Filomat 37:25 (2023), 8537–8542
https://doi.org/10.2298/FIL2325537V

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Invariants for equitorsion geometric mappings

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Abstract. New method of obtaining invariants for torsion-preserving mappings is the main subject of this research. This research is consisted of introduction and main part of research. After introduction, where we remind to some of obtained invariants, we will develop general formula (Vesić, 2020) for obtaining invariants for mappings. As special cases, we will remind to some invariants for symmetric affine connection spaces which are not obtained yet.

1. Introduction

An N-dimensional manifold $\mathcal{M}_N$ equipped with a non-symmetric affine connection ∇ whose coefficients are L^i_{jk}, $L^i_{jk} \neq L^i_{kj}$ for at least one pair (j,k), $j,k = 1,\dots,N$, is the non-symmetric affine connection space in sense of Eisenhart's definition.

The symmetric and anti-symmetric parts of affine connection coefficients L^i_{jk} are

$$L^i_{\underline{jk}} = \frac{1}{2}\Big(L^i_{jk} + L^i_{kj}\Big) \quad \text{and} \quad L^i_{\underset{\vee}{jk}} = \frac{1}{2}\Big(L^i_{jk} - L^i_{kj}\Big). \tag{1}$$

The manifold $\mathcal{M}_N$ equipped with torsion-free affine connection $\overset{0}{\nabla}$, whose coefficients are $L^i_{\underline{jk}}$, is the (symmetric) associated (affine connection) space of space $\mathbb{GA}_N$.

The doubled anti-symmetric parts $L^i_{\underset{\vee}{jk}}$ of coefficients L^i_{jk} are components of torsion tensor of the space $\mathbb{GA}_N$.

With respect to the symmetric affine connection $\overset{0}{\nabla}$, one kind of covariant derivative is defined [2, 11]

$$a^i_{j|k} = a^i_{j,k} + L^i_{\underline{\alpha k}} a^\alpha_j - L^\alpha_{\underline{jk}} a^i_\alpha, \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 53A55; Secondary 53B05.
Keywords. Invariant; Non-symmetric affine connection; Tensor; Parameter.
Received: 14 December 2022; Accepted: 22 December 2022
Communicated by Zoran Rakić
The first author supported by Serbian Ministry of Science through Mathematical Institute of the Serbian Academy of Sciences and Arts. The third author acknowledge the grant of the Ministry of Education, Science and Technological Development of Serbia 451-03-68/2022-14/200124 for carrying out this research.
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where a^i_j is tensor of the type $(1,1)$ and partial derivative of a^i_j by x^k is denoted by comma.

With respect to alternation of double covariant derivative of a^i_j by x^m and x^n, one curvature tensor is obtained $\Big($for details, see [2, 11]$\Big)$

$$\overset{0}{R}{}^i_{jmn} = L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + L^\alpha_{\underline{jm}} L^i_{\underline{\alpha n}} - L^\alpha_{\underline{jn}} L^i_{\underline{\alpha m}}. \tag{3}$$

S. M. Minčić defined four kinds of covariant derivatives with respect to non-symmetric affine connection ∇ $\Big($see [3–8]$\Big)$. Four curvature tensors, eight derived curvature tensors and fifteen curvature pseudotensors of the space $\mathbb{GA}_N$ are obtained with respect to these four kinds of covariant derivative. Many years later, it was proved [9, 15] that curvature pseudotensors are not coefficients of alternation of double covariant derivatives. Hence, the family of curvature tensors of space $\mathbb{GA}_N$ is

$$K^i_{jmn} = \overset{0}{R}{}^i_{jmn} + uL^i_{\underset{\vee}{jm}|n} + u'L^i_{\underset{\vee}{jn}|m} + vL^\alpha_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} + v'L^\alpha_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}} + wL^\alpha_{\underset{\vee}{mn}} L^i_{\underset{\vee}{\alpha j}}, \tag{4}$$

for real scalar invariants u, u', v, v', w.

