Article

# Linear Diophantine Fuzzy Fairly Averaging Operator for Suitable Biomedical Material Selection 

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#### Abstract

Nowadays, there is an ever-increasing diversity of materials available, each with its own set of features, capabilities, benefits, and drawbacks. There is no single definitive criteria for selecting the perfect biomedical material; designers and engineers must consider a vast array of distinct biomedical material selection qualities. The goal of this study is to establish fairly operational rules and aggregation operators (AOs) in a linear Diophantine fuzzy context. To achieve this goal, we devised innovative operational principles that make use of the notion of proportional distribution to provide an equitable or fair aggregate for linear Diophantine fuzzy numbers (LDFNs). Furthermore, a multi-criteria decision-making (MCDM) approach is built by combining recommended fairly AOs with evaluations from multiple decision-makers (DMs) and partial weight information under the linear Diophantine fuzzy paradigm. The weights of the criterion are determined using incomplete data with the help of a linear programming model. The enhanced technique might be used in the selection of compounds in a variety of applications, including biomedical programmes where the chemicals used in prostheses must have qualities similar to those of human tissues. The approach presented for the femoral component of the hip joint prosthesis may be used by orthopaedists and practitioners who will choose bio-materials. This is due to the fact that biomedical materials are employed in many sections of the human body for various functions.


Keywords: multi-criteria decision making; aggregation operators; optimization model; fairly operations; material selection; biomedical material

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## 1. Introduction

Numerous academic fields, including sociology, epistemology, intellectual technology, and machine learning, investigate how humans come to their conclusions and make decisions in response to the myriad of challenges that they confront on a daily basis. In general, several quantitative and analytical models are used in an effort to characterise these processes. The difficulty of making decisions is a challenge that develops during this procedure. The process of choosing one or more of the alternate forms of behaviour faced by individuals or an organisation in order to achieve a particular goal is referred to as "decision making", and it is distinguished as the procedure of selecting one or more of the available options. According to research, while it is possible to get by with making many of your day-to-day judgments based just on gut instinct, this method is not sufficient for making significant and important choices on its own. MCDM refers to a group of analytic methods that analyse the benefits and drawbacks of potential options based on a number of different factors. Methods from MCDM are utilised to provide assistance for the decisionmaking process as well as to pick one or more alternatives from a group of alternatives
featuring varying features based on criteria that are in conflict with one another or to rank these options. In different terms, while using MCDM approaches, decision-makers evaluate the various options based on a number of criteria in order to rank them according to the attributes that are most important to them.

The choice of materials is one of the most important parts of the process of designing, researching, and making products. The outcomes of the material procurement process have a direct impact on both the product's quality and its budget [1]. When used in a particular product, the optimum material will allow the product to have the highest possible performance while also having the lowest possible cost. It is thus of utmost importance to determine how to choose the best material from among the available alternatives [2]. Because of developments in materials science and improvements in production processes, the types of materials that may be selected are increasing in number, and the product requirements that must be taken into account during the selection procedure are becoming more in depth. The difficulty of selecting the appropriate material is made more difficult by the extensive prerequisites as well as the extensive range of options. Therefore, it is of utmost importance to investigate suitable methods to handle the problem of material selection [3]. During the practise of selecting materials, several different product criteria, including productivity, affordability, stability, dependability, market dynamics, fashion, and so on, need to be taken into consideration in order to choose the best material [4]. These product needs can be thought of as selection criteria, and different materials can be assessed using the data on the factors [5].

A bio-material, which is defined as a material meant to come into contact with biological structures, can be used to develop, treat, or modify a human tissue, muscle, or physiological function. An implant is a device that is entirely or partly implanted under the epithelial surface. Implants are defined as devices that are made of one or more bio-materials. The body's vastly different perimeter necessitates the use of biomedical materials. During the course of your daily activities, the bones in your body are subjected to a range of stressors. Similarly, when the body is in motion, orthopaedic materials are subjected to billions of loading cycles. Fatigue resistance and mechanical toughness are other important considerations. A number of natural and synthetic materials, referred to together as biomedical materials, can fulfil or assist the functions of live tissues in the human body. Because the human body is made up of proteins and oxygenated salt solutions, it is reasonable to expect that these materials will not bloat, deform, or corrode as a result of absorbing biological fluids. Under these conditions, some implant materials are accepted by the body, while others are rejected. Materials used in bio-medicine should not be corrosive, toxic, or carcinogenic; they should also have adequate mechanical strength; they should not cause reactions that are not naturally occurring in the body; and they should not decay. Biomedical materials are employed in orthopaedic applications such as joint prostheses and skeletal appropriate substitute materials. Other applications for these materials include frontal and reattachment surgery, dentistry, heart valves, artificial heart parts, catheters, backbone instrumentation, fixator material, metal parts, perforated screws, screw washers, fixator wires, nails, hip plates, angled plates, anatomical plates, and implantable devices. Metals are the preferred material for use in the biomedical sector due to their extended lifespan, malleability, and abrasion resistance. Metals, on the other hand, have limited bio-compatibility, exceedingly difficult corrosion in human fluids compared to bodily tissues, a high density, and the potential to elicit allergic tissue responses. Ceramics are dense, hard, brittle, and difficult to manufacture materials with low mechanical properties and excellent bio-compatibility. Furthermore, ceramics offer high bio-compatibility and corrosion resistance. Composite materials have been developed as a replacement for less attractive materials that were previously utilised. Polymers are utilised in general plastic surgery materials for the circulatory system and general plastic surgery operations, as opposed to metallic bio-materials and bio-ceramics used in orthopaedic and dental implants. Hip replacement surgery, commonly known as total hip arthroplasty, is the
medical term for the procedure of replacing a damaged or worn out hip joint. The patient's native joint will be replaced with an artificial joint during this treatment.

The information regarding criteria for alternative materials is typically hazy and imprecise. This is because human cognition is inherently cloudy, and the qualities of materials are not always completely clear. As a result, the criteria information is better suited for depiction by a fuzzy set (FS) [6], and the fuzzy MCDM approaches have been utilised in order to tackle a variety of material decision issues. Despite this, the fuzzy MCDM approach is still susceptible to the following drawbacks: the features of the material grow increasingly complicated and unclear, and the information needed to evaluate them cannot be adequately represented by FS. As a result, material attribute knowledge must be expressed using a more powerful quantitative tool because: (1) the consistency of the attribute knowledge expressed by FS is uncertain, and if the attribute knowledge is not precise, the final outcome of the material selection will be incorrect; and (2) the consistency of the attribute knowledge expressed by FS is uncertain.

Atanassov expanded FSs by developing the innovative concept of "intuitionistic fuzzy sets" (IFS) [7], which is described as having a "membership degree (MSD)" and a "nonmembership degree (NMSD)" that is less than or equal to 1 . When dealing with ambiguous, unclear, or insufficient data, one of the most powerful and effective techniques has been demonstrated to be the IFS hypothesis. This element of the IFS is important for a large number of professionals that work with real-world scenarios within the context of an IFS framework. In 2019, Riaz and Hashmi carried out an exhaustive evaluation of the constraints associated with MSDs and NMSDs in the structures of FS, IFS, "Pythagorean fuzzy set" (PFS) [8], and "q-rung orthopair fuzzy set" (q-ROFS) [9], and such limitations were defined. They came up with the "linear Diophantine fuzzy set" (LDFS) [10] as a solution to these problems by including IFS-specific reference parameters (RPs) into its design. They contend that the LDFS concept will eliminate the restrictions placed on the selection of features in exercise by the methodologies that are now in use for the various sets, and that this will make it possible to pick features with no limits. By utilising the arbitrary quality of the RPs, they were also able to prove that the universe of this set has a greater number of occurrences than the FS, IFS, PFS, and q-ROFS did. Table 1 shows a quick comparison of the proposed approach with current notions.

Table 1. Comparative analysis of LDFS with other extensions.

| Concepts | Remarks |
| :--- | :--- |
| FSs [6] | It does not take into account NMSDs. |
| IFSs [7] | It is inapplicable if $1<\xi_{\mathcal{I}}(\omega)+\rho_{\mathcal{I}}(\omega) \leq 2$ for some $\omega$. |
| PFSs [8] | It is inapplicable if $1<\xi_{\mathcal{P}}^{2}(\omega)+\rho_{\mathcal{P}}^{2}(\omega) \leq 2$ for some $\omega$. |
| q-ROFSs [9] | It is inapplicable for smaller "q" values. with $1<\xi_{\mathcal{O}}^{q}(\omega)+\rho_{\mathcal{O}}^{q}(\omega) \leq 2$, or if <br> $\xi_{\mathcal{O}}(\omega)=\rho_{\mathcal{O}}(\omega)=1$ for some $\omega$. |
|  | (1) it is far capable of coping with all eventualities wherein FS, IFS, PFS, and <br> LDFSs [10] <br> q-ROFS can't be employed; (2) it takes a parameterisation technique and <br> operates underneath the manipulate of reference parameters; (3) MSD and <br> NMSD may be taken in a free way from $[0,1]$. |

Despite the fact that much study has been done on material selection in the past, there is a requirement for a clear and comprehensive scientific technique or computational tool to aid user organisations in making the right material selection conclusion. A material selection procedure's goal is to discover material selection qualities and acquire the best acceptable mix of material selection attributes in conjunction with actual demand. We utilised LDFS and fairly operations to construct fairly AOs for this purpose.

