL (for example C1–C8 form [38]) are theorems of our logic, and that the standard Generalization rule "if α is a theorem, from α infer $G\alpha$" is a derived rule in $Ax_{PL_{LTL}}$. As an illustration, let us derive Generalization. If $\vdash \alpha$, applying rule R2 we obtain $\vdash \bigcirc^n \alpha$ for every $n \in \omega$. Using A3, we conclude $\vdash \neg \bigcirc^n \neg\alpha$ for every $n \in \omega$. Note that $\neg \bigcirc^n \neg\alpha$ can be written as $\neg(\top \overline{U}_n \neg\alpha)$. Finally, applying R4 we obtain $\vdash \neg(\top U \neg\alpha)$, or, equivalently, $\vdash G\alpha$.
(2) Follows directly form R3.
(3) Apply R3, then A8.
(4) Follows directly from Definition 11. $\square$

Let us comment the above results. By (1), we can use all the standard theorems of LTL in our reasoning in PL_{LTL}. (2) is an axiom for probabilistic reasoning from [14]. (3) plays the crucial role in the construction of the canonical model in the next section. If we choose α and β to be propositional formulas and $T = \emptyset$, we obtain another axiom by [14]. Thus, by (1)–(3), $AX_{PL_{LTL}}$ extends both temporal and probabilistic logic.

We use (4) in the proof of Theorem 5. We already pointed out that the same property doesn't hold for the LTL formulas.

4. The completeness of PL_{LTL}

In this section we prove strong completeness: "every consistent set of formulas has a model". We use a Henkin-like construction. First we extend a consistent set T of formulas to a maximal consistent set T^*, then we use T^* to define the corresponding structure M_{T^*}, and finally we prove that M_{T^*} is a model of T. For a given T^*, we say that M_{T^*} is its canonical model.

4.1. Lindenbaum's lemma

First we show that any set of formulas that is consistent under our axiomatization can be extended to a maximal consistent superset.

Theorem 3 *(Lindenbaum's lemma). Every consistent set of formulas can be extended to a maximal consistent set.*

Proof. Let T be a consistent set of formulas, and let $\psi_0, \psi_1, \dots$ be an enumeration of all formulas from For_P. Recall that we used the expression $Cn_{LTL}(T)$ to denote the set of all LTL formulas derivable from T. We define the sequence of sets T_i, $i = 0, 1, 2, \dots$ and the set T^* recursively as follows:

1. $T_0 = T \cup Cn_{LTL}(T) \cup \{w(\alpha) = 1 \mid \alpha \in Cn_{LTL}(T)\}$,
2. for every $i \geq 0$,
   (a) if $T_i \cup \{\psi_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\psi_i\}$, otherwise
   (b) if ψ_i is of the form $\phi \rightarrow f \geq r$, then $T_{i+1} = T_i \cup \{\phi \rightarrow f < r - \frac{1}{m}\}$, where m is the smallest positive integer such that T_{i+1} is consistent, otherwise
   (c) if ψ_i is of the form $\phi \rightarrow w(\alpha U \beta) \leq r$, then $T_{i+1} = T_i \cup \{\phi \rightarrow w(\alpha U_n \beta) > r\}$, where n is the smallest nonnegative integer such that T_{i+1} is consistent, otherwise
   (d) $T_{i+1} = T_i$.
3. $T^\star = \bigcup_{i=0}^{\infty} T_i$.

First, using Theorem 2 one can prove that the set T^* is correctly defined, i.e., there exist m and n from the parts 2(b) and 2(c) of the construction. Each T_i, $i > 0$ is consistent. The steps (1) and (2) of the construction ensure that $T^\star$ is maximal. Also, $T^\star$ obviously doesn't contain all formulas. Finally, one can show that $T^\star$ is a deductively closed set, and as a consequence we obtain that $T^\star$ is consistent (otherwise it would contain $\perp$).

We can prove that $T^\star$ is a deductively closed set, i.e., that $T^* \vdash \Phi$ implies $\Phi \in T^*$, by considering temporal and probabilistic formulas separately. The case when $\Phi \in For_{LTL}$ holds trivially from the construction of T_0. Note that the step 1 also ensures that T^* is closed under the inference rule R3. For $\Phi \in For_P$, we can prove that $T^* \vdash \Phi$ implies $\Phi \in T^*$ by the induction on the length of inference. The only non-trivial part of the proof is to show closure under the infinitary inference rules. Note that closure under R4 is ensured by the first step of construction. Here we only give a proof for the infinitary rule R6, since R5 can be considered in a similar way.

Suppose that $\phi \rightarrow w(\alpha U_n \beta) \leq r \in T^\star$ for every $n \in \omega$. We need to prove that $\phi \rightarrow w(\alpha U \beta) \leq r \in T^\star$. Assume that $\phi \rightarrow w(\alpha U \beta) \leq r \notin T^\star$. In that case, from maximality of $T^\star$ we have $\neg(\phi \rightarrow w(\alpha U \beta) \leq r) \in T^\star$, or, equivalently, $\phi \wedge \neg w(\alpha U \beta) \leq r \in T^\star$. That implies both $\phi \in T^\star$ and $\neg w(\alpha U \beta) \leq r \in T^\star$. Let i be a non-negative integer such that $\phi \in T_i$. If j is a non-negative integer such that $\Phi_j = w(\alpha U \beta) \leq r$, by the construction of $T^\star$, step 2(d), we have $w(\alpha U_k \beta) > r \in T_{j+1}$ for some k. Furthermore, let m be a non-negative integer such that $\phi \rightarrow w(\alpha U_k \beta) \leq r \in T_m$. Then $T_{\max\{i,m\}} \vdash w(\alpha U_k \beta) \leq r$. This means that

$T_{\max\{i,j+1,m\}} \vdash w(\alpha U_k \beta) \leq r \wedge w(\alpha U_k \beta) > r$, which contradicts consistency of $T_{\max\{i,j+1,m\}}$. Therefore, $T^\star$ is closed under the application of R6. □

4.2. Canonical model

Definition 12 *(Canonical model).* For a maximal consistent set T^*, we define a PL_{LTL} structure for T^* as a tuple $M_{T^*} = \langle W, H, \mu, \pi \rangle$, such that:

1. $W = \{\sigma \in \overline{\Sigma} \mid v(\sigma, \alpha) = 1$ for all $\alpha \in T^* \cap For_{LTL}\}$,
2. $H = \{[\alpha] \mid \alpha \in For_{LTL}\}$, where $[\alpha] = \{w \in W \mid v(w, \alpha) = 1\}$,
3. $\mu([\alpha]) = \sup\{r \in \mathcal{Q} \mid T^* \vdash w(\alpha) \geq r\}$, for every $\alpha \in For_{LTL}$,
4. $\pi(w) = w$ for every $w \in W$.

Now we show that M_{T^*} is a measurable PL_{LTL} structure. In the proof, we will use the following result. It states that the temporal part of our axiom system is strongly complete for LTL.