1.1. Mappings of non-symmetric affine connection spaces

Infinitely many affine connections may be defined on the manifold $\mathcal{M}_N$. If affine connections ∇ and $\bar{\nabla}$ are defined on $\mathcal{M}_N$, the difference of corresponding affine connection coefficients, L^i_{jk} and $\bar{L}^i_{jk}$ is deformation tensor $P^i_{jk} = \bar{L}^i_{jk} - L^i_{jk}$. The difference of symmetric parts $L^i_{\underline{jk}}$ and $\bar{L}^i_{\underline{jk}}$, $P^i_{\underline{jk}} = \bar{L}^i_{\underline{jk}} - L^i_{\underline{jk}}$, is tensor as well.

The deformation tensors P^i_{jk} and $P^i_{\underline{jk}}$ uniquely determine mappings $f : \mathbb{GA}_N \to \mathbb{G}\bar{\mathbb{A}}_N$ and $f : \mathbb{A}_N \to \bar{\mathbb{A}}_N$, respectively.

Different forms of deformation tensors P^i_{jk} and $P^i_{\underline{jk}}$ are corresponding to different kinds of mappings between affine connection spaces. Invariants of these mappings have been a research subject of different authors.

N. S. Sinyukov [11], J. Mikeš and his research group [1, 2], V. Berezovski [1] and many other authors have obtained invariants for mappings of a symmetric affine connection space.

The geodesic mapping $f : \mathbb{A}_N \to \bar{\mathbb{A}}_N$ is determined with the deformation tensor $P^i_{\underline{jk}} = \psi_k \delta^i_j + \psi_j \delta^i_k$. After contracting this equality by i and k, the Thomas projective parameter for the geodesic mapping f was obtained [2, 11]

$$T^i_{\underline{jk}} = L^i_{\underline{jk}} - \frac{1}{N+1}\Big(L^\alpha_{\underline{k\alpha}} \delta^i_j + L^\alpha_{\underline{j\alpha}}\Big)\delta^i_k. \tag{5}$$

After involving $\psi_{ij} = \psi_{i|j} - \psi_i \psi_j$, it was proved that the Weyl projective tensor W^i_{jmn} is an invariant for the geodesic mapping f $\Big($see [2, 11]$\Big)$,

$$W^i_{jmn} = \overset{0}{R}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j \overset{0}{R}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\overset{0}{R}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\overset{0}{R}_{n]j}, \tag{6}$$

for Ricci tensor $\overset{0}{R}_{ij} = \overset{0}{R}{}^\alpha_{ij\alpha}$ and anti-symmetrization without division denoted by square brackets.

The previous procedure for obtaining invariants with respect to transformation has been continued when the corresponding invariants for geodesic mappings of a non-symmetric affine connection space were obtained [13, 16]. One family of invariants which generalize the Weyl projective tensor was obtained in any of these papers.

The main purpose of this research is to complete research for invariants of mappings whose deformation tensor is of the form $P^i_{\underline{jk}} = \psi_k \delta^i_j + \psi_j \delta^i_k + \sigma^i_{jk}$, $\sigma^i_{jk} = \sigma^i_{kj}$. This research was started in [14]. In this paper, we will continue and complete this study.

2. Main results

At the start, let us consider an equitorsion mapping (mapping which preserves the anti-symmetric parts $L^i_{\underset{\vee}{jk}}$ of non-symmetric affine connection coefficients L^i_{jk}) $f : \mathbb{GA}_N \to \mathbb{G}\bar{\mathbb{A}}_N$ characterized by the equation of form

$$\bar{L}^i_{jk} = L^i_{jk} + \psi_k\delta^i_j + \psi_j\delta^i_k + \sigma^i_{jk} + \xi^i_{jk}, \tag{7}$$

for tensors σ^i_{jk} of the type (1, 2) symmetric by j and k, and ξ^i_{jk} which is anti-symmetric by these indices. Because $\bar{L}^i_{jk} = \bar{L}^i_{\underline{jk}} + \bar{L}^i_{\underset{\vee}{jk}}$, $L^i_{jk} = L^i_{\underline{jk}} + L^i_{\underset{\vee}{jk}}$, and $\bar{L}^i_{\underset{\vee}{jk}} = L^i_{\underset{\vee}{jk}}$, the equation (7) reduces to

$$\bar{L}^i_{\underline{jk}} = L^i_{\underline{jk}} + \psi_k\delta^i_j + \psi_j\delta^i_k + 2\sigma^i_{jk}. \tag{7'}$$