The remainder of the article is structured as follows: Section 2 conducts a literature study on the material selection process and AOs. Section 3 discusses some fundamental LDFS definitions. Section 4 introduces some fairly operations and verifies the corresponding
theorems. Section 5 discusses certain LDF-fairly AOs and their properties. Section 6 elaborates on the MCDM material selection approach and discusses the computation procedure. The case study and decision-making procedure are proposed in Section 7. Section 8 is a summary of the whole study.

## 2. Literature Review

This section discusses some relevant research on the MCDM approach for the selection of materials and various AOs.

### 2.1. Literature Review on MCDM Method for Material Selection

The material selection technique focuses most of its research on the MCDM approach, and it has been used a lot to solve a wide range of material selection problems. Shanian and Savadogo [11,12] initially used the traditional MCDM techniques "Elimination and Choice Expressing REality" (ELECTRE) and "Technique for Order Preference by Similarity to an Ideal Solution)" (TOPSIS) to the problem of material selection. Using these approaches, they selected the optimal materials for certain products. Liu et al. [13] introduced a unique hybrid MCDM approach based on an enhanced version of "VIsekriterijumska Optimizacija I Kompromisno Resenje" (VIKOR) to handle the issue of interrelated multiple factors and industry standard in material selection.

The interval-valued VIKOR method was presented by Jahan and Edwards [14] as a solution to the target-dependent materials selection difficulties. Peng and Xiao [15] developed a unique MCDM approach that combines the "Preference Ranking Organisation Methods for Enrichment Evaluations " (PROMETHEE) and "analytic network process" (ANP) to determine the optimal material for a given application. Rao and Davim [16] used TOPSIS and the "analytical hierarchy process" (AHP) to handle the problem of selecting materials for a given technological design. Moreover, Rao [4] offered a systematic procedure for the selection of materials by combining the VIKOR and AHP. Combining AHP, grey correlation, and TOPSIS, Tian et al. [17] developed a hybrid MCDM strategy to address the issues involved with the selection of ecologically sustainable ornamental materials. Zhang et al. [18] developed a hybrid MCDM approach by combining the tactical decision and assessment laboratories, grey relational analysis, ANP, and TOPSIS to determine the most effective environmentally friendly material. Jahan et al. [19] created an aggregation process in an effort to address the problem of material selection. The inputs for this approach are the material ranking orders generated by various MCDM techniques, and the outputs are the aggregate rankings.

When there is a degree of unpredictability in the knowledge about the requirements for materials, fuzzy MCDM is the method that is most suited for finding solutions to material selection issues. Steel alloys materials selection difficulties were the focus of the presentation of Wang and Chang [20] on the fuzzy MCDM technique, which proposes that the ideal material may be identified through the aggregate and ordering of fuzzy numbers. Rathod and Kanzaria [21] came up with the idea for an assessment model that is based on fuzzy AHP and TOPSIS so that they could choose a suitable phase transition material. An incorporated fuzzy MCDM technique was given by Sa et al. [22] to address the issue of the green materials segment. This method integrates the fuzzy AHP and fuzzy TOPSIS approaches. Girubha and Vinodh [23] came up with the idea of using the fuzzy VIKOR approach to choose the best material for the electricity instrument cluster. Khabbaz et al. [24] came up with a technique that is a simplified form of fuzzy logic. This approach is able to readily cope with the qualitative character of materials and the fuzzy space that corresponds to it, which allows for improved material selection. Material selection was one of the applications Yang and Ju [25] used for their newly constructed fuzzy variable, which they named uncertain member language variable. An evaluation framework was presented by Roy et al. [26] for the purpose of solving the material selection problem. Farid and Riaz [27] introduced Einstein interactive AOs related to a neutrosophic
set with its use to material selection in engineering design. Zhang et al. [28] proposed the MCDM approach for material selection based on group generalised AOs related to PFSs.

In terms of implementation possibilities, fuzzy MCDM is superior when it comes to material selection. When it comes to material selection, expanded variants of FS such as LDFS are utilised rather infrequently. In the same vein, the issue of determining whether or not criteria information has been accurately represented is seldom ever researched. As a result, the purpose of this study is to attempt to address these gaps.

### 2.2. Literature Review on Fuzzy AOs

AOs are capable of processing and making use of data effectively. Different AOs will be produced as a result of different sets of rules or by focusing on various components of the available information. Jana et al. [29] proposed a new MCDM dynamic approach with the help of some complex AOs. Feng et al. [30] introduced some notions about IFSs. Ashraf et al. [31] developed some sine trigonometric AOs for a single valued neutrosophic. Liu et al. [32] introduced "power Maclaurin symmetric mean" AOs for q-ROFNs with applications to MCDM. Xing et al. [33] presented the concept of point weighted AOs for q-ROFNs. Liu and Wang [34] gave the idea of "Archimedean Bonferroni AOs" for q-ROFNs. Liu et al. [35] developed a heterogeneous relationship among the criteria for q-ROFNs. Mahmood and Ali [36] proposed complex q-ROF Hamacher AOs for MCDM. Hussain et al. [37] proposed AOs for hesitant q-ROFSs, and Jana et al. [38] initiated the concept of AOs for the MCDM method using a bipolar fuzzy soft set. Saha et al. [39] gave the concept of fairly AOs for $q$-ROFSs.

Liu and Shi [40] presented linguistic Heronian mean AOs, Lu and Ye [41] proposed some exponential AOs and Li et al. [42] gave generalised Einstein AOs for SVNNs. Saha et al. [43], Wei and Zhang, and Wei and Wei [44] gave some different AOs related to different extensions of FSs. Alcantud [45] presented some extensive results related to soft sets. Karaaslan and Ozlu [46] developed some work related to dual type-2 hesitant FSs. Senapati et al. [47] proposed Aczel-Alsina geometric AOs for interval-valued IFSs. Wang and Zhang [48] introduced MCDM based on rough sets and fuzzy measures. Gergin et al. [49] proposed some extensive MCDM approch for supplier selection. Narang et al. [50] introduced the decision-making approach based on Heronian mean AOs. Karamaşa et al. [51] gave the idea of neutrosophic operational sciences techniques. Some brilliant work related to some extension of fuzzy sets can be seen in [52-54].

While dominating AOs intervene to settle MCDM difficulties within the LDF framework, they seldom investigate the impartiality of their peers when addressing with MSD and NMSD. For instance, the values derived by AOs that previously exist in pieces of literature cannot be distinguished when a DM supervises all MSD and NMSD for a comparable work. This indicates that the ultimate conclusion is unquestionably biased. Thus, new procedures are necessary to handle MSD and NMSD properly and to ensure that LDFN operates fairly or neutrally. Using the notion of proportionate distributing rules for all functions, we develop two neutral or fair procedures in order to attain genuine pleasure while assessing MSD and NMSD.

As a result of this issue, the key aims of the article are as follows:

1. LDFNs are very good at tackling challenges with a two-degree complexity scale. LDFSs are employed to develop new AOs.
2. Construct several distinct fairly or neutral procedures for handling the MSD and NMSG using the interaction coefficient in an acceptable manner.
3. We proposed two new AOs, namely, the "linear Diophantine fuzzy fairly weighted averaging (LDFFWA) operator" and the "linear Diophantine fuzzy fairly ordered weighted averaging (LDFFOWA) operator".
4. Several innovative concepts linked with freshly generated AOs for data fusion are demonstrated by a sufficient number of illustrative cases. The suggested operators provide information that is more general, trustworthy, and accurate than previous methods.
5. Using the provided AOs, a novel MCDM approach for modelling uncertainty in material selection is developed.

## 3. Certain Fundamental Concepts

This portion of the article will provide various essential notions connected to LDFSs, and it will do so in the context of the universal set $X$.