Lemma 2. *The axioms A1–A4 and the inference rules R1, R2 and R4 form a sound and strongly complete axiomatization for LTL.*

Proof. The proof of soundness is straightforward. In order to prove strong completeness, we need to show that every consistent set Γ of LTL formulas has a model, i.e., that there is σ such that $v(\sigma, \alpha) = 1$ for every $\alpha \in \Gamma$. Reasoning similarly as above, we can prove that Deduction theorem holds for this restricted system. Now we work with LTL formulas only, and in this system we define the notion of maximal consistency in a different way than before: we say that Γ^* is a maximal consistent set of LTL formulas iff it is consistent and for every α either $\alpha \in \Gamma^*$ or $\neg\alpha \in \Gamma^*$. The proof that Γ can be extended to a maximal consistent set Γ^* of LTL formulas is also along the lines of the proof construction above, so we omit some details. We assume an enumeration $\alpha_1, \alpha_2, \ldots$ of all LTL formulas, and define a maximal consistent set Γ^* recursively, by setting $\Gamma_0 = \Gamma$ and, in each step i of the construction, defining $\Gamma_{i+1} = \Gamma_i \cup \{\alpha_i\}$ if α_i is consistent with Γ_i, while if α_i is not consistent with Γ_i, we check whether α_i is of the form $\gamma \to \neg(\alpha U \beta)$, in which case we define

$$\Gamma_{i+1} = \Gamma_i \cup \{\gamma \to (\alpha \overline{U}_k \beta)\},$$

where k is the smallest nonnegative integer such that Γ_{i+1} is consistent (Deduction theorem for LTL guarantees that there exist such k); otherwise $\Gamma_{i+1} = \Gamma_i$. Finally, $\Gamma^* = \bigcup_{n \in \omega} \Gamma_n$. As in the previous result, the non-trivial part of the proof is to show that Γ^* is closed under applications of the infinitary inference rule R4. Assume that $\gamma \to \neg(\alpha \overline{U}_n \beta) \in \Gamma^*$ for every $n \in \omega$. Suppose that $\gamma \to \neg(\alpha U \beta) \notin \Gamma^*$. Since Γ^* is a maximal consistent set, $\neg(\gamma \to \neg(\alpha U \beta)) \in \Gamma^*$, so $\gamma \wedge (\alpha U \beta) \in \Gamma^*$. Consequently, $\gamma \in \Gamma^*$ and $\alpha U \beta \in \Gamma^*$, so there are $m, n \in \omega$ such that $\gamma \in \Gamma_m$ and $\alpha U \beta \in \Gamma_n$. Let ℓ be the integer such that $\alpha_l = \gamma \to \neg(\alpha U \beta)$. By the construction of Γ^*, there is k such that $\gamma \to (\alpha \overline{U}_k \beta) \in \Gamma_l$. By temporal reasoning we obtain $\Gamma_l \vdash \alpha U \beta$. Then $\alpha U \beta \in \Gamma_{\max\{l,m,n\}}$, which contradicts the consistency of $\Gamma_{\max\{l,m,n\}}$. Also, using the axiomatization it is straightforward to show that if Γ^* is a maximal consistent set, then the set $\Gamma^*_n = \{\alpha \mid \bigcirc^n \alpha \in \Gamma^*\}$ is also maximal consistent.

For a given Γ^*, we define the path $\sigma = s_0, s_1, \ldots$ by $s_i = \{p \in \mathcal{P} \mid \Gamma^*_i \vdash p\}$. It is sufficient to prove that $v(\sigma, \gamma) = 1$ iff $\Gamma^* \vdash \gamma$, for every LTL formula γ. We use induction on the complexity of the formula. The only interesting case is when γ is of the form $\alpha U \beta$. Consider the following sequence of equivalent statements:
$v(\sigma, \gamma) = 0$ iff $v(\sigma, \neg(\alpha U \beta)) = 1$
iff for all $n \in \omega$, it is not the case that both $v(\sigma_{\geq n}, \beta) = 1$ and for all $k < n$, $v(\sigma_{\geq k}, \alpha) = 1$
iff for all $n \in \omega$, it is not the case that both $\Gamma^*_n \vdash \beta$ and for all $k < n$, $\Gamma^*_k \vdash \alpha$ (by induction hypothesis)

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iff for all $n \in \omega$, it is not the case that both $\Gamma^* \vdash \bigcirc^n \beta$ and for all $k < n$, $\Gamma^* \vdash \bigcirc^k \alpha$
iff for all $n \in \omega$, $\Gamma^* \vdash \neg(\alpha \overline{U}_n \beta)$ (by the maximal consistency of Γ^*)
iff $\Gamma^* \vdash \neg(\alpha U \beta)$ (by R4). □

We will use the notation $\Gamma \vdash_{LTL} \alpha$ to denote that α is deducible from the set Γ of LTL formulas. From Remark 2 we obtain that for every consistent set of formulas $T \subseteq For$ and every formula $\alpha \in For_{LTL}$, we have

$$T \vdash \alpha \text{ iff } \mathrm{T} \cap \mathrm{For}_{\mathrm{LTL}} \vdash_{\mathrm{LTL}} \alpha.$$

Let us denote by $\models_{LTL}$ the standard consequence relation of LTL: $\Gamma \models_{LTL} \alpha$ iff for every σ the following implication holds: if $v(\sigma, \beta) = 1$ for every $\beta \in \Gamma$, then $v(\sigma, \alpha) = 1$. As it is well known, we can state the alternative formulation of Completeness theorem for LTL (Lemma 2) as follows:

$$\Gamma \models_{LTL} \alpha \text{ iff } \Gamma \vdash_{\mathrm{LTL}} \alpha. \tag{3}$$

Theorem 4. *For every maximal consistent set T^*, $M_{T^*} \in PL^{Meas}_{LTL}$.*

Proof. First we need to show that the definition of M_{T^*} is correct. The set $\{[\alpha] \mid \alpha \in For_{LTL}\}$ is an algebra of subsets of W, since $W = [\top]$, $W \backslash [\alpha] = [\neg\alpha]$ and $[\alpha] \cup [\beta] = [\alpha \vee \beta]$. We also need to check that μ is correctly defined, i.e., that if $[\alpha] = [\beta]$ then $\mu([\alpha]) = \mu([\beta])$. From $[\alpha] = [\beta]$ we conclude that if σ is a path such that $v(\sigma, \gamma) = 1$ for all $\gamma \in T^* \cap For_{LTL}$, then $v(\sigma, \alpha \leftrightarrow \beta) = 1$. In other words, $T^* \cap For_{LTL} \models_{LTL} \alpha \leftrightarrow \beta$. From Lemma 2 (see also the Equation (3)) we obtain $T^* \cap For_{LTL} \vdash_{LTL} \alpha \leftrightarrow \beta$, so $T^* \vdash \alpha \leftrightarrow \beta$ holds as well. Consequently, $T^* \vdash w(\alpha) = w(\beta)$ by Lemma 1(3), so $\mu([\alpha]) = \mu([\beta])$. Obviously $\mu(W) = \mu([\top]) = 1$ by Lemma 1(2). Similarly, using A6 we conclude that μ is nonnegative, and using A7 we conclude that μ is a finitely additive probability measure on H. We need to prove that μ is σ-additive.

Let $H_{\overline{\Sigma}} = \{[\alpha]_{\overline{\Sigma}} \mid \alpha \in For_{LTL}\}$, where $[\alpha]_{\overline{\Sigma}} = \{\sigma \in \overline{\Sigma} \mid v(\sigma, \alpha) = 1\}$. By $For^{\bigcirc}_{LTL}$ we denote the set of all LTL formulas in which $\bigcirc$ is the only temporal operator (i.e. there are no appearances of U). We also introduce the set $A = \{[\alpha]_{\overline{\Sigma}} \mid \alpha \in For^{\bigcirc}_{LTL}\}$. Using the same argument as above, we can show that the sets $H_{\overline{\Sigma}}$ and A are two algebras of subsets of $\overline{\Sigma}$. Similarly as in the definition of M_{T^*}, we define μ^* on $H_{\overline{\Sigma}}$ by

$$\mu^*([\alpha]_{\overline{\Sigma}}) = \sup\{r \in \mathcal{Q} \mid T^* \vdash w(\alpha) \geq r\}.$$

Reasoning as above, we conclude that μ^* is a finitely additive measure. We also use the same symbol μ^* to denote the restriction of μ^* to A. We actually want to show that μ^* is σ-additive on A. It is sufficient to show that if $B = \bigcup_{i \in \omega} B_i$, where $B, B_i \in A$, then there is n such that $B = \bigcup_{n=0}^{n} B_i$.