2.1. Basic and derived invariants

In [14], the research for invariants was started of a mapping which transforms $L^i_{\underline{jk}}$ to $\bar{L}^i_{\underline{jk}}$ by the rule

$$\bar{L}^i_{\underline{jk}} = L^i_{\underline{jk}} + \bar{\omega}^i_{jk} - \omega^i_{\underline{jk}}, \tag{8}$$

for geometrical objects $\hat{\bar{\omega}}$ and $\hat{\omega}$, whose components are symmetric by covariant indices, i.e. $\bar{\omega}^i_{jk} = \bar{\omega}^i_{kj}$, $\omega^i_{jk} = \omega^i_{kj}$.

The equation (8) characterizes a mapping from $\mathbb{A}_N$ to $\bar{\mathbb{A}}_N$, or an equitorsion mapping from $\mathbb{GA}_N$ to $\mathbb{G}\bar{\mathbb{A}}_N$, as well.

Let us assume that the geometrical objects $\psi_k\delta^i_j + \psi_j\delta^i_k$ and σ^i_{jk} are linearly independent. Otherwise, the equation (7′) will determine the equitorsion geodesic mapping of space $\mathbb{GA}_N$.

We need to define the tensor $\bar{\sigma}^i_{jk} = -\sigma^i_{jk}$. These components satisfy the equality $\bar{\sigma}^i_{jk}\bar{\sigma}^p_{qr} = \left(-\sigma^i_{jk}\right)\cdot\left(-\sigma^p_{qr}\right) = \sigma^i_{jk}\sigma^p_{qr}$. That means that the geometrical object $\sigma^i_{jk}\sigma^p_{qr}$ is an invariant for the considered mapping.

Because $2\sigma^i_{jk} = \sigma^i_{jk} - \bar{\sigma}^i_{jk}$, it was concluded that is [14]

$$\omega^i_{jk} = -\sigma^i_{jk} + \frac{1}{N+1}\Big(\big(L^\alpha_{\underline{j\alpha}} + \sigma^\alpha_{j\alpha}\big)\delta^i_k + \big(L^\alpha_{\underline{k\alpha}} + \sigma^\alpha_{k\alpha}\big)\delta^i_j\Big), \quad \bar{\omega}^i_{jk} = -\bar{\sigma}^i_{jk} + \frac{1}{N+1}\Big(\big(\bar{L}^\alpha_{\underline{j\alpha}} + \bar{\sigma}^\alpha_{j\alpha}\big)\delta^i_k + \big(\bar{L}^\alpha_{\underline{k\alpha}} + \bar{\sigma}^\alpha_{k\alpha}\big)\delta^i_j\Big). \tag{9}$$

The basic associated invariants of Thomas and Weyl type for the mapping f are [14] $\tilde{\mathcal{T}}^i_{\underline{jk}} = L^i_{\underline{jk}} - \omega^i_{jk}$ and $\tilde{\mathcal{W}}^i_{jmn} = \overset{0}{R}{}^i_{jmn} - \omega^i_{jm|n} + \omega^i_{jn|m} + \omega^\alpha_{jm}\omega^i_{\alpha n} - \omega^\alpha_{jn}\omega^i_{\alpha m}$. After using the invariance of geometrical object $\sigma^i_{jk}\sigma^p_{qr}$ under the considered mapping, the basic associated invariant of Weyl type for the considered mapping is reduced to (11), and these invariants are

$$\overset{0}{\tilde{\mathcal{T}}}{}^i_{\underline{jk}} = L^i_{\underline{jk}} + \sigma^i_{\underline{jk}} - \frac{1}{N+1}\Big(\big(L^\alpha_{\underline{k\alpha}} + \sigma^\alpha_{k\alpha}\big)\delta^i_j + \big(L^\alpha_{\underline{j\alpha}} + \sigma^\alpha_{j\alpha}\big)\delta^i_k\Big), \tag{10}$$