Definition 1 ([10]). A "linear Diophantine fuzzy set (LDFS)" $\Re$ in $X$ is defined as

$$
\Re=\left\{\left(\omega,\left\langle\xi_{\Re}(\omega), \rho_{\Re}(\omega)\right\rangle,\left\langle\eta_{\Re}(\omega), \beta_{\Re}(\omega)\right\rangle\right): \omega \in X\right\},
$$

where $\xi_{\Re}(\omega), \rho_{\Re}(\omega), \eta_{\Re}(\omega), \beta_{\Re}(\omega) \in[0,1]$ are the MSD, the NMSD and the corresponding reference parameters (RPs), respectively. Moreover,

$$
0 \leq \omega_{\Re}(\omega)+\beta_{\Re}(\omega) \leq 1
$$

and

$$
0 \leq \eta_{\Re}(\omega) \xi_{\Re}(\omega)+\beta_{\Re}(\omega) \rho_{\Re}(\omega) \leq 1
$$

for all $\omega \in X$. The LDFS

$$
\Re_{X}=\{(\omega,\langle 1,0\rangle,\langle 1,0\rangle): \omega \in X\}
$$

is known as the "absolute LDFS" in X. The LDFS

$$
\Re_{\phi}=\{(\omega,\langle 0,1\rangle,\langle 0,1\rangle): \omega \in X\}
$$

is known as the "null LDFS" in X.
Specific components can be modeled or classified using the RPs. We can categorise distinct systems by changing the physical significance of the RPs. In addition, $\eta_{\Re}(\omega) \pi_{\Re}(\omega)=$ $1-\left(\eta_{\Re}(\omega) \xi_{\Re}(\omega)+\beta_{\Re}(\omega) \rho_{\Re}(\omega)\right)$ is called the "indeterminacy degree" and its corresponding reference parameter of $\omega$ to $\Re$.

It is clear that our proposed approach is more acceptable and superior to others, and it incorporates a diverse set of reference variables. This technique may be used for a wide range of technological, medical, intelligent systems, and MADM applications.

Definition 2 ([10]). A "linear Diophantine fuzzy number (LDFN)" is a tuple $\alpha^{\gamma}=\left(\left\langle\xi_{\alpha \gamma}, \rho_{\alpha \gamma}\right\rangle,\left\langle\eta_{\alpha \gamma}, \beta_{\alpha^{\gamma}}\right\rangle\right)$ satisfying the following conditions:
(1) $0 \leq \xi_{\alpha^{\gamma}}, \rho_{\alpha^{\gamma}}, \eta_{\alpha^{\gamma}}, \beta_{\alpha^{\gamma}} \leq 1$;
(2) $0 \leq \eta_{\alpha \gamma}+\beta_{\alpha \gamma} \leq 1$;
(3) $0 \leq \eta_{\alpha \gamma} \xi_{\alpha \gamma}+\beta_{\alpha \gamma} \rho_{\alpha \gamma} \leq 1$.

Definition 3 ([10]). Let $\alpha^{\gamma}=\left(\left\langle\xi_{\alpha^{\gamma}}, \rho_{\alpha^{\gamma}}\right\rangle,\left\langle\eta_{\alpha^{\gamma}}, \beta_{\alpha \gamma}\right\rangle\right)$ be a LDFN; then, the "score function" $\breve{\mathrm{Y}}\left(\alpha^{\gamma}\right)$ can be defined by the mapping $\left.\breve{\mathrm{Y}} \alpha^{\gamma}\right): \operatorname{LDFN}(X) \rightarrow[-1,1]$ and given by

$$
\breve{Y}\left(\alpha^{\gamma}\right)=\frac{1}{2}\left[\left(\xi_{\alpha \gamma}-\rho_{\alpha \gamma}\right)+\left(\eta_{\alpha \gamma}-\beta_{\alpha \gamma}\right)\right]
$$

where $\operatorname{LDFN}(X)$ is a collection of LDFNs on $X$.
Definition 4 ([10]). Let $\alpha^{\gamma}=\left(\left\langle\xi_{\alpha \gamma}, \rho_{\alpha \gamma}\right\rangle,\left\langle\eta_{\alpha \gamma}, \beta_{\alpha \gamma}\right\rangle\right)$ be a LDFN; then, the "accuracy function" can be defined by the mapping $\Theta: \operatorname{LDFN}(X) \rightarrow[0,1]$ and given as

$$
\Theta\left(\alpha^{\gamma}\right)=\frac{1}{2}\left[\left(\frac{\xi_{\alpha \gamma}+\rho_{\alpha \gamma}}{2}\right)+\left(\eta_{\alpha \gamma}+\beta_{\alpha \gamma}\right)\right]
$$

Definition 5 ([10]). Let $\alpha^{\gamma}{ }_{1}$ and $\alpha^{\gamma}{ }_{2}$ be two LDFNs; then, by using the score function and accuracy function, we have:
(i) If $\breve{Y}\left(\alpha^{\gamma}{ }_{1}\right)<\breve{Y}\left(\alpha^{\gamma}{ }_{2}\right)$ then $\alpha^{\gamma}{ }_{1}<\alpha^{\gamma}{ }_{2}$,
(ii) If $\left.\breve{Y}\left(\alpha^{\gamma}{ }_{2}\right)<\breve{Y} \alpha^{\gamma}{ }_{1}\right)$ then $\alpha^{\gamma}{ }_{2}<\alpha^{\gamma}{ }_{1}$,
(ii) If $\left.\breve{Y}\left(\alpha^{\gamma}{ }_{2}\right)=\breve{Y} \alpha^{\gamma}{ }_{1}\right)$ then,
(a) If $\Theta\left(\alpha^{\gamma}{ }_{1}\right)<\Theta\left(\alpha^{\gamma}{ }_{2}\right)$ then $\alpha^{\gamma}{ }_{1}<\alpha^{\gamma}{ }_{2}$,
(b) If $\Theta\left(\alpha^{\gamma}{ }_{2}\right)<\Theta\left(\alpha^{\gamma}{ }_{1}\right)$ then $\alpha^{\gamma}{ }_{2}<\alpha^{\gamma}{ }_{1}$,
(c) If $\Theta\left(\alpha^{\gamma}{ }_{1}\right)=\Theta\left(\alpha^{\gamma}{ }_{2}\right)$ then $\alpha^{\gamma}{ }_{1}=\alpha^{\gamma}{ }_{2}$.

Definition 6 ([10]). Let $\alpha^{\gamma}{ }_{1}=\left(\left\langle\xi_{1}, \rho_{1}\right\rangle,\left\langle\eta_{1}, \beta_{1}\right\rangle\right)$ be a LDFN and $\mathfrak{X}>0$. Then

- $\alpha_{1}^{\gamma}=\left(\left\langle\rho_{1}, \xi_{1}\right\rangle,\left\langle\beta_{1}, \eta_{1}\right\rangle\right)$;
- $\mathfrak{X} \alpha^{\gamma}{ }_{1}=\left(\left\langle 1-\left(1-\xi_{1}\right)^{\mathfrak{X}}, \rho_{1}^{\mathfrak{X}}\right\rangle,\left\langle 1-\left(1-\eta_{1}\right)^{\mathfrak{X}}, \beta_{1}^{\mathfrak{X}}\right\rangle\right)$;
- $\alpha_{1}^{\mathfrak{X}}=\left(\left\langle\xi_{1}^{\mathfrak{X}}, 1-\left(1-\rho_{1}\right)^{\mathfrak{X}}\right\rangle,\left\langle\eta_{1}^{\mathfrak{X}}, 1-\left(1-\beta_{1}\right)^{\mathfrak{X}}\right\rangle\right)$.

Definition 7 ([10]). Let $\alpha^{\gamma}=\left(\left\langle\xi_{i}, \rho_{i}\right\rangle,\left\langle\eta_{i}, \beta_{i}\right\rangle\right)$ be two LDFNs with $i=1,2$. Then

- $\alpha^{\gamma}{ }_{1} \subseteq \alpha^{\gamma} \Leftrightarrow \xi_{1} \leq \xi_{2}, \rho_{2} \leq \rho_{1}, \eta_{1} \leq \eta_{2}, \beta_{2} \leq \beta_{1}$;
- $\alpha^{\gamma}{ }_{1}=\alpha^{\gamma}{ }_{2} \Leftrightarrow \xi_{1}=\xi_{2}, \rho_{1}=\rho_{2}, \eta_{1}=\eta_{2}, \beta_{1}=\beta_{2}$;
- $\alpha^{\gamma}{ }_{1} \oplus \alpha^{\gamma}{ }_{2}=\left(\left\langle\xi_{1}+\xi_{2}-\xi_{1} \xi_{2}, \rho_{1} \rho_{2}\right\rangle,\left\langle\eta_{1}+\eta_{2}-\eta_{1} \eta_{2}, \beta_{1} \beta_{2}\right\rangle\right)$;
- $\alpha^{\gamma}{ }_{1} \otimes \alpha^{\gamma}{ }_{2}=\left(\left\langle\xi_{1} \xi_{2}, \rho_{1}+\rho_{2}-\rho_{1} \rho_{2}\right\rangle,\left\langle\eta_{1} \eta_{2}, \beta_{1}+\beta_{2}-\beta_{1} \beta_{2}\right\rangle\right)$.