If $2^{\mathcal{P}}$ denotes the set of subsets of $\mathcal{P}$, note that $\overline{\Sigma} = 2^{\mathcal{P}} \times 2^{\mathcal{P}} \times 2^{\mathcal{P}} \times \ldots$ If we assume discrete topology on the finite set $2^{\mathcal{P}}$ and the induced product topology on $\overline{\Sigma}$, then $\overline{\Sigma}$ is a compact space as a product of compact spaces.[1] By definition of evaluation function v, we obtain that for every $\alpha \in For^{\bigcirc}_{LTL}$ there exist $n \in \omega$ (for example n is the number of appearances of $\bigcirc$) and $S \subseteq (2^{\mathcal{P}})^n$ such that $[\alpha]_{\overline{\Sigma}} = S \times 2^{\mathcal{P}} \times 2^{\mathcal{P}} \times \ldots$ Note that the sets of the form $S \times 2^{\mathcal{P}} \times 2^{\mathcal{P}} \times \ldots$, where $S \subseteq (2^{\mathcal{P}})^n$ for some $n \in \omega$, are clopen (both closed and open) sets in product topology. Thus, each $[\alpha]_{\overline{\Sigma}} \in A$ is a clopen set in $\overline{\Sigma}$. Now assume that $[\alpha]_{\overline{\Sigma}} = \bigcup_{n \in \omega} [\alpha_n]_{\overline{\Sigma}}$, where $\alpha \in For^{\bigcirc}_{LTL}$ and $\alpha_n \in For^{\bigcirc}_{LTL}$ for every $n \in \omega$. The set $\{[\alpha_n]_{\overline{\Sigma}} \mid n \in \omega\}$ is an open cover of the closed subset $[\alpha]_{\overline{\Sigma}}$ of the compact space $\overline{\Sigma}$, so there is a finite subcover $\{[\alpha_{n_1}]_{\overline{\Sigma}}, \ldots, [\alpha_{n_1}]_{\overline{\Sigma}}\}$ of $[\alpha]_{\overline{\Sigma}}$. Thus, μ^* is σ-additive on A.

Let F be the σ-algebra generated by A. Since $[\alpha U \beta]_{\overline{\Sigma}} = \bigcup_{n \in \omega} [\alpha U_n \beta]_{\overline{\Sigma}}$, we can show that $[\alpha]_{\overline{\Sigma}} \in F$ for every $\alpha \in For_{LTL}$, using the induction on the number of appearances of U in α. Thus, $H_{\overline{\Sigma}} \subseteq F$. By

[1] For the basic notions and results about the topology used here we refer the reader to [27].

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Caratheodory's extension theorem (see [1]), there is a unique σ-additive probability measure ν on F which coincides with μ^* on A. As $W = \bigcap_{\alpha \in T^* \cap For_{LTL}} [\alpha]_{\overline{\Sigma}}$, with each $[\alpha]_{\overline{\Sigma}} \in W$, so also is $W \in F$. Then $H \subseteq F$, since for each $\alpha \in For_{LTL}$, $[\alpha] = [\alpha]_{\overline{\Sigma}} \cap W$.

Our strategy for showing that μ is σ-additive on H is to show that μ and ν agree on H, via the following chain of equalities:

$$\mu([\alpha]) = \mu^*([\alpha]_{\overline{\Sigma}}) = \nu([\alpha]_{\overline{\Sigma}}) = \nu([\alpha]).$$

The first equality follows directly from the definitions of μ^* and μ. We will prove the second equality and the third equality in the following two paragraphs.

We will now show that μ^* is the restriction of ν to $H_{\overline{\Sigma}}$, i.e., that $\mu^*([\alpha]_{\overline{\Sigma}}) = \nu([\alpha]_{\overline{\Sigma}})$ for all $\alpha \in For_{LTL}$, using the induction on the number of appearances of U in α. Indeed, $\nu([\alpha U \beta]_{\overline{\Sigma}}) = \nu(\bigcup_{n\in\omega} [\alpha U_n \beta]_{\overline{\Sigma}}) = \lim_{k\to+\infty} \nu(\bigcup_{n=1}^{k} [\alpha U_n \beta]_{\overline{\Sigma}}) = \lim_{k\to+\infty} \mu^*(\bigcup_{n=1}^{k} [\alpha U_n \beta]_{\overline{\Sigma}}) = \mu^*([\alpha U \beta]_{\overline{\Sigma}})$. Here we used σ-additivity of ν, the induction hypothesis and, in the last step, the definition of μ^* and R6. Thus, for all $\alpha \in For_{LTL}$, $\mu^*([\alpha]_{\overline{\Sigma}}) = \nu([\alpha]_{\overline{\Sigma}})$.

Note that we have that $\mu^*([\alpha]_{\overline{\Sigma}}) = 1$ whenever $T^* \vdash \alpha$, by R3. From $\mu^*([\alpha]_{\overline{\Sigma}}) = \nu([\alpha]_{\overline{\Sigma}})$ we obtain $\mu^*([\alpha]_{\overline{\Sigma}}) = 1$ whenever $T^* \vdash \alpha$. Thus, $\nu(W) = \nu(\bigcap_{\alpha: T^*\vdash\alpha} [\alpha]_{\overline{\Sigma}}) = 1$, by σ-additivity of ν. Note that $[\alpha] = [\alpha]_{\overline{\Sigma}} \cap W$, so $\nu([\alpha]) = \nu([\alpha]_{\overline{\Sigma}} \cap W)$. Also, from $\nu(W) = 1$ we obtain $\nu([\alpha]_{\overline{\Sigma}} \cap W) = \nu([\alpha]_{\overline{\Sigma}})$. Therefore we have $\nu([\alpha]) = \nu([\alpha]_{\overline{\Sigma}})$.

Thus, $\mu([\alpha]) = \mu^*([\alpha]_{\overline{\Sigma}}) = \nu([\alpha]_{\overline{\Sigma}}) = \nu([\alpha])$, so μ is σ-additive.

We showed that M_{T^*} is a PL_{LTL} structure. Finally, note that $[\alpha] = [\alpha]_{M_{T^*}}$, by the choice of π, so $M_{T^*} \in PL_{LTL}^{Meas}$. □

4.3. Completeness theorem

Now we can prove the main result of this section.

Theorem 5 *(Strong completeness). A set of formulas $T \subseteq For$ is consistent iff it is satisfiable.*

Proof. The direction from right to left follows from the soundness of the axiomatization $AX_{PL_{LTL}}$. For the other direction, we need to show that a consistent set of formulas T has a model. First we extend T to a maximal consistent set T^*, and we construct the canonical model M_{T^*}. We will show that M_{T^*} is a model of T^*, and, consequently, a model of T. It is sufficient to prove that for all $\Phi \in For$, $T^* \vdash \Phi$ iff $M_{T^*} \models \Phi$.

If $\Phi = \alpha \in For_{LTL}$. If $\alpha \in T^*$, then by the definition of W from M_{T^*}, $M_{T^*} \models \alpha$. Conversely, if $M_{T^*} \models \alpha$, we have $T^* \cap For_{LTL} \models_{LTL} \alpha$ (by the construction of the canonical model). From Lemma 2 (see also the Equation (3)) we obtain $T^* \cap For_{LTL} \vdash_{LTL} \alpha$, so $\alpha \in T^*$.

If $\Phi \in For_P$, we proceed by induction on the complexity of Φ.

Let $\Phi = \mathbf{f} \geq 0$. Then we can show, using the properties of supremum, that

$$\mathbf{f}^M = \sup\{s \mid T^* \vdash \mathbf{f} \geq s\}.$$

If we suppose that $\mathbf{f} \geq 0 \in T^*$, then $0 \leq \sup\{s \mid T^* \vdash \mathbf{f} \geq s\}$, so $M_{T^*} \models \mathbf{f} \geq 0$. For the other direction, assume that $M_{T^*} \models \mathbf{f} \geq 0$. Then $M_{T^*} \not\models \mathbf{f} < 0$. If $\mathbf{f} < 0 \in T^*$, then, reasoning as above, we conclude $M_{T^*} \models \mathbf{f} < 0$, a contradiction. By maximality of T^*, we obtain $\mathbf{f} \geq 0 \in T^*$.

If $\Phi = \neg\phi$, then $M_{T^*} \models \neg\phi$ iff $M_{T^*} \not\models \phi$ iff $\phi \notin T^*$ iff $\neg\phi \in T^*$, by maximality of T^*.
If $\Phi = \phi \wedge \psi$, then $M_{T^*} \models \phi \wedge \psi$ iff $M_{T^*} \models \phi$ and $M_{T^*} \models \phi$ iff $\phi, \psi \in T^*$ iff $\phi \wedge \psi \in T^*$, by maximality of T^*. □

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As it is well known, the alternative formulation of Completeness theorem, stated below, follows directly from the previous result.