$$\begin{aligned}\overset{0}{\tilde{\mathcal{W}}}{}^i_{jmn} &= \overset{0}{R}{}^i_{jmn} + \sigma^i_{j[m|n]} + \frac{1}{N+1}\delta^i_j\Big(\overset{0}{R}_{[mn]} - \sigma^\alpha_{[m\alpha|n]}\Big)\\ &\quad - \frac{1}{(N+1)^2}\delta^i_m\Big((N+1)\big(L^\alpha_{\underline{j\alpha}|n} + \sigma^\alpha_{j\alpha|n} - L^\beta_{\underline{\alpha\beta}}\sigma^\alpha_{jn}\big) + L^\alpha_{\underline{j\alpha}}L^\beta_{\underline{n\beta}} + L^\alpha_{\underline{j\alpha}}\sigma^\beta_{n\beta} + L^\alpha_{\underline{n\alpha}}\sigma^\beta_{j\beta}\Big)\\ &\quad + \frac{1}{(N+1)^2}\delta^i_n\Big((N+1)\big(L^\alpha_{\underline{j\alpha}|m} + \sigma^\alpha_{j\alpha|m} - L^\beta_{\underline{\alpha\beta}}\sigma^\alpha_{jm}\big) + L^\alpha_{\underline{j\alpha}}L^\beta_{\underline{m\beta}} + L^\alpha_{\underline{j\alpha}}\sigma^\beta_{m\beta} + L^\alpha_{\underline{m\alpha}}\sigma^\beta_{j\beta}\Big).\end{aligned} \tag{11}$$

The corresponding transformed geometrical objects (10, 11), $\overset{0}{\tilde{\bar{\mathcal{T}}}}{}^i_{\underline{jk}}$, $\overset{0}{\tilde{\bar{\mathcal{W}}}}{}^i_{jmn}$, are of the same form as the original ones expect that the covariant derivative | used in $\overset{0}{\tilde{\mathcal{W}}}{}^i_{jmn}$ is transformed to the covariant derivative || with respect to affine connection $\overset{0}{\bar{\nabla}}$ of the associated space $\bar{\mathbb{A}}_N$.

As in [14], we will obtain the derived associated invariant for mapping f by contracting difference $0 = \overset{0}{\tilde{\bar{\mathcal{W}}}}{}^i_{jmn} - \overset{0}{\tilde{\mathcal{W}}}{}^i_{jmn}$ by i and n. From this contacting, we will express summands which are multiplied by Kronecker's delta symbols in (11) as functions of other contracted summands in this equation. In this way, we will obtain another invariant for the mapping f which is called the derived associated invariant for this mapping [14].

The difference $0 = \overset{0}{\tilde{\bar{\mathcal{W}}}}{}^\alpha_{jm\alpha} - \overset{0}{\tilde{\mathcal{W}}}{}^\alpha_{jm\alpha}$ is

$$0 = \Big(\overset{0}{\bar{R}}_{jm} + \bar{\sigma}^\alpha_{jm\|\alpha}\Big) - \Big(\overset{0}{R}_{jm} + \sigma^\alpha_{jm|\alpha}\Big) - \frac{1}{N+1}\Big\{\Big(\overset{0}{\bar{R}}_{[jm]} - \bar{\sigma}^\alpha_{[j\alpha\|m]}\Big) - \Big(\overset{0}{R}_{[jm]} - \sigma^\alpha_{[j\alpha|m]}\Big)\Big\} + (N-1)X_{jm}, \tag{12}$$