Definition 8 ([10]). Let $\alpha^{\gamma}{ }_{i}=\left(\left\langle\xi_{i}, \rho_{i}\right\rangle,\left\langle\eta_{i}, \beta_{i}\right\rangle\right)$ be the assemblage of $L D F N s$ with $i \in \Delta$. Then

- $\bigcup_{i \in \Delta} \alpha^{\gamma}{ }_{i}=\left(\left\langle\sup _{i \in \Delta} \xi_{i}, \inf _{i \in \Delta} \rho_{i}\right\rangle,\left\langle\sup _{i \in \Delta} \eta_{i}, \inf _{i \in \Delta} \beta_{i}\right\rangle\right)$;
- $\bigcap_{i \in \Delta} \alpha^{\gamma}{ }_{i}=\left(\left\langle\inf _{i \in \Delta} \xi_{i}, \sup _{i \in \Delta} \rho_{i}\right\rangle,\left\langle\inf _{i \in \Delta} \eta_{i}, \sup _{i \in \Delta} \beta_{i}\right\rangle\right)$.

Example 1. Consider $\alpha^{\gamma}=(\langle 0.810,0.470\rangle,\langle 0.520,0.390\rangle)$ and
$\alpha^{\gamma}{ }_{2}=(\langle 0.910,0.360\rangle,\langle 0.640,0.270\rangle)$ be two LDFNs. Then, it is clear that $\alpha^{\gamma}{ }_{1} \subseteq \alpha^{\gamma}{ }_{2}$. One can verify that

- $\alpha^{\gamma}{ }_{1}^{c}=(\langle 0.470,0.810\rangle,\langle 0.390,0.520\rangle)$;
- $\alpha^{\gamma}{ }_{1} \cup \alpha^{\gamma}{ }_{2}=(\langle 0.910,0.360\rangle,\langle 0.640,0.270\rangle)=\alpha^{\gamma}$;
- $\alpha^{\gamma}{ }_{1} \cap \alpha^{\gamma}{ }_{2}=(\langle 0.810,0.470\rangle,\langle 0.520,0.390\rangle)=\alpha^{\gamma}{ }_{1} ;$
- $\alpha^{\gamma}{ }_{1} \oplus \alpha^{\gamma}{ }_{2}=(\langle 0.9829,0.1692\rangle,\langle 0.8272,0.1053\rangle)$;
- $\alpha^{\gamma}{ }_{1} \otimes \alpha^{\gamma}{ }_{2}=(\langle 0.7371,0.6608\rangle,\langle 0.3328,0.5547\rangle)$.

In addition, if $\mathfrak{X}=0.1$, then we have

- $\mathfrak{X} \alpha{ }^{\gamma}{ }_{1}=(\langle 0.1530,0.9272\rangle,\langle 0.0707,0.9101\rangle)$;
- $\alpha_{1}^{\gamma \mathfrak{X}}=(\langle 0.9791,0.0615\rangle,\langle 0.9366,0.0482\rangle)$.

Definition 9. Let $\alpha^{\gamma}{ }_{1}=\left(\left\langle\xi_{1}, \rho_{1}\right\rangle,\left\langle\eta_{1}, \beta_{1}\right\rangle\right)$ and $\alpha^{\gamma}=\left(\left\langle\xi_{2}, \rho_{2}\right\rangle,\left\langle\eta_{2}, \beta_{2}\right\rangle\right)$ be two LDFNs and $\beth, \beth_{1}, \beth_{2}>0$ be the real numbers, then we have,

1. $\quad \alpha^{\gamma}{ }_{1} \oplus \alpha^{\gamma}{ }_{2}=\alpha^{\gamma}{ }_{2} \oplus \alpha^{\gamma}{ }_{1}$;
2. $\alpha^{\gamma}{ }_{1} \otimes \alpha^{\gamma}{ }_{2}=\alpha^{\gamma}{ }_{2} \otimes \alpha^{\gamma}{ }_{1}$;
3. $\left.\quad \beth\left(\alpha^{\gamma}{ }_{1} \oplus \alpha^{\gamma}{ }_{2}\right)=\left(\beth \alpha^{\gamma}{ }_{1}\right) \oplus\left(\beth \alpha^{\gamma}\right)_{2}\right)$;
4. $\left(\alpha^{\gamma}{ }_{1} \otimes \alpha^{\gamma}\right)^{\beth}=\alpha^{\gamma} \frac{\beth}{1} \otimes \alpha^{\gamma} \frac{\beth}{2}$;
5. $\quad\left(\beth_{1}+\beth_{2}\right) \alpha^{\gamma}{ }_{1}=\left(\beth_{1} \alpha^{\gamma}{ }_{1}\right) \oplus\left(\beth_{2} \alpha^{\gamma}{ }_{2}\right)$;
6. $\quad \alpha{ }_{1}^{\gamma} \beth_{1}+\beth_{2}=\alpha{ }_{1}^{\gamma} \otimes \alpha \gamma \beth_{2}^{2}$.

If $\xi_{\alpha \gamma_{1}}=\rho_{\alpha \gamma_{1}}$ and $\xi_{\alpha \gamma_{2}}=\rho_{\alpha \gamma_{2}}$ then we obtain $\quad \xi_{\alpha \gamma_{1} \oplus \alpha \gamma_{2}} \neq \rho_{\alpha \gamma_{1} \oplus \alpha \gamma_{2}}, \xi_{\alpha \gamma_{1} \otimes \alpha \gamma_{2}} \neq \rho_{\alpha \gamma_{1} \otimes \alpha \gamma_{2}}$, $\xi_{\beth \alpha \gamma_{1}} \neq \rho_{\beth \alpha \gamma_{1}}, \xi_{\alpha \gamma \beth} \neq \rho_{\alpha \gamma_{1}^{\beth}}$. Thus, none of the operations $\alpha^{\gamma_{1}} \oplus \alpha^{\gamma}{ }_{2}, \alpha^{\gamma}{ }_{1} \otimes \alpha^{\gamma}{ }_{2}, \beth \alpha{ }_{1}, \alpha^{\gamma}{ }_{1}$ are located to be impartial or honest, indeed. So, at the very start, our attention has to be in the direction of expanding a few fairly operations among IFNs.

## 4. Fairly Operations on LDFNs

In this section, we develop some fairly operations between LDFNs and study their primary properties.

Definition 10. Consider $\alpha^{\gamma}{ }_{1}=\left(\left\langle\xi_{\alpha \gamma_{1}}, \rho_{\alpha \gamma_{1}}\right\rangle,\left\langle\eta_{\alpha \gamma_{1}}, \beta_{\alpha \gamma_{1}}\right\rangle\right)$ and $\alpha^{\gamma_{2}}=\left(\left\langle\xi_{\alpha \gamma_{2}}, \rho_{\alpha \gamma_{2}}\right\rangle,\left\langle\eta_{\alpha \gamma_{2}}, \beta_{\alpha \gamma_{2}}\right\rangle\right)$ are the two LDFNs and $\beth>0$. Then, we define.

$$
\alpha \gamma_{1} \tilde{\oplus} \alpha \gamma_{2}=\left(\begin{array}{l}
\left\langle\frac{\xi_{\alpha}\left(\frac{\xi_{\alpha} \gamma_{1} \xi_{\alpha \gamma_{2}}}{2}\right.}{\xi_{\alpha} \gamma_{1} \xi_{\alpha} \gamma_{2}+\rho_{\alpha} \gamma_{1} \rho_{\alpha} \gamma_{2}}\right) \times\left(1+\left(2-\xi_{\alpha \gamma_{1}}-\rho_{\alpha \gamma_{1}}\right)\left(2-\xi_{\alpha \gamma_{2}}-\rho_{\alpha \gamma_{2}}\right)\right), \\
\left.\frac{1}{2}\left(\frac{\rho_{\alpha} \gamma_{1} \rho_{\alpha} \gamma_{2}}{\xi_{\alpha} \gamma_{1} \xi_{\alpha} \gamma_{2}+\rho_{\alpha} \gamma_{1} \rho_{\alpha} \gamma_{2}}\right) \times\left(1+\left(2-\xi_{\alpha \gamma_{1}}-\rho_{\alpha \gamma_{1}}\right)\left(2-\xi_{\alpha \gamma_{2}}-\rho_{\alpha \gamma_{2}}\right)\right)\right\rangle, \\
\left\langle\left(\frac{\eta_{\alpha} \gamma_{1} \eta_{\alpha} \gamma_{2}}{\eta_{\alpha} \gamma_{1} \eta_{\alpha} \gamma_{2}+\beta_{\alpha} \gamma_{1} \beta_{\alpha} \gamma_{2}}\right) \times\left(1-\left(1-\eta_{\alpha \gamma_{1}}-\beta_{\alpha \gamma_{1}}\right)\left(1-\eta_{\alpha \gamma_{2}}-\beta_{\alpha \gamma_{2}}\right)\right),\right. \\
\left.\left(\frac{\beta_{\alpha} \gamma_{1} \beta_{\alpha} \gamma_{2}}{\eta_{\alpha} \gamma_{1} \eta_{\alpha} \gamma_{2}+\beta_{\alpha} \gamma_{1} \beta_{\alpha} \gamma_{2}}\right) \times\left(1-\left(1-\eta_{\alpha \gamma_{1}}-\beta_{\alpha \gamma_{1}}\right)\left(1-\eta_{\alpha \gamma_{2}}-\beta_{\alpha \gamma_{2}}\right)\right)\right\rangle
\end{array}\right)
$$