Theorem 6. *If $T \subseteq For$ and $\Phi \in For$, then $T \models \Phi$ iff $T \vdash \Phi$.*

5. The decidability of PL_{LTL}

In this section we prove that the problem of deciding satisfiability of formulas of our logic PL_{LTL} is decidable, and that there is a decidability procedure for the problem that runs in polynomial space. Since PL_{LTL} has two disjoint classes of formulas, we need to consider two cases.

First, let us consider LTL formulas. Let us recall that Sistla and Clarke [40] proved that the logic LTL is decidable, and they showed that the problem of deciding whether an LTL formula is satisfiable in a path is $PSPACE$-complete. Therefore, it is sufficient to show that satisfiability of LTL formulas under the standard LTL semantics coincides with satisfiability under the class of measurable structures PL_{LTL}^{Meas}. Note that if α is not satisfiable in any path, then by Definition 7 it is not satisfiable in the logic PL_{LTL}. On the other hand, if there is a path σ such that $v(\sigma, \alpha) = 1$, then we can define a measurable structure $M = \langle W, H, \mu, \pi \rangle$, such that $W = \{w\}$ is a singleton and $\pi(w) = \sigma$. Note that in that case the range of μ is simply the doubleton $\{0, 1\}$. Obviously, $v(\pi(w), \alpha) = 1$ for every $w \in W$, so $M \models \alpha$. In this way, we have shown that the satisfiability problem of LTL formulas for the logic PL_{LTL} is $PSPACE$-complete.

Next we consider the satisfiability of a formula $\varphi \in For_P$. We will first show that if φ is satisfiable, then it is satisfied in a measurable structure with a small number of worlds. This step does not automatically imply decidability, since there are infinitely many possibilities to assign real valued probabilities even with finitely many worlds. Now, let us introduce some notation. Let

$$Term(\varphi) = \{\mathbf{f}_1, \dots, \mathbf{f}_m\}$$

be the set of all probabilistic terms that appear in φ, and let $For_{LTL}(\varphi)$ denote the set of all LTL formulas which appear under the scope of probability operator w in at least one element of $Term(\varphi)$ (in other words, $\alpha \in For_{LTL}(\varphi)$ iff $w(\alpha)$ is a sub-expression of φ). Next, let

$$Subfor = \bigcup_{\alpha \in For_{LTL}(\varphi)} Subfor(\alpha),$$

where $Subfor(\alpha)$ denotes the set of all subformulas of α. Let us consider the formulas of the form

$$\bigwedge_{k=1}^{|Subfor|} \beta_k, \tag{4}$$

where each β_k belongs to $Subfor \cup \{\neg\beta \mid \beta \in Subfor\}$, and each subformula of α appears exactly once (negated or not). Obviously the conjunction of any two different formulas of the form (4) is a contradiction, while the disjunction of all such formulas is a tautology. Those facts will enable us to distribute the probability of each element of $For_{LTL}(\varphi)$ as the sum of probabilities of formulas of the form (4) in any structure. Note that overall we have $2^{|Subfor|}$ formulas of the form (4). The problem in assigning the probabilities to such formulas is that not all of them are consistent (for example, consider the case when both $\bigcirc\neg\alpha$ and $G\alpha$ are elements of $For_{LTL}(\varphi)$). For that reason, we first eliminate every formula of the form (4) that is not satisfiable in LTL, using the procedure from [40] (for the complexity of our logic, it is important to recall that this procedure runs in PSPACE). Suppose that there are ℓ formulas which are satisfiable ($\ell \leq 2^{|Subfor|}$). We denote those formulas by $\alpha_1, \dots, \alpha_\ell$.

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Since $\alpha \in Subfor$ holds for every formula $\alpha \in For_{LTL}(\varphi)$, we have that such an α appears in each conjunction α_k, either negated or not. Since $\bigvee_{k=1}^{\ell} \alpha_k$ is a tautology, there is a unique set of indices $I_\alpha \subseteq \{1, \dots, \ell\}$ such that

$$\alpha \leftrightarrow \bigvee_{i \in I_\alpha} \alpha_i$$

is a tautology. Let Γ_α be the corresponding set of disjuncts, i.e.,

$$\Gamma_\alpha = \{\alpha_i \mid i \in I_\alpha\}.$$

Using the probabilistic axioms and Lemma 1(3), we obtain

$$\vdash w(\alpha) = \sum_{\alpha_i \in \Gamma_\alpha} w(\alpha_i), \tag{5}$$

so, by completeness, in every measurable structure the probability value of α will coincide with the sum of the probabilities of α_i.

Now suppose that there exists a measurable structure $M = \langle W, H, \mu, \pi \rangle$ such that $M \models \varphi$. A standard filtration approach would be to chose one world from the class $[\alpha_i]$ of α_i, for each of the formulas $\alpha_1, \dots, \alpha_\ell$, to replace the entire $[\alpha_i]$ with that world, and to assign probability $\mu([\alpha_i])$ to the corresponding singleton. That would lead to a model of size $\ell \leq 2^{|Subfor|}$. We now show that we can do a bit better than that, namely that we can reduce the size of a model to be at most the length of φ. For that, let us denote by y_i the probability assigned to the formula α_i by that potential small model M', in such a way that

$$\mathbf{f}_k^M = \mathbf{f}_k^{M'}, \tag{6}$$

for every $\mathbf{f}_k \in Term(\varphi)$. The variables $y_1, \dots, y_n$ must satisfy of the system of equations and inequalities consisting of the equation

$$y_1 + \cdots + y_\ell = 1, \tag{7}$$

the inequalities

$$y_1 \geq 0,\ y_2 \geq 0,\ \dots\ y_\ell \geq 0, \tag{8}$$

and the set of equations (one equation for every $\alpha \in Form_{LTL}(\varphi)$) of the form

$$\sum_{\alpha_i \in \Gamma_\alpha} y_i = \mu([\alpha]). \tag{9}$$

The system (7)–(9) has a solution, namely $y_i = \mu([\alpha_i])$. It is known that if a system of k linear equations has a non-negative solution, then it has a non-negative solution where at most k values are different than zero (see [4]). Thus, the system of equations (7)&(9) has a non-negative solution with at most $|Form_{LTL}(\varphi)| + 1$ values different than zero. Without any loss of generality, assume that the solution is $(y_1, \dots, y_\ell) = (p_1, \dots, p_\ell)$, where $p_i = 0$ for every $i > |Form_{LTL}(\varphi)| + 1$. Now we define the measurable structure $M' = \langle W', H', \mu', \pi' \rangle$ such that

- $W' = \{w'_1, \dots, w'_{|Form_{LTL}(\varphi)|+1}\}$
- Every subset of W' belongs to H'

- μ' is uniquely defined by $\mu'(\{w'_i\}) = p_i$
- For every $i \leq |Form_{LTL}(\varphi)| + 1$, we chose one element w_i from $[\alpha_i]_M$ and then define $\pi'(w'_i) := \pi(w_i)$.

Obviously (6) holds. Thus, from $M \models \varphi$ we obtain $M' \models \varphi$. Thus, it is sufficient to consider the structures with $|Form_{LTL}(\varphi)| + 1$ worlds.