After solving the equation (12) by X_{jm}, and substituting the solution in the difference $0 = \overset{0}{\tilde{\bar{\mathcal{W}}}}{}^i_{jmn} - \overset{0}{\tilde{\mathcal{W}}}{}^i_{jmn}$, one gets

$$\overset{0}{\tilde{\bar{W}}}{}^i_{jmn} = \overset{0}{\tilde{W}}{}^i_{jmn},$$

for

$$\begin{aligned}\overset{0}{\tilde{W}}{}^i_{jmn} = {} & \overset{0}{R}{}^i_{jmn} + \sigma^i_{jm|n} + \frac{1}{N+1}\delta^i_j\Big(\overset{0}{R}_{[mn]} - \sigma^\alpha_{[m\alpha|n]}\Big) + \frac{N}{N^2-1}\delta^i_{[m}\overset{0}{R}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\overset{0}{R}_{n]j} \\ & + \frac{1}{N-1}\delta^i_{[m}\sigma^\alpha_{jn]|\alpha} + \frac{1}{N^2-1}\Big(\delta^i_m\sigma^\alpha_{[j\alpha|n]} - \delta^i_n\sigma^\alpha_{[j\alpha|m]}\Big),\end{aligned} \tag{13}$$

and the corresponding $\overset{0}{\tilde{\bar{W}}}{}^i_{jmn}$.

Because the mapping f is equitorsion, the difference $\bar{L}^i_{\underset{\vee}{jm}\|n} - L^i_{\underset{\vee}{jm}|n}$ is expressed as

$$\bar{L}^i_{\underset{\vee}{jm}\|n} - L^i_{\underset{\vee}{jm}|n} = \Big(\bar{\omega}^i_{(a).\alpha n} - \omega^i_{(a).\alpha n}\Big)L^\alpha_{\underset{\vee}{jm}} - \Big(\bar{\omega}^\alpha_{(b).jn} - \omega^\alpha_{(b).jn}\Big)L^i_{\underset{\vee}{\alpha m}} - \Big(\bar{\omega}^\alpha_{(c).mn} - \omega^\alpha_{(c).mn}\Big)L^i_{\underset{\vee}{j\alpha}}, \tag{14}$$

for $a, b, c \in \{1, 2\}$, $\omega^i_{(1).jk} = L^i_{\underline{jk}}$, $\bar{\omega}^i_{(1).jk} = \bar{L}^i_{\underline{jk}}$, $\omega^i_{(2).jk} = \omega^i_{jk}$, and $\bar{\omega}^i_{(2).jk} = \bar{\omega}^i_{jk}$, for ω^i_{jk} and $\bar{\omega}^i_{jk}$ given by (9). Because $\bar{L}^i_{\underset{\vee}{jk}} = L^i_{\underset{\vee}{jk}}$, the equation (14) is equivalent to

$$\tilde{\bar{\mathcal{B}}}{}^i_{(l).jmn} = \tilde{\mathcal{B}}{}^i_{(l).jmn},$$

for $(l) = (a, b, c)$,

$$\tilde{\mathcal{B}}{}^i_{(l).jmn} = L^i_{\underset{\vee}{jm}|n} - \omega^i_{(a).\alpha n}L^\alpha_{\underset{\vee}{jm}} + \omega^\alpha_{(b).jn}L^i_{\underset{\vee}{\alpha m}} + \omega^\alpha_{(c).mn}L^i_{\underset{\vee}{j\alpha}}, \tag{15}$$

and the corresponding $\tilde{\bar{\mathcal{B}}}{}^i_{(l).jmn}$ and $\tilde{\Omega}^i_{(l).jmn}$.

The next theorem is proved above.

Theorem 2.1. *Let $f : \mathbb{GA}_N \to \mathbb{G}\bar{\mathbb{A}}_N$ be an equitorsion mapping of non-symmetric affine connection space $\mathbb{GA}_N$.*

The geometrical objects $\overset{0}{\tilde{\mathcal{T}}}{}^i_{\underline{jk}}$ and $\overset{0}{\tilde{\mathcal{W}}}{}^i_{jmn}$, given by (10, 11), are the basic associated invariants of the Thomas and the Weyl type for the mapping f, respectively.

The geometrical object $\overset{0}{\tilde{W}}{}^i_{jmn}$ given by (13) is the derived associated invariant for the mapping f.

The geometrical object $\tilde{\mathcal{B}}^i_{(l).jmn}$, given by (15) is an invariant for the equitorsion mapping f.