$$
\begin{aligned}
& \beth * \alpha^{\beth} \gamma_{1} \\
& =\left(\begin{array}{l}
\left\langle\frac{1}{2}\left(\frac{\tilde{\zeta}_{\alpha}^{\beth} \gamma_{1}}{\zeta_{\alpha}^{\beth} \gamma_{1}+\rho_{\alpha \gamma_{1}}^{\beth}}\right) \times\left(1+\left(2-\tilde{\zeta}_{\alpha \gamma_{1}}-\rho_{\alpha \gamma_{1}}\right)^{\beth}\right), \frac{1}{2}\left(\frac{\rho_{\alpha \gamma_{1}}^{\beth}}{\zeta_{\alpha}^{\beth} \gamma_{1}+\rho_{\alpha \gamma_{1}}^{\beth}}\right) \times\left(1+\left(2-\xi_{\alpha \gamma_{1}}-\rho_{\alpha \gamma_{1}}\right)^{\beth}\right)\right\rangle, \\
\\
\\
\left\langle\left(\frac{\eta_{\alpha \gamma_{1}}^{\beth}}{\eta_{\alpha \gamma_{1}}^{\beth}+\beta_{\alpha \gamma_{1}}^{\beth}}\right) \times\left(1-\left(1-\eta_{\alpha \gamma_{1}}-\beta_{\alpha \gamma_{1}}\right)^{\beth}\right),\left(\frac{\beta_{\alpha}^{\beth} \gamma_{1}}{\eta_{\alpha \gamma_{1}}^{\beth}+\beta_{\alpha}^{\beth}}\right) \times\left(1-\left(1-\eta_{\alpha \gamma_{1}}-\beta_{\alpha \gamma_{1}}\right)^{\beth}\right)\right\rangle
\end{array}\right)
\end{aligned}
$$

It can be easily verified that $\alpha^{\gamma}{ }_{1} \widetilde{\oplus} \alpha^{\gamma}{ }_{2}, \beth * \alpha^{\gamma}{ }_{1}$ are the LDFNs.
Theorem 1. Consider ${ }^{\gamma}{ }_{1}=\left(\left\langle\xi_{\alpha \gamma_{1}}, \rho_{\alpha \gamma_{1}}\right\rangle,\left\langle\eta_{\alpha \gamma_{1}}, \beta_{\alpha \gamma_{1}}\right\rangle\right)$ and $\alpha^{\gamma}{ }_{2}=\left(\left\langle\xi_{\alpha \gamma_{2}}, \rho_{\alpha \gamma_{2}}\right\rangle,\left\langle\eta_{\alpha} \gamma_{2}, \beta_{\alpha \gamma_{2}}\right\rangle\right)$ are the two LDFNs. If $\xi_{\alpha \gamma_{1}}=\rho_{\alpha \gamma_{1}}, \xi_{\alpha \gamma_{2}}=\rho_{\alpha \gamma_{2}}, \eta_{\alpha \gamma_{1}}=\beta_{\alpha \gamma_{1}}$ and $\eta_{\alpha \gamma_{2}}=\beta_{\alpha \gamma_{2}}$ then we have
(i) $\xi_{\alpha \gamma_{1} \tilde{\oplus} \gamma_{2}}=\rho_{\alpha \gamma_{1} \tilde{\oplus} \gamma_{2}}$, and $\eta_{\alpha \gamma_{1} \tilde{\oplus} \gamma_{2}}=\beta_{\alpha \gamma_{1} \tilde{\oplus} \gamma_{2}}$,
(ii) $\xi_{\beth * \alpha \gamma_{1}}=\rho_{\beth * \alpha \gamma_{1}}$, and $\eta_{\beth * \alpha \gamma_{1}}=\beta_{\beth * \alpha \gamma_{1}}$

Proof. (i) As given $\xi_{\alpha \gamma_{1}}=\rho_{\alpha \gamma_{1}}$ and $\xi_{\alpha \gamma_{2}}=\rho_{\alpha \gamma_{2}}$

$$
\begin{aligned}
\frac{\xi_{\alpha \gamma_{1} \tilde{\oplus} \gamma_{2}}}{\rho_{\alpha \gamma_{1} \tilde{\oplus} \gamma_{2}}} & =\frac{\frac{1}{2}\left(\frac{\xi_{\alpha} \gamma_{1} \xi_{\alpha} \gamma_{2}}{\xi_{\alpha} \xi_{1} \xi_{\alpha} \gamma_{2}+\rho_{\alpha} \gamma_{1} \rho_{\alpha \gamma_{2}}}\right) \times\left(1+\left(2-\xi_{\alpha \gamma_{1}}-\rho_{\alpha \gamma_{1}}\right)\left(2-\xi_{\alpha \gamma_{2}}-\rho_{\alpha \gamma_{2}}\right)\right)}{\frac{1}{2}\left(\frac{\rho_{\alpha \gamma_{1}} \rho_{\alpha \gamma_{2}}}{\xi_{\alpha} \gamma_{1} \xi_{\alpha} \gamma_{2}+\rho_{\alpha} \gamma_{1} \rho_{\alpha \gamma_{2}}}\right) \times\left(1+\left(2-\xi_{\alpha \gamma_{1}}-\rho_{\alpha \gamma_{1}}\right)\left(2-\xi_{\alpha \gamma_{2}}-\rho_{\alpha \gamma_{2}}\right)\right)} \\
& =1
\end{aligned}
$$

Consequently, $\xi_{\alpha \gamma_{1} \tilde{\oplus} \gamma_{2}}=\rho_{\alpha \gamma_{1} \tilde{\oplus} \gamma_{2}}$. If $\xi_{\alpha \gamma_{1}}=\rho_{\alpha \gamma_{1}}$ and $\xi_{\alpha \gamma_{2}}=\rho_{\alpha \gamma_{2}}$.
As given, $\eta_{\alpha \gamma_{1}}=\beta_{\alpha \gamma_{1}}$ and $\eta_{\alpha \gamma_{2}}=\beta_{\alpha \gamma_{2}}$

$$
\begin{aligned}
\frac{\eta_{\alpha \gamma_{1} \tilde{\oplus} \alpha \gamma_{2}}}{\beta_{\alpha \gamma_{1} \tilde{\oplus} \alpha \gamma_{2}}} & =\frac{\left(\frac{\eta_{\alpha} \gamma_{1} \eta_{\alpha} \gamma_{2}}{\eta_{\alpha} \gamma_{1} \eta_{\alpha} \gamma_{2}+\beta_{\alpha} \gamma_{1} \beta_{\alpha} \gamma_{2}}\right) \times\left(1-\left(1-\eta_{\alpha \gamma_{1}}-\beta_{\alpha \gamma_{1}}\right)\left(1-\eta_{\alpha \gamma_{2}}-\beta_{\alpha \gamma_{2}}\right)\right)}{\left(\frac{\beta_{\alpha} \gamma_{1} \beta_{\alpha} \gamma_{2}}{\eta_{\alpha}}\right) \times\left(1-\left(1-\eta_{\alpha \gamma_{1}}-\beta_{\alpha \gamma_{1}}\right)\left(1-\eta_{\alpha \gamma_{2}}-\beta_{\alpha \gamma_{2}}\right)\right)} \\
& =1
\end{aligned}
$$