Our procedure now runs as follows: it systematically cycles through sets of subsets of $\{\alpha_1, \dots, \alpha_\ell\}$ of size $|Form_{LTL}(\varphi)| + 1$. Note that polynomial space is sufficient for that, since each α_i is a consistent LTL formula of the form (4), and, as we have already mentioned, checking consistency of each such formula is decidable in PSPACE. Then, for each subset $\Gamma = \{\alpha_{n_1}, \dots, \alpha_{n_{|Form_{LTL}(\varphi)|+1}}\}$, we check if we can assign the probability values $\exists x_1 \dots \exists x_{|Form_{LTL}(\varphi)}$ to them such that φ is satisfied. We consider the following formula of RCF:

$$\begin{aligned} \exists x_1 \dots \exists x_{|Form_{LTL}(\varphi)|+1} \quad & \Big(\bigwedge_{k=1}^{|Form_{LTL}(\varphi)|+1} (x_k \geq 0) \\ & \wedge \sum_{k=1}^{|Form_{LTL}(\varphi)|+1} x_k = 1 \\ & \wedge\ Ineq(\varphi), \end{aligned}$$

where $Ineq(\varphi)$ is obtained by replacing in each term $\mathbf{f}_k$ of the formula φ, each occurrence of every $w(\alpha)$ (for every $\alpha \in Form_{LTL}(\varphi)$) with the sum

$$\sum_{\alpha_i \in \Gamma_\alpha \cup \Gamma} x_i. \tag{10}$$

It is clear that φ is satisfiable in a measurable structure iff for some Γ the formula above is satisfied in RCF. Since the theory of real closed fields is decidable, our logic is decidable as well. Moreover, that the above sentence is an existential sentence so we can use Canny's procedure from [3], which decides satisfiability of existential sentences of RCF in PSPACE.

Thus, in both the probabilistic and LTL cases there is a procedure which decides satisfiability of the formula in $PSPACE$. Therefore, we proved the following result.

Theorem 7. *There is a procedure that decides whether a formula of the logic PL_{LTL} is satisfiable in a measurable structure from PL_{LTL}^{Meas} which runs in polynomial space.*

6. The logic MPL_{LTL}

In this section we present a novel logic, MPL_{LTL}, in which probabilistic and temporal operators can be nested in an arbitrary way. In addition, we consider a set of agents, $Agt = \{1, \dots, N\}$ where different agents can place different probabilities to the same events. Following [13], we first introduce the notion of i-probabilistic term.

Definition 13 *(i-probabilistic term).* For every $i \in Agt$ and a given set F of formulas (of an arbitrary language) we define the set $Term_i(F)$ of all i-probabilistic terms over F recursively as follows:

- $Term_i^0(F) = \{w_i(\alpha) \mid \alpha \in F\} \cup \{0, 1\}$,
- $Term_i^{n+1}(F) = Term_i^n(F) \cup \{(\mathbf{f} + \mathbf{g}), (\mathbf{f} \cdot \mathbf{g}), (-\mathbf{f}) \mid \mathbf{f}, \mathbf{g} \in Term_i^n(F)\}$,
- $Term_i(F) = \bigcup_{n=0}^{\infty} Term_i^n(F)$.

Note that this time terms are defined in a more abstract way. This is the consequence of the fact that the probabilities can be iterated in a formula, and it allows us to define the set of formulas of MPL_{LTL} recursively as follows.

Definition 14 *(MPL_{LTL} formula).* The set $Form$ of all MPL_{LTL} formulas is the smallest set satisfying the conditions:

- $\mathcal{P} \subseteq Form$,
- if $\{\alpha, \beta\} \subseteq Form$, then $\{\alpha \wedge \beta, \neg\alpha, \bigcirc\alpha, \alpha U \beta\} \subseteq Form$, and
- if $i \in Agt$ and $\mathbf{f} \in Term_i(Form)$, then $\mathbf{f} \geq 0 \in Form$.

The logic MPL_{LTL} will use the same set of abbreviations as the logic PL_{LTL}. In particular, recall that we can assume that all rational numbers are terms, and that we can express conditional probabilities in the language (by simply clearing the denominators).

Example 4. The expression

$$w_i(\bigcirc w_j(p) > 0) \leq \frac{1}{2}$$

is a well defined formula of MPL_{LTL}. Its meaning is "according to agent i, the probability that the agent j will place a positive probability to p in the next moment is at most one half".

Now we define the semantics of our logic.

Definition 15 *(MPL_{LTL} structure).* An MPL_{LTL} structure Σ^P is a set of *probabilistic paths* of the form $\sigma = s_0, s_1, s_2, \ldots$ where each s_j contains:

- $\mathcal{P}(s_j)$, a set of propositional letters that hold in s_j, and
- $Prob(s_j, i) = \langle \Sigma^P(s_j, i), H(s_j, i), \mu(s_j, i)\rangle$, a probability space such that $\Sigma^P(s_j, i) \subseteq \Sigma^P$, for every s_j and every agent $i \in Agt$.

Recall that for a probabilistic path (we will also call it just "path") $\sigma = s_0, s_1, s_2, \ldots$, the state (or the time instant) s_i is denoted by σ_i. Probabilistic paths are primitive semantic notions, so we assume that each of them has its own states. This means that there is no state that belongs to two different paths (and also there is no state that appears twice in one path).

Now let us define the satisfiability relation. Since in MPL_{LTL} we can freely iterate probability and temporal operators, there are two issues that need to be taken care of:

- First, we need to assign a probability space to each time instance of a path (and for each agent). The problem with a direct adaptation of assigning values to all terms as in the case of PL_{LTL} is that now we cannot know in advance which sets of paths will be measurable. For that reason, we follow the approach of [13] and first define the satisfiability relation $\models$ using inner and outer measures. Then we will restrict our attention to the measurable structures, in which we know that formulas correspond to the measurable sets. Let us recall that, for a given probability space $\langle W, H, \mu\rangle$, the *inner measure* μ_* defined as

$$\mu_*(A) = \sup\{\mu(B) \mid B \subset A,\ B \in H\}$$

  assigns a value to every subset A of W. It is known that μ_* and μ coincide on measurable sets.

- And second, as we will emphasize in Remark 3, the satisfiability relation is defined between time instants and formulas.

Definition 16 *(Satisfiability).* For an MPL_{LTL} structure Σ^P, a path $\sigma \in \Sigma^P$, a time instant σ_j and a formula α, we define when $\sigma_j \models \alpha$, recursively as follows:

1. $\Sigma^P, \sigma_j \models p$ iff $p \in \mathcal{P}(\sigma_j)$,
2. $\Sigma^P, \sigma_j \models \mathbf{f} \geq 0$ iff $\mathbf{f}^{\Sigma^P,\sigma_j} \geq 0$, whenever $\mathbf{f}$ is a probabilistic i-term, where $\mathbf{f}^{\Sigma^P,\sigma_j}$ is defined recursively as follows:
   (a) $0^{\Sigma^P,\sigma_j} = 0$, $1^{\Sigma^P,\sigma_j} = 1$.
   (b) $w_i(\alpha)^{\Sigma^P,\sigma_j} = \mu_\star(\sigma_j, i)([\alpha]^i_{\Sigma^P,\sigma_j})$, where
   $$[\alpha]^i_{\Sigma^P,\sigma_j} = \{\pi \in \Sigma^P(\sigma_j, i) \mid \Sigma^P, \pi_0 \models \alpha\}. \tag{11}$$
   (c) $(\mathbf{f} + \mathbf{g})^{\Sigma^P,\sigma_j} = \mathbf{f}^{\Sigma^P,\sigma_j} + \mathbf{g}^{\Sigma^P,\sigma_j}$.
   (d) $(\mathbf{f} \cdot \mathbf{g})^{\Sigma^P,\sigma_j} = \mathbf{f}^{\Sigma^P,\sigma_j} \cdot \mathbf{g}^{\Sigma^P,\sigma_j}$.
   (e) $(-\mathbf{f})^{\Sigma^P,\sigma_j} = -(\mathbf{f}^{\Sigma^P,\sigma_j})$.
3. $\Sigma^P, \sigma_j \models \bigcirc\alpha$ iff $\Sigma^P, \sigma_{j+1} \models \alpha$,
4. $\Sigma^P, \sigma_j \models \alpha U \beta$ iff $\Sigma^P, \sigma_{j+k} \models \beta$ for some $k \geq 0$ and $\Sigma^P, \sigma_{\geq j+l} \models \alpha$ for every l such that $0 \leq l < k$
5. $\Sigma^P, \sigma_j \models \neg\alpha$ iff $\Sigma^P, \sigma_j \not\models \alpha$,
6. $\Sigma^P, \sigma_j \models \alpha \wedge \beta$ iff $\Sigma^P, \sigma_j \models \alpha$ and $\Sigma^P, \sigma_j \models \beta$.