The families

$$\overset{0}{\mathcal{W}}{}^i_{(l_1).(l_2).jmn} = \overset{0}{\tilde{\mathcal{W}}}{}^i_{jmn} + u\tilde{\mathcal{B}}^i_{(l_1).jmn} + u'\tilde{\mathcal{B}}^i_{(l_2).jnm}, \tag{16}$$

$$\overset{0}{W}{}^i_{(l_1).(l_2).jmn} = \overset{0}{\tilde{W}}{}^i_{jmn} - u\tilde{\mathcal{B}}^i_{(l_1).jmn} - u'\tilde{\mathcal{B}}^i_{(l_2).jnm}, \tag{17}$$

for $(l_1) = (a_1, b_1, c_1)$, $(l_2) = (a_2, b_2, c_2)$, are basic and derived invariants for the equitorsion mapping f obtained with respect to transformation of family K^i_{jmn} of curvature tensors given by (4). □

2.2. Invariants for special mappings

In this part of paper, we will present invariants for some special mappings. Because the geometrical object σ^i_{jk} is the crucial one for obtaining invariants for an equitorsion mapping $f : \mathbb{GA}_N \to \mathbb{G}\bar{\mathbb{A}}_N$, we will present the invariants for equitorsion geodesic mapping (the case of $\sigma^i_{jk} = 0$), but invariants for other studied mappings we will characterized by the corresponding σ^i_{jk}, $\omega^i_{(1).jk} = L^i_{\underline{jk}}$, and $\omega^i_{(2).jk}$, below.

2.2.1. Invariants for equitorsion geodesic mappings

Let $f : \mathbb{GA}_N \to \mathbb{G}\bar{\mathbb{A}}_N$ be equitorsion geodesic mapping.

For this mapping, we have

$$\sigma^i_{jk} = 0, \quad \omega^i_{(1).jk} = L^i_{\underline{jk}}, \quad \omega^i_{(2).jk} = \frac{1}{N+1}\Big(L^\alpha_{\underline{j\alpha}}\delta^i_k + L^\alpha_{\underline{k\alpha}}\delta^i_j\Big). \tag{18}$$

The basic associated invariants and the derived associated invariant for this mapping are

$$\mathcal{G}\overset{0}{\tilde{\mathcal{T}}} = L^i_{\underline{jk}} - \frac{1}{N+1}\Big(L^\alpha_{\underline{k\alpha}}\delta^i_j + L^\alpha_{\underline{j\alpha}}\delta^i_k\Big), \tag{19}$$

$$\begin{aligned}\mathcal{G}\overset{0}{\tilde{\mathcal{W}}}{}^i_{jmn} = \overset{0}{R}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j\overset{0}{R}_{[mn]} - \frac{1}{(N+1)^2}\delta^i_m\Big((N+1)L^\alpha_{\underline{j\alpha}|n} + L^\alpha_{\underline{j\alpha}}L^\beta_{\underline{n\beta}}\Big)\\ + \frac{1}{(N+1)^2}\delta^i_n\Big((N+1)L^\alpha_{\underline{j\alpha}|m} + L^\alpha_{\underline{j\alpha}}L^\beta_{\underline{m\beta}}\Big).\end{aligned} \tag{20}$$

The derived associated invariant for the equitorsion geodesic mapping f is

$$\mathcal{G}\overset{0}{W}{}^i_{jmn} = \overset{0}{R}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j\overset{0}{R}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\overset{0}{R}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\overset{0}{R}_{n]j}. \tag{21}$$

The invariants $\mathcal{G}\overset{0}{\tilde{\mathcal{T}}}{}^i_{\underline{jk}}$ and $\mathcal{G}\overset{0}{\tilde{W}}{}^i_{jmn}$ coincide with the Thomas projective parameter and with the Weyl projective tensor, respectively given by (5, 6).