Consequently, $\eta_{\alpha \gamma_{1} \tilde{\oplus} \alpha \gamma_{2}}=\beta_{\alpha \gamma_{1} \tilde{\oplus} \alpha \gamma_{2}}$. If $\eta_{\alpha \gamma_{1}}=\beta_{\alpha \gamma_{1}}$ and $\eta_{\alpha \gamma_{2}}=\beta_{\alpha \gamma_{2}}$.
(ii) As given $\xi_{\alpha \gamma_{1}}=\rho_{\alpha \gamma_{1}}$ and $\xi_{\alpha \gamma_{2}}=\rho_{\alpha \gamma_{2}}$

$$
\begin{aligned}
\frac{\xi_{\beth * \alpha \gamma_{1}}}{\rho_{\beth * \alpha \gamma_{1}}} & =\frac{\frac{1}{2}\left(\frac{\xi_{\alpha \gamma_{1}}^{\beth}}{\xi_{\alpha}^{\beth}+\rho_{1}}\right) \times\left(1+\left(2-\xi_{\alpha \gamma_{1}}-\rho_{\alpha \gamma_{1}}\right)^{\beth}\right)}{\frac{1}{2}\left(\frac{\rho_{\alpha \gamma_{1}}^{\beth}}{\xi_{\alpha}^{\beth} \gamma_{1}+\rho_{\alpha \gamma_{1}}^{\beth}}\right) \times\left(1+\left(2-\xi_{\alpha \gamma_{1}}-\rho_{\alpha \gamma_{1}}\right)^{\beth}\right)} \\
& =1
\end{aligned}
$$

Consequently, $\xi_{\beth * \alpha \gamma_{1}}=\rho_{\beth_{* \alpha \gamma_{1}}}$. If $\xi_{\alpha \gamma_{1}}=\rho_{\alpha \gamma_{1}}$ and $\xi_{\alpha \gamma_{2}}=\rho_{\alpha \gamma_{2}}$.
As given, $\eta_{\alpha \gamma_{1}}=\beta_{\alpha \gamma_{1}}$ and $\eta_{\alpha \gamma_{2}}=\beta_{\alpha \gamma_{2}}$

$$
\begin{aligned}
\frac{\eta_{\beth * \alpha \gamma_{1}}}{\beta_{\beth * \alpha \gamma_{1}}} & =\frac{\left(\frac{\eta_{\alpha \gamma_{1}}^{\beth}}{\eta_{\alpha \gamma_{1}}^{\beth}+\beta_{\alpha \gamma_{1}}^{\beth}}\right) \times\left(1-\left(1-\eta_{\alpha \gamma_{1}}-\beta_{\alpha \gamma_{1}}\right)^{\beth}\right)}{\left(\frac{\beta_{\alpha \gamma_{1}}^{\beth}}{\eta_{\alpha \gamma_{1}}^{\beth}+\beta_{\alpha \gamma_{1}}^{\beth}}\right) \times\left(1-\left(1-\eta_{\alpha \gamma_{1}}-\beta_{\alpha \gamma_{1}}\right)^{\beth}\right)} \\
& =1
\end{aligned}
$$

Consequently, $\eta_{\beth * \alpha \gamma_{1}}=\beta_{\beth * \alpha \gamma_{1}}$. If $\eta_{\alpha \gamma_{1}}=\beta_{\alpha \gamma_{1}}$ and $\eta_{\alpha \gamma_{2}}=\beta_{\alpha \gamma_{2}}$.
The above theorem shows that the operations $\alpha^{\gamma}{ }_{1} \tilde{\oplus} \alpha^{\gamma}{ }_{2}, \beth * \alpha^{\gamma}$ show the fairly or neutral nature to the DMs, when MSG, NMSG and RPs are equal initially. This is why we call the operations $\tilde{\otimes},{ }^{*}$ fairly operations.

Theorem 2. Consider ${ }^{\gamma} \gamma_{1}=\left(\left\langle\xi_{\alpha \gamma_{1}}, \rho_{\alpha \gamma_{1}}\right\rangle,\left\langle\eta_{\alpha \gamma_{1}}, \beta_{\alpha \gamma_{1}}\right\rangle\right)$ and $\alpha^{\gamma_{2}}=\left(\left\langle\xi_{\alpha \gamma_{2}}, \rho_{\alpha \gamma_{2}}\right\rangle,\left\langle\eta_{\alpha \gamma_{2}}, \beta_{\alpha \gamma_{2}}\right\rangle\right)$ are the LDFNs and $\beth, \beth_{1}$ and $\beth_{2}$ are any three real numbers, then we have
(i) $\alpha^{\gamma}{ }_{1} \tilde{\oplus} \alpha^{\gamma}{ }_{2}=\alpha^{\gamma}{ }_{2} \tilde{\oplus} \alpha^{\gamma}{ }_{1}$
(ii) $\beth *\left(\alpha^{\gamma}{ }_{1} \tilde{\oplus} \alpha^{\gamma}{ }_{2}\right)=\left(\beth * \alpha^{\gamma}{ }_{1}\right) \tilde{\oplus}\left(\beth * \alpha^{\gamma}{ }_{2}\right)$
(iii) $\left(\beth_{1}+\beth_{2}\right) * \alpha^{\gamma}{ }_{1}=\left(\beth_{1} * \alpha^{\gamma}{ }_{1}\right) \tilde{\oplus}\left(\beth_{2} * \alpha^{\gamma}{ }_{1}\right)$

Proof. (i) This one is trivial.
(ii) $\beth *\left(\alpha^{\gamma}{ }_{1} \tilde{\oplus} \alpha^{\gamma}{ }_{2}\right)$

$$
\begin{aligned}
& \left.\left(1+\left(1+\left(2-\tilde{\xi}_{\alpha \gamma_{2}}-\rho_{a \gamma_{2}}\right)^{2}\right)\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
& \left.\left.\left(1+\left(1+\left(2-\tilde{\xi}_{\alpha \gamma_{2}}-\rho_{\alpha \gamma_{2}}\right)^{2}\right)\right)\right)\right\rangle,
\end{aligned}
$$

$$
\begin{aligned}
& \left.\left.\left(1-\left(1-\left(1-\eta_{\alpha \gamma_{2}}-\beta_{\alpha \gamma_{2}}\right)^{I}\right)\right)\right)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
& \text { Hence, } \beth *\left(\alpha^{\gamma}{ }_{1} \tilde{\oplus} \alpha^{\gamma}{ }_{2}\right)=\left(\beth * \alpha^{\gamma}{ }_{1}\right) \tilde{\oplus}\left(\beth * \alpha^{\gamma}{ }_{2}\right) \text {. } \\
& \text { (iii) }\left(\beth_{1} * \alpha^{\gamma}{ }_{1}\right) \tilde{\oplus}\left(\beth_{2} * \alpha^{\gamma}{ }_{1}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \left(\beth_{1}+\beth_{2}\right) * \alpha^{\gamma}{ }_{1}
\end{aligned}
$$

$$
\begin{aligned}
& \text { Hence, }\left(\beth_{1}+\beth_{2}\right) * \alpha^{\gamma}{ }_{1}=\left(\beth_{1} * \alpha^{\gamma}{ }_{1}\right) \tilde{\oplus}\left(\beth_{2} * \alpha^{\gamma}{ }_{1}\right)
\end{aligned}
$$

## 5. Fairly Aggregation Operators for LDFNs

This section discusses fairly AOs for LDFNs and their characteristics.

### 5.1. LDFFWA Operator

Definition 11. Let $\alpha^{\gamma}{ }_{h}=\left(\left\langle\tilde{\zeta}_{h}, \rho_{h}\right\rangle,\left\langle\eta_{h}, \beta_{h}\right\rangle\right)$ be the accumulation of LDFNs, and LDFFWA: $\mathscr{F}^{n} \rightarrow \mathscr{F}$, be a $n$ dimension mapping. If

$$
\begin{equation*}
\operatorname{LDFFWA}\left(\alpha^{\gamma}{ }_{1}, \alpha^{\gamma}{ }_{2}, \ldots \alpha^{\gamma}{ }_{e}\right)=\left(\vartheta^{\gamma}{ }_{1} * \alpha^{\gamma}{ }_{1} \tilde{\oplus}^{\gamma}{ }_{2} * \alpha^{\gamma}{ }_{2} \tilde{\oplus} \ldots, \tilde{\oplus} \vartheta^{\gamma}{ }_{e} * \alpha^{\gamma}{ }_{e}\right) \tag{1}
\end{equation*}
$$

then the mapping LDFFWA is called "linear Diophantine fuzzy fairly weighted averaging (LDFFWA) operator", here, $\vartheta^{\gamma}{ }_{i}$ is the weight vector $(W V)$ of $\alpha \gamma_{i}$ with $\vartheta^{\gamma}>0$ and $\sum_{i=1}^{e} \vartheta \gamma_{i}=1$.

Moreover, as demonstrated in the theorem similarly below, we can consider LDFFWA using fairly operational laws.