Remark 3. There are two widely used definitions of satisfiability of LTL formulas in the literature. The first one, anchored version of satisfiability, assigns special significance to the initial state and uses subpaths in the definition. We have followed that approach in Section 2 (for example, we defined that $v(\sigma, \bigcirc\alpha) = 1$ iff $v(\sigma_{\geq 1}, \alpha) = 1$). The second, floating version of satisfiability, treats all states in a path equally, and defines satisfiability of a formula in a state of a path (for example, it states that $\bigcirc\alpha$ holds in the n-th state of a path, if α holds in the $(n+1)$-th state of the same path). It is well-known [30] that anchored and floating versions of satisfiability are equivalent, in the sense that a formula is satisfiable in the first approach iff it is satisfiable in the second one. That means that the class of satisfiable formulas does not change.

In the above definition, we have followed the floating version of satisfiability of temporal formulas (for example, the third item of the previous definition states that $\bigcirc\alpha$ holds in the i-th state of σ if α holds in the $(i+1)$-th state of σ). The reason for this change with respect to the previous sections of this paper is practical – it turned that proving statements related to decidability would be more tedious under the anchored version of satisfiability.

Definition 17 *(MPL_{LTL}- measurable structure).* An MPL_{LTL}-structure Σ^P is MPL_{LTL}-measurable iff $[\alpha]^i_{\Sigma^P,\sigma_j} = \{\pi \in \Sigma^P(\sigma_j, i) \mid \Sigma^P, \pi_0 \models \alpha\} \in H(\sigma_j, i)$ for every path $\sigma \in \Sigma^P$, every time instant σ_j, every $i \in Agt$ and every $\alpha \in Form$. The set of all MPL_{LTL}-measurable structures is denoted by MPL^{meas}_{LTL}.

In the rest of this paper we focus on MPL_{LTL}-measurable structures. For this class of structures we do not need inner measures in the definition of satisfiability relation. Indeed, since $\mu_\star(\sigma_j, i)$ and $\mu(\sigma_j, i)$ coincide on measurable sets, in Definition 16 we can replace the condition 2 (b) with:

(b') $w_i(\alpha)^{\Sigma^P,\sigma_j} = \mu(\sigma_j, i)([\alpha]^i_{\Sigma^P,\sigma_j})$.

A formula α is satisfiable if there is an MPL_{LTL} structure Σ^P, a path $\sigma \in \Sigma^P$, and a time instant σ_j such that $\Sigma^P, \sigma_j \models \alpha$.

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7. Decidability of MPL_{LTL}

Let us assume that α is satisfiable, i.e., that there is an MPL_{LTL} structure Σ^P, a path $\sigma \in \Sigma^P$ and a time instant σ_j such that $\Sigma^P, \sigma_j \models \alpha$. Similarly as above, we consider the set $Subfor(\alpha)$ of all subformulas of α, and all classically consistent formulas (called atoms) of the form

$$\bigwedge_{k=1}^{|Subfor(\alpha)|} \beta_k$$

where each β_k belongs to $Subfor(\alpha) \cup \{\neg\beta \mid \beta \in Subfor(\alpha)\}$ and each subformula of α appears exactly once (negated or not). We say that the formulas β_k occur in the top conjunction of an atom $at = \bigwedge_{k=1}^{|Subfor(\alpha)|} \beta_k$. Furthermore, by a classically consistent atom $at = \bigwedge_{k=1}^{|Subfor(\alpha)|} \beta_k$ we mean that each formula β_k of the form $\mathbf{f} \geq 0$, $\bigcirc\gamma$ or $\gamma U \delta$ is considered as a propositional letter, each formula β_k of the form $\neg(\mathbf{f} \geq 0)$, $\neg\bigcirc\gamma$ or $\neg(\gamma U \delta)$, is considered as a negated propositional letter, and that the conjunction at is not a classical contradiction.

Then:

- $\mathbf{At}(\alpha)$ denotes the set of all atoms of α,
- for a formula $\beta \in Subfor(\alpha)$ and an atom $at \in \mathbf{At}(\alpha)$, $\beta \in at$ means that β occurs in the top conjunction of at, i.e., $at = \bigwedge_{k=1}^{|Subfor(\alpha)|} \beta_k$, and for some k, $\beta = \beta_k$,
- $\mathbf{At}_\beta(\alpha)$ denotes the set $\{at \in \mathbf{At}(\alpha) : \beta \in at\}$, and
- if $at \in \mathbf{At}(\alpha)$, then $[at] = \{\sigma \;:\; \Sigma^P, \sigma_0 \models at\}$, where σ is a probabilistic path in a structure Σ^P.

Note that:

- $|Subfor(\alpha)| \leq |\alpha|$, i.e., the number of subformulas of α is not greater than the length of α,
- $|\mathbf{At}(\alpha)| \leq 2^{|\alpha|}$,
- for every probabilistic path σ in Σ^P, every time instant σ_j and exactly one atom $at \in \mathbf{At}(\alpha)$, $\Sigma^P, \sigma_j \models at$; that atom will be denoted $at(\alpha, \sigma_j)$,
- different atoms of α are mutually exclusive, and
- for every $\beta \in Subfor(\alpha)$, $\models \beta \leftrightarrow \bigvee_{\{at \in \mathbf{At}(\alpha) : \beta \in at\}} at$.

We say that a formula $\beta U \gamma$, satisfied in the time instant σ_j, is fulfilled before the time instant σ_{j+k} if there is an integer $l \in [j, j+k-1]$ such that:

- $\Sigma^P, \sigma_l \models \gamma$, and
- $\Sigma^P, \sigma_i \models \beta$ for every integer $i \in [j, l-1]$.

Finally, since satisfiability of a formula α depends only on members of $Subfor(\alpha)$, in the remaining part of this section we consider valuations on $Subfor(\alpha) \cap \mathcal{P}$, algebras generated by sets of the form $[at]$, and the probabilities defined on such sets. Note that it holds that, thanks to Corollary 3.3.4 from [2], measurable spaces in structures discussed in the sequel can be extended appropriately.

7.1. Ultimately periodic runs

Theorem 4.7 from [40] guarantees that to check satisfiability of an LTL-formula α it is enough to consider so-called ultimately periodic paths, where a path $\sigma = \sigma_0, \sigma_1, \ldots$ is ultimately periodic with starting index i and period m if:

- for every $k \geq i$, σ_k and σ_{k+m} coincide,
- both i and m are bounded by the functions of the size of α ($i \leq 2^{|\alpha|}$, $m \leq 4^{1+|\alpha|}$), and
- every $\beta U \gamma \in Subfor(\alpha)$ satisfied in Σ^P, σ_i is fulfilled before σ_{i+m}.

Here we can apply the same idea, but since the set of MPL_{LTL} formulas contains also formulas of the form $\mathbf{f} \geq 0$, the corresponding procedure should be adapted in the following way. Let us consider an MPL_{LTL}-measurable structure Σ^P, and an MPL_{LTL}-formula α, and let $\Sigma^{P'}$ denote the structure obtained after the Sistla-Clarke-procedure is applied on a path σ from Σ^P. Since satisfiability of a formula $\mathbf{f} \geq 0$ in a time instant σ_j depends only on probabilities $\mu(\sigma_j, i)$ defined on sets of paths, in the procedure of transforming the path σ into a ultimately periodic path we can consider $\mathbf{f} \geq 0$ as a propositional letter. Furthermore, since a path π belongs to the set $[\beta]^i_{\Sigma^P,\sigma_j}$ if $\Sigma^P, \pi_0 \models \beta$, and the procedure of transforming a path into a ultimately periodic path does not change satisfiability of the considered subformulas, the correctness of the properties of the structure $\Sigma^{P'}$ with respect to the definition of MPL_{LTL}-measurable structures is preserved:

- if σ' is the ultimately periodic path obtained from the path σ, then σ belongs to a measurable set in the starting structure Σ^P iff σ' belongs to the corresponding measurable set in $\Sigma^{P'}$,
- by transforming σ into the ultimately periodic paths σ', nonempty measurable sets from Σ^P remain nonempty measurable sets from $\Sigma^{P'}$, and
- the same values of probabilities of measurable sets in Σ^P and the corresponding measurable sets in $\Sigma^{P'}$ are kept.