Based on the equations (16, 17), the families of basic and derived invariants for the equitorsion geodesic mapping f are

$$\mathcal{G}\overset{0}{\mathcal{W}}{}^i_{(l_1).(l_2).jmn} = \mathcal{G}\overset{0}{\tilde{\mathcal{W}}}{}^i_{jmn} + u\tilde{\mathcal{B}}^i_{(l_1).jmn} + u'\tilde{\mathcal{B}}^i_{(l_2).jnm}, \tag{22}$$

$$\mathcal{G}\overset{0}{W}{}^i_{(l_1).(l_2).jmn} = \mathcal{G}\overset{0}{\tilde{W}}{}^i_{jmn} + u\tilde{\mathcal{B}}^i_{(l_1).jmn} + u'\tilde{\mathcal{B}}^i_{(l_2).jnm}, \tag{23}$$

for $\tilde{\mathcal{B}}^i_{(l).jmn}$ given by (18).

2.2.2. Invariants for third type almost geodesic mappings

The equitorsion mapping $f : \mathbb{G}\mathbb{A}_N \to \mathbb{G}\bar{\mathbb{A}}_N$ whose basic equations are [10, 12]

$$\begin{cases} \bar{L}^i_{\underline{jk}} = L^i_{\underline{jk}} + \psi_k \delta^i_j + \psi_j \delta^i_k + 2\sigma_{jk}\varphi^i, \\ \varphi^i_{\underset{u}{|}j} = \psi_j \varphi^i + \nu \delta^i_j, \quad u = 1, 2, \end{cases} \tag{24}$$

for the scalar ν, 1-form ψ_j, contravariant vector φ^i and covariant symmetric tensor σ_{jk} of the type $(0, 2)$, is the equitorsion third type almost geodesic mapping of the u-th kind.

From the first of basic equations (24), one gets

$$\sigma^i_{jk} = \sigma_{jk}\varphi^i, \quad \omega^i_{(1).jk} = L^i_{\underline{jk}}, \quad \omega^i_{(2).jk} = -\sigma_{jk}\varphi^i + \frac{1}{N+1}\Big(\big(L^\alpha_{\underline{j\alpha}} + \sigma_{j\alpha}\varphi^\alpha\big)\delta^i_k + \big(L^\alpha_{\underline{k\alpha}} + \sigma_{k\alpha}\varphi^\alpha\big)\delta^i_j\Big). \tag{25}$$

The corresponding invariants $\mathcal{AG}_3 \overset{0}{\tilde{\mathcal{T}}}{}^i_{jk}$, $\mathcal{AG}_3 \overset{0}{\tilde{\mathcal{W}}}{}^i_{jmn}$, $\mathcal{AG}_3 \overset{0}{\tilde{W}}{}^i_{jmn}$, $\mathcal{AG}_3 \overset{0}{\mathcal{W}}{}^i_{(l_1).(l_2).jmn}$, $\mathcal{AG}_3 \overset{0}{W}{}^i_{(l_1).(l_2).jmn}$, are obtained after substituting the expressions (25) in (10, 11, 13, 16, 17).

The forms of invariants $\mathcal{AG}_3 \overset{0}{\tilde{\mathcal{W}}}{}^i_{jmn}$, $\mathcal{AG}_3 \overset{0}{\tilde{W}}{}^i_{jmn}$, $\mathcal{AG}_3 \overset{0}{\mathcal{W}}{}^i_{(l_1).(l_2).jmn}$, $\mathcal{AG}_3 \overset{0}{W}{}^i_{(l_1).(l_2).jmn}$ are not identical to the forms of their images in general. For this reason, the terms of valued invariant (invariant whose transformed version has the same value but different form of the original) and total invariant (invariant whose transformed version has the same both value and form) are involved in [10]. As it was proved in that article, the invariant $\mathcal{AG}_3 \overset{0}{\tilde{\mathcal{T}}}{}^i_{\underline{jk}}$ is total invariant for the equitorsion almost geodesic mapping f. The other four invariants for this mapping are valued ones in general. The valued invariants for the almost geodesic mapping f are total if and only if the inverse mapping of mapping f is the third type almost geodesic mapping of the u-th kind.

2.3. Future research

We will present the families of invariants (16, 17) as functions of family K^i_{jmn} of curvature tensors and the corresponding family of Ricci tensors. After that, the general invariants for mappings of different kinds of non-symmetric affine connection spaces will be obtained.

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