Theorem 3. Let $\alpha^{\gamma}{ }_{h}=\left(\left\langle\xi_{h}, \rho_{h}\right\rangle,\left\langle\eta_{h}, \beta_{h}\right\rangle\right)$ be the accumulation of LDFNs; we can also find LDFFWA by
$\operatorname{LDFFWA}\left(\alpha^{\gamma}{ }_{1}, \alpha^{\gamma}{ }_{2}, \ldots, \alpha^{\gamma}{ }_{e}\right)$

$$
=\left(\begin{array}{c}
\left\langle\frac{1}{2} \frac{\prod_{i=1}^{e}\left(\xi_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\xi_{i}\right)^{\vartheta \gamma_{i}}+\prod_{i=1}^{e}\left(\rho_{i}\right)^{\vartheta \gamma_{i}}} \times\left(1+\prod_{i=1}^{e}\left(2-\xi_{i}-\rho_{i}\right)^{\vartheta \gamma_{i}}\right),\right. \\
\left.\frac{1}{2} \frac{\prod_{i=1}^{e}\left(\rho_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\xi_{i}\right)^{\vartheta \gamma}{ }_{i}+\prod_{i=1}^{e}\left(\rho_{i}\right)^{\vartheta \gamma_{i}}} \times\left(1+\prod_{i=1}^{e}\left(2-\xi_{i}-\rho_{i}\right)^{\vartheta \gamma_{i}}\right)\right\rangle, \\
\left\langle\frac{\prod_{i=1}^{e}\left(\eta_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\eta_{i}\right)^{\vartheta \gamma_{i}}+\prod_{i=1}^{e}\left(\beta_{i}\right)^{\vartheta \gamma_{i}}} \times\left(1-\prod_{i=1}^{e}\left(1-\eta_{i}-\beta_{i}\right)^{\vartheta \gamma_{i}}\right),\right. \\
\left.\frac{\prod_{i=1}^{e}\left(\beta_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\eta_{i}\right)^{\vartheta \gamma_{i}}+\prod_{i=1}^{e}\left(\beta_{i}\right)^{\vartheta \gamma_{i}}} \times\left(1-\prod_{i=1}^{e}\left(1-\eta_{i}-\beta_{i}\right)^{\vartheta \gamma_{i}}\right)\right\rangle
\end{array}\right)
$$

where $\vartheta^{\gamma_{i}}$ is the WV of $\alpha^{\gamma_{i}}$ with $\vartheta^{\gamma_{i}}>0$ and $\sum_{i=1}^{e} \vartheta^{\gamma}{ }_{i}=1$.

Proof. This proof will begin with mathematical induction.
For $e=1$, we have $\alpha^{\gamma}{ }_{1}=\left\langle\xi_{1}, \rho_{1}\right\rangle$ and $\vartheta^{\gamma}=1$.
$\operatorname{LDFFWA}\left(\alpha^{\gamma}\right)=\vartheta^{\gamma}{ }_{1} * \alpha^{\gamma_{1}}$

$$
=\binom{\left\langle\frac{1}{2} \frac{\left(\xi_{1}\right)^{\vartheta \gamma_{1}}}{\left(\xi_{1}\right)^{\vartheta \gamma} 1+\left(\rho_{1}\right)_{1}^{\theta \gamma}} \times\left(1+\left(2-\xi_{1}-\rho_{1}\right)^{\vartheta \gamma_{1}}\right), \frac{1}{2} \frac{\left(\rho_{1}\right)^{\vartheta \gamma_{1}}}{\left(\xi_{1}\right)^{\vartheta \gamma} 1+\left(\rho_{1}\right)^{\theta \gamma_{1}}} \times\left(1+\left(2-\xi_{1}-\rho_{1}\right)^{\vartheta \gamma_{1}}\right)\right\rangle}{\left.\frac{\left(\eta_{1}\right)^{\vartheta \gamma_{1}}}{\left(\eta_{1}\right)^{\vartheta \gamma_{1}}+\left(\beta_{1}\right)_{1}^{\theta \gamma}} \times\left(1-\left(1-\eta_{1}-\beta_{1}\right)^{\vartheta \gamma_{1}}\right), \frac{\left(\beta_{1}\right)^{\vartheta \gamma_{1}}}{\left(\eta_{1}\right)^{\vartheta \gamma_{1}}+\left(\beta_{1}\right)^{\vartheta \gamma_{1}}} \times\left(1-\left(1-\eta_{1}-\beta_{1}\right)^{\vartheta \gamma_{1}}\right)\right\rangle}
$$

Even as theorem holds true for $e=1$, we now anticipate it to hold proper for $e=g$, i.e.,

We will prove for $e=g+1$.

$$
\operatorname{LDFFWA}\left(\alpha^{\gamma}{ }_{1}, \alpha^{\gamma}{ }_{2}, \ldots, \alpha^{\gamma}{ }_{g+1}\right)=\operatorname{LDFFWA}\left(\alpha^{\gamma}{ }_{1}, \alpha^{\gamma}{ }_{2}, \ldots, \alpha^{\gamma}{ }_{g}\right) \tilde{\oplus}\left(\theta^{\gamma}{ }_{g+1} * \alpha^{\gamma}{ }_{g+1}\right)
$$

$$
\begin{aligned}
& =\left(\begin{array}{c}
\left\langle\frac{1}{2} \frac{\prod_{i=1}^{g}\left(\xi_{i} \xi^{\theta \gamma} i_{i}\right.}{\prod_{i=1}^{g}\left(\xi_{i}\right)^{\theta_{i}}+\prod_{i=1}^{g}\left(\rho_{i}\right)^{\theta \gamma_{i}}} \times\left(1+\prod_{i=1}^{g}\left(2-\xi_{i}-\rho_{i}\right)^{\theta \gamma_{i}}\right),\right. \\
\left.\frac{1}{2} \frac{\prod_{i=1}^{g}\left(\rho_{i}\right)^{\theta \gamma_{i}}}{\prod_{i=1}^{g}\left(\xi_{i}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{g}\left(\rho_{i}\right)^{\theta \gamma_{i}}} \times\left(1+\prod_{i=1}^{g}\left(2-\xi_{i}-\rho_{i}\right)^{\theta \gamma_{i}}\right)\right\rangle, \\
\left\langle\frac{\prod_{i=1}^{g}\left(\eta_{i}\right)^{\theta \gamma_{i}}}{\prod_{i=1}^{g}\left(\eta_{i}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{g}\left(\beta_{i}\right)^{\theta \gamma_{i}}} \times\left(1-\prod_{i=1}^{g}\left(1-\eta_{i}-\beta_{i}\right)^{\theta \gamma_{i}}\right),\right. \\
\left.\frac{\prod_{i=1}^{g}\left(\beta_{i}\right)^{\theta \gamma_{i}}}{\prod_{i=1}^{g}\left(\eta_{i}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{g}\left(\beta_{i}\right)^{\theta \gamma_{i}}} \times\left(1-\prod_{i=1}^{g}\left(1-\eta_{i}-\beta_{i}\right)^{\theta \gamma_{i}}\right)\right\rangle
\end{array}\right) \tilde{\oplus}
\end{aligned}
$$

$$
\begin{aligned}
& \left.\frac{1}{2}\left(\frac{\rho_{\alpha \gamma} \gamma_{g+1}}{\xi_{\alpha} \gamma_{\alpha} \gamma_{g+1}+\rho_{\alpha \alpha} \gamma_{g+1}}\right) \times\left(1+\left(2-\tilde{\xi}_{\alpha} \gamma_{g+1}-\rho_{\alpha} \gamma_{g+1}\right)^{\theta \gamma_{g+1}}\right)\right\rangle,
\end{aligned}
$$

$$
\begin{aligned}
& \left.\left.\left(\frac{\beta_{\alpha}^{\theta \gamma} \gamma_{g+1}}{\eta_{\alpha \alpha}^{\theta \gamma} \gamma_{g+1}+\beta_{\alpha+1} \gamma_{g+1} \gamma_{g+1}}\right) \times\left(1-\left(1-\eta_{\alpha} \gamma_{g+1}-\beta_{\alpha \gamma_{g+1}}\right)^{\theta \gamma_{g+1}}\right)\right\rangle\right)
\end{aligned}
$$

Consequently, the statement holds genuine for $e=g+1$ as well. Therefore, the belief is authentic for every $e$ using the principle of induction.