Thus, $\Sigma^{P'}$ is an MPL_{LTL}-measurable structure, and satisfiability of formulas from $Subfor(\alpha)$ does not change when Σ^P is transformed to $\Sigma^{P'}$. Since the same procedure can be applied on every path σ from Σ^P, to check satisfiability of α it is enough to consider MPL_{LTL}-measurable structures containing only ultimately periodic paths.

7.2. Finitely represented structures

Let Σ^P be an MPL_{LTL}-measurable structure containing only ultimately periodic paths. Let $\sim_\alpha$ be a relation on the paths in Σ^P such that $\sigma \sim_\alpha \pi$ iff

- the starting indices of σ and π coincide, and the periods of σ and π coincide, and
- for every $at \in \mathbf{At}(\alpha)$, $\Sigma^P, \sigma_j \models at$ iff $\Sigma^P, \pi_j \models at$, $j \leqslant i + m$.

Since we consider MPL_{LTL}-measurable structures containing only ultimately periodic paths, the second constraint guarantees that for every $j \geqslant 0$, $\Sigma^P, \sigma_j \models at$ iff $\Sigma^P, \pi_j \models at$. Obviously, $\sim_\alpha$ is an equivalence relation, and there is only a finite number of equivalence classes bounded by $|\mathbf{At}(\alpha)|$.

Now, we consider the following structure Σ^P_{fin}:

- Σ^P_{fin} contains one representative from every equivalence class of $\sim_\alpha$,
- for every path σ in Σ^P_{fin} and every time instant σ_j the set $\mathcal{P}(\sigma_j)$ remains the same as in Σ^P,
- for every path σ in Σ^P_{fin} and every time instant σ_j the probability space $Prob^{\Sigma^P_{fin}}(\sigma_j, i)$ is obtained by the restriction of $Prob^{\Sigma^P}(\sigma_j, i)$ to the algebra generated by sets of the form $[at] \cap \Sigma^P_{fin}$.

The last condition implies that the measures $\mu^{\Sigma^P}(\sigma_j, i)$ and $\mu^{\Sigma^P_{fin}}(\sigma_j, i)$ coincide on sets of the form $[\beta]$ for every $\beta \in Subfor(\alpha)$.

The next lemma shows that satisfiability of subformulas of α in Σ^P is preserved in Σ^P_{fin}.

Lemma 3. *For every $\beta \in Subfor(\alpha)$, every path σ in Σ^P_{fin} and every σ_j*

$$\Sigma^P, \sigma_j \models \beta \quad \textit{iff} \quad \Sigma^P_{fin}, \sigma_j \models \beta.$$

Proof. The statement can be proved using induction on the structure of formulas. Since differences between paths in Σ^P and Σ^P_{fin} are related to the corresponding probability spaces, the statement holds trivially for propositional letters and every $\beta \in Subfor(\alpha)$ with the leading temporal or classical operator. It remains to consider subformulas of α of the form $\mathbf{f} \geq 0$. Then, a check based on the complexity of subterms in $\mathbf{f}$ gives that $\mathbf{f}^{\Sigma^P, \sigma_j} = \mathbf{f}^{\Sigma^P_{fin}, \sigma_j}$. Thus, $\Sigma^P, \sigma_j \models \mathbf{f} \geq 0$ iff $\Sigma^P_{fin}, \sigma_j \models \mathbf{f} \geq 0$.

7.3. Decision procedure

The previous subsections imply that to check satisfiability of α it is enough to consider MPL_{LTL}-measurable structures containing only at most $|\mathbf{At}(\alpha)|$ ultimately periodic paths. So, let us consider a structure Σ^P:

- with $n \leq |\mathbf{At}(\alpha)| \leq 2^{|\alpha|}$ ultimately periodic probabilistic paths, and
- for every $\sigma = \sigma_0, \ldots, \sigma_{i_\sigma}, \ldots, \sigma_{i_\sigma + m_\sigma - 1}, \ldots$ in Σ^P the lengths of staring index i_σ and period m_σ are at most $2^{|\alpha|}$ and $4^{1+|\alpha|}$, respectively.

For such a structure Σ^P the set

$$\underline{\Sigma^P} = \{\{\sigma^1_0, \ldots, \sigma^1_{i_{\sigma^1}}, \ldots, \sigma^1_{i_{\sigma^1} + m_{\sigma^1} - 1}\}, \ldots, \{\sigma^n_0, \ldots, \sigma^n_{i_{\sigma^n}}, \ldots, \sigma^n_{i_{\sigma^n} + m_{\sigma^n} - 1}\}\}$$

is called carrier. Note that, although the number of considered structures related to α is not bounded (because of probability spaces attached to states), the number of the corresponding carriers is finite. Since the number of atoms of α is also finite, there is only a finite number of different ways to distribute those atoms to states in carriers.

So, let us consider a carrier $\underline{\Sigma^P}$ with atoms denoted

$$at(\sigma^1_0), \ldots, at(\sigma^1_{i_{\sigma^1}}), \ldots, at(\sigma^1_{i_{\sigma^1} + m_{\sigma^1} - 1}), \ldots, at(\sigma^n_{i_{\sigma^n} + m_{\sigma^n} - 1})$$

attached to the corresponding states. Particularly, to at least one σ_j an atom at is attached such that $\alpha \in at$.

Now, the idea of the procedure is to check whether it is possible that all of the above atoms hold in the corresponding time instants. The affirmative answer implies that α is satisfiable, since it is required that for at least one atom at, $\alpha \in at$. Assuming that:

- for every σ_j in a path σ and for every propositional letter $p \in \mathcal{P} \cap Subfor(\alpha)$, $p \in \mathcal{P}(\sigma_j)$ iff $p \in at(\sigma_j)$

we can check whether $\Sigma^P, \sigma_j \models at(\sigma_j)$ in the following way by recursively analyzing formulas that appear in the top conjunctions of atoms attached to the time instants from the path σ:

- for every propositional letter $p \in at(\sigma_j)$, $\Sigma^P, \sigma_j \models p$,
- for $\neg\beta \in at(\sigma_j)$, $\Sigma^P, \sigma_j \models \neg\beta$ iff $\Sigma^P, \sigma_j \not\models \neg\beta$,
- for $\beta \wedge \gamma \in at(\sigma_j)$, $\Sigma^P, \sigma_j \models \beta \wedge \gamma$ iff $\Sigma^P, \sigma_j \models \beta$ and $\Sigma^P, \sigma_j \models \gamma$,
- for $\bigcirc\beta \in at(\sigma_j)$, $\Sigma^P, \sigma_j \models \bigcirc\beta$ iff $\Sigma^P, \sigma_{j+1} \models \beta$, with the proviso that if $j = i_\sigma + m_\sigma - 1$, $j + 1$ denotes i_σ,

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- for $\beta U \gamma \in at(\sigma_j)$, thanks to the requirement that U-formulas satisfied in σ_{i_σ}, is fulfilled before $\sigma_{i_\sigma + m_\sigma}$, to check whether $\Sigma^P, \sigma_j \models \beta U \gamma$ means to check whether β and γ are satisfied in a finite number of time instants from the path σ.

Finally, to consider probabilistic constraints, i.e., formulas of the forms $\mathbf{f} \geq 0$, $\neg\mathbf{f} \geq 0 \in at(\sigma_j)$ we have to check if there are probability spaces such that those formulas are satisfied in $\sigma_{\geq j}$. Recall that atoms are mutually exclusive and that every $\beta \in Subfor(\alpha)$ is equivalent to the disjunctions of all atoms in which β appears in the top conjunctions. For every σ_j in a path σ and for the atom $at(\sigma_j)$ we consider the corresponding system of (in)equalities:

$$\sum_{at \in \mathbf{At}(\alpha)} \mu(\sigma_j, i)([[at]]) = 1, \text{ for every agent } i \in Agt \tag{12}$$

$$\mu(\sigma_j, i)([[at]]) \geqslant 0, \text{ for all } at \in \mathbf{At}(\alpha), i \in Agt \tag{13}$$

$$\mathbf{f}^{\Sigma^P, \sigma_j} \geqslant 0, \text{ for every } \mathbf{f} \geq 0 \in at(\sigma_j) \tag{14}$$

$$\mathbf{f}^{\Sigma^P, \sigma_j} < 0, \text{ for every } \neg\mathbf{f} \geq 0 \in at(\sigma_j) \tag{15}$$

where $[[at]]$ denotes the set of all probabilistic paths σ such that $at = at(\sigma_0)$, and the notion $\mathbf{f}^{\Sigma^P, \sigma_j}$ is the same as it is introduced in Definition 16 with the additional condition that:

- $w_i(\beta)^{\Sigma^P, \sigma_j} = \sum_{at \in \mathbf{At}(\alpha), \beta \in at} \mu(\sigma_j, i)([[at]])$.

The (in)equalities (12) and (13) correspond to the standard properties of probability measures: the probability of the set of all paths is 1 and the probability of every measurable set of paths is nonnegative. The equalities (14) and (15) correspond to positive and negative probabilistic constraints that appear in $at(\sigma_j)$. For example, the inequalities (14) mean that for $\mathbf{f} \geq 0 \in at(\sigma_j)$. Note that, since we consider polynomial weight formulas, in the above system polynomials in $w_i(\beta)^{\Sigma^P, \sigma_j}$ appear.

For the considered carrier $\underline{\Sigma^P}$ the union LS of all such systems corresponding to states is finite and, thanks to the decision procedure from [3], it can be examined in a finite number of steps. LS is solvable iff there are probability measures which satisfies formulas of the form $\mathbf{f} \geq 0$ and $\neg\mathbf{f} \geq 0$ that appear in the top conjunction of atoms attached to states.

Thus, we can prove:

Theorem 8 *(Decidability). Satisfiability of MPL_{LTL}-formulas in the class of MPL_{LTL}- measurable structure is decidable.*

Proof. Decidability follows from the above described procedure since:

- for any α there is only a finite number of carriers,
- there is a finite number of ways in which atoms can be attached to time instants,
- satisfiability of atoms in the corresponding time instants can be checked in a finite number of steps,
- if atoms are satisfiable, since to at least one σ_j an atom at is attached such that $\alpha \in at$, α is also satisfiable, and
- if there is no carrier such that atoms can be attached to its time instants so that the atoms are satisfiable, α is not satisfiable. □

8. Conclusion

In this paper, we introduced two logics for probabilistic temporal reasoning. The languages of both logics contain both LTL formulas and polynomial weight formulas in the style of [14].

The first logic, PL_{LTL} is designed for probabilistic reasoning about temporal information, and its language allows the probabilistic operator w to be applied to LTL formulas. The corresponding semantics consists of a probability spaces over worlds, where each world contains one path of LTL. We propose an axiomatization for the logic and prove strong completeness, modifying some of our earlier methods [5,7,31,35,41]. Since the semantical relationship between the operators "next" and "until" explicitly requires σ-additive semantics, the axiomatization contains infinitary rules of inference. We show that the satisfiability problem is decidable in PSPACE, which is neither more complex than satisfiability of LTL nor the satisfiability of PWF alone.

Our second logic MPL_{LTL} uses a non-restricted modal approach to probabilistic temporal logic, and allows all temporal and probabilistic modalities to be combined in an arbitrary way. The semantics of MPL_{LTL} generalize paths of linear-time temporal logic by assigning probability spaces to all time instances of paths. We proved that the problem of satisfiability of formulas of MPL_{LTL} is decidable. We propose development of a complete axiom system for MPL_{LTL} as a topic for future work. We hope that this can be obtained by combining the techniques presented here with those presented in [26]. Ognjanović et al. [36] presented the logic $PTEL$, which has some similarity with MPL_{LTL}, and proved its decidability. The language of $PTEL$ involves epistemic, temporal and probability operators, so in that sense its semantics is more complicated than the semantics for MPL_{LTL} where epistemic aspects are not considered. On the other hand, only probabilistic formulas equivalent to the form $w_i(\alpha) \geq s$ are allowed in $PTEL$ (i.e., there are no arithmetical operations built into syntax), so expressiveness of MPL_{LTL} (where polynomial weight formulas are allowed) is much broader in that domain. Also, as a consequence, the decision procedure for MPL_{LTL} is more general than the decision procedure for $PTEL$ since consideration of polynomial weight formulas must be involved.

The two most common interpretations of probabilities are: objective probabilities, where the numbers represent relative frequencies, and subjective probability, where the numbers reflect subjective assessments of likelihood [21]. We believe that our first logic, PL_{LTL}, can be useful to model both the cases when frequence-based probabilities are assigned on temporal events (e.g. prediction where a cell phone will be in the future may be derived probabilistically from past logs showing the object's location [18]), and in the cases where subjective probabilities are used (e.g., a subjective probability of an agent that represents the chance that his favorite football team will win tomorrow). Indeed, the same formal mathematical definition of probability measures holds for both interpretations.

For our second logic, MPL_{LTL}, the situation is essentially different. The semantics of MPL_{LTL} contain multiple agents that are allowed to place different probabilities to an event. Typically, such frameworks are used to model situations where probability values represent the subjective probabilities the agents assign to events [12,9].[2]

In the logic MPL_{LTL} we can also apply temporal operators to probabilistic formulas, so one question that naturally arises is: how do probabilities change over time? We are not aware that the interaction between the probabilistic and the temporal modalities is systematically studied in the literature. A possible starting point for such a research would be the study on the interrelationship between knowledge and time presented in [22]. They analyzed semantically the properties of knowledge based on different assumptions

[2] Actually, Fagin and Halpern [13] emphasized that those semantics can be restricted to model also objective probability, by restricting the models in the way that in each state the probability spaces of all agents coincide. They have considered that restriction in the context of an epistemic probabilistic framework, to model the situations where all the probabilistic events are common knowledge (for example, if there is a global coin). However, we believe that in our logic such a restriction would not be of interest from multi-agent perspective, as we do not have any other modality that would help to distinguish between agents.

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of the underlying system (for example, whether the agents have *perfect recall*, which intuitively means that an agent's local state encodes everything that has happened - from her point of view - thus far in the run). It seems that a comparison of this approach with probabilistic temporal frameworks can be only done if we assume that knowledge is equated with certainty, i.e., an agent knows α if she places probability 1 to α (this approach is studied by van Eijck and Schwarzentruber [9]). We should emphasize that this interpretation of probability would require additional constraints on probability spaces, that would ensure desirable properties of the implied indistinguishability relation (like reflexivity and transitivity). However, there are also situations in which probability of an agent does not represent approximation of knowledge. For example, an agent might place one probability value to the event "it will rain on Friday" on Wednesday, and another one on Thursday, if she forms her probabilistic belief based on some weather forecast website. In addition, an agent can update her probability on the hypotheses as the effect of the observations (which is formalized by Halpern and Pucella [23]), or probabilities can change as the effect of actions of agents [17]. Each of those situations requires an extension of the semantics (with observations over time and actions, respectively). In this paper, our focus is not on any specific scenario (which would require further extensions of our semantics), but on formal properties like completeness for σ-additive semantics and decidability. We especially hope that our axiomatization could be used in various extensions of our general framework.

Some probabilistic LTL's were motivated by the need to analyze probabilistic programs and stochastic systems [8,15,24,28,29]. In some of them, probabilistic operators are not explicitly mentioned in the formulas, while in the others it is possible to directly express probabilities. Our logic allows one to quantify runs satisfying some properties. In this paper we restrict our attention to theoretical issues, while the possible applications (e.g., heuristic procedures for satisfiability checking) are left for the future work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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