Example 2. Assume $\alpha^{\gamma}{ }_{1}=(\langle 0.86,0.76\rangle,\langle 0.45,0.25\rangle), \alpha^{\gamma}{ }_{2}=(\langle 0.67,0.98\rangle,\langle 0.53,0.29\rangle)$ and $\alpha^{\gamma}=(\langle 0.58,0.68\rangle,\langle 0.65,0.15\rangle)$ are three LDFNs with $W V \vartheta^{\gamma}=(0.30,0.45,0.25)$, then

$$
\begin{aligned}
\frac{1}{2} \frac{\prod_{i=1}^{3}\left(\xi_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{3}\left(\xi_{i}\right)^{\vartheta \gamma_{i}}+\prod_{i=1}^{3}\left(\rho_{i}\right)^{\theta \gamma_{i}}} \times\left(1+\prod_{i=1}^{3}\left(2-\xi_{i}-\rho_{i}\right)^{\theta \gamma_{i}}\right) & =0.327101 \\
\frac{1}{2} \frac{\prod_{i=1}^{3}\left(\rho_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{3}\left(\xi_{i}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{3}\left(\rho_{i}\right)^{\theta \gamma_{i}}} \times\left(1+\prod_{i=1}^{3}\left(2-\xi_{i}-\rho_{i}\right)^{\theta \gamma_{i}}\right) & =0.389193 \\
\frac{\prod_{i=1}^{3}\left(\eta_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{3}\left(\eta_{i}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{3}\left(\beta_{i}\right)^{\theta \gamma_{i}}} \times\left(1-\prod_{i=1}^{3}\left(1-\eta_{i}-\beta_{i}\right)^{\vartheta \gamma_{i}}\right) & =0.543735 \\
\frac{\prod_{i=1}^{3}\left(\beta_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{3}\left(\eta_{i}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{3}\left(\beta_{i}\right)^{\theta \gamma_{i}}} \times\left(1-\prod_{i=1}^{3}\left(1-\eta_{i}-\beta_{i}\right)^{\theta \gamma_{i}}\right) & =0.240855
\end{aligned}
$$

$\operatorname{LDFFWA}\left(\alpha^{\gamma}{ }_{1,} \alpha^{\gamma}{ }_{2}, \alpha^{\gamma}{ }_{3}\right)$

$$
\begin{aligned}
& =\left(\begin{array}{l}
\left\langle\frac{1}{2} \frac{\prod_{i=1}^{3}\left(\xi_{i}\right)^{\theta \gamma_{i}}}{\prod_{i=1}^{3}\left(\xi_{i}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{3}\left(\rho_{i}\right)^{\theta \gamma_{i}}} \times\left(1+\prod_{i=1}^{3}\left(2-\xi_{i}-\rho_{i}\right)^{\vartheta \gamma_{i}}\right),\right. \\
\left.\frac{1}{2} \frac{\prod_{i=1}^{3}\left(\rho_{i}\right)^{\theta \gamma} i_{i}}{\prod_{i=1}^{3}\left(\xi_{i}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{3}\left(\rho_{i}\right)^{\theta \gamma_{i}}} \times\left(1+\prod_{i=1}^{3}\left(2-\xi_{i}-\rho_{i}\right)^{\theta \gamma_{i}}\right)\right\rangle, \\
\left\langle\frac{\prod_{i=1}^{3}\left(\eta_{i}\right)^{\theta \gamma_{i}}}{\prod_{i=1}^{3}\left(\eta_{i}\right)^{\theta \gamma} i_{i}+\prod_{i=1}^{3}\left(\beta_{i}\right)^{\theta \gamma_{i}}} \times\left(1-\prod_{i=1}^{3}\left(1-\eta_{i}-\beta_{i}\right)^{\vartheta \gamma_{i}}\right),\right. \\
\left.\frac{\prod_{i=1}^{3}\left(\beta_{i}\right)^{\theta \gamma_{i}}}{\prod_{i=1}^{3}\left(\eta_{i}\right)^{\theta \gamma} i+\prod_{i=1}^{3}\left(\beta_{i}\right)^{\theta \gamma_{i}}} \times\left(1-\prod_{i=1}^{3}\left(1-\eta_{i}-\beta_{i}\right)^{\vartheta \gamma_{i}}\right)\right\rangle
\end{array}\right) \\
& =(\langle 0.327101,0.389193\rangle,\langle 0.543735,0.240855\rangle)
\end{aligned}
$$

The proposed AO meets a number of special prerequisites, which are described in the following theorems.

Theorem 4. Let $\alpha^{\gamma}{ }_{i}=\left(\left\langle\xi_{i}, \rho_{i}\right\rangle,\left\langle\eta_{i}, \beta_{i}\right\rangle\right\rangle$ be the accumulation of LDFNs and $\alpha^{\gamma}{ }_{\diamond}=\left(\left\langle\xi_{\diamond}, \rho_{\diamond}\right\rangle\right.$, $\left.\left\langle\eta_{\diamond}, \beta_{\diamond}\right\rangle\right)$ be the LDFN such that, $\alpha^{\gamma}{ }_{i}=\alpha^{\gamma}{ }_{\diamond}$. Then

$$
\begin{equation*}
\operatorname{LDFFWA}\left(\alpha^{\gamma}{ }_{1}, \alpha^{\gamma}{ }_{2}, \ldots, \alpha^{\gamma}{ }_{e}\right)=\alpha^{\gamma} \tag{2}
\end{equation*}
$$

Proof. Given that $\alpha^{\gamma}{ }_{i}=\alpha^{\gamma}{ }_{\diamond} \forall i$, by this, $\xi_{i}=\xi_{\diamond,} \rho_{i}=\rho_{\diamond,} \eta_{i}=\eta_{\diamond}$ and $\beta_{i}=\beta_{\diamond} \forall i$
$\operatorname{LDFFWA}\left(\alpha^{\gamma}{ }_{1}, \alpha^{\gamma}{ }_{2}, \ldots, \alpha^{\gamma}{ }_{e}\right)$

$$
\begin{aligned}
& =\left(\begin{array}{c}
\left\langle\frac{1}{2} \frac{\prod_{i=1}^{e}\left(\xi_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\xi_{i}\right)^{\vartheta \gamma_{i}}+\prod_{i=1}^{e}\left(\rho_{i}\right)^{\theta \gamma_{i}}} \times\left(1+\prod_{i=1}^{e}\left(2-\xi_{i}-\rho_{i}\right)^{\vartheta \gamma_{i}}\right),\right. \\
\left.\frac{1}{2} \frac{\prod_{i=1}^{e}\left(\rho_{i}\right)^{\theta \gamma_{i}}}{\prod_{i=1}^{e}\left(\xi_{i}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{e}\left(\rho_{i}\right)^{\theta \gamma_{i}}} \times\left(1+\prod_{i=1}^{e}\left(2-\xi_{i}-\rho_{i}\right)^{\vartheta \gamma_{i}}\right)\right\rangle, \\
\left\langle\frac{\prod_{i=1}^{e}\left(\eta_{i}\right)^{\theta \gamma_{i}}}{\prod_{i=1}^{e}\left(\eta_{i}\right)^{\theta \gamma} \eta_{i}+\prod_{i=1}^{e}\left(\beta_{i}\right)^{\theta \gamma_{i}}} \times\left(1-\prod_{i=1}^{e}\left(1-\eta_{i}-\beta_{i}\right)^{\vartheta \gamma_{i}}\right),\right. \\
\left.\frac{\prod_{i=1}^{e}\left(\beta_{i}\right)^{\theta \gamma_{i}}}{\prod_{i=1}^{e}\left(\eta_{i}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{e}\left(\beta_{i}\right)^{\theta \gamma_{i}}} \times\left(1-\prod_{i=1}^{e}\left(1-\eta_{i}-\beta_{i}\right)^{\vartheta \gamma_{i}}\right)\right\rangle
\end{array}\right) \\
& =\left(\begin{array}{c}
\left\langle\frac{1}{2} \frac{\prod_{i=1}^{e}\left(\xi_{\diamond}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\xi_{\diamond}\right)^{\vartheta \gamma}{ }_{i}+\prod_{i=1}^{e}\left(\rho_{\diamond}\right)^{\theta \gamma_{i}}} \times\left(1+\prod_{i=1}^{e}\left(2-\xi_{\diamond}-\rho_{\diamond}\right)^{\vartheta \gamma_{i}}\right),\right. \\
\left.\frac{1}{2} \frac{\prod_{i=1}^{e}\left(\rho_{\diamond}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\xi_{\diamond}\right)^{\theta \gamma} \gamma_{i}+\prod_{i=1}^{e}\left(\rho_{\diamond}\right)^{\vartheta \gamma_{i}}} \times\left(1+\prod_{i=1}^{e}\left(2-\xi_{\diamond}-\rho_{\diamond}\right)^{\vartheta \gamma_{i}}\right)\right\rangle, \\
\left\langle\frac{\prod_{i=1}^{e}\left(\eta_{\diamond}\right)^{\vartheta \gamma}{ }_{i}}{\prod_{i=1}^{e}\left(\eta_{\diamond}\right)^{\vartheta \gamma_{i}}+\prod_{i=1}^{e}\left(\beta_{\diamond}\right)^{\theta \gamma} \gamma_{i}} \times\left(1-\prod_{i=1}^{e}\left(1-\eta_{\diamond}-\beta_{\diamond}\right)^{\vartheta \gamma_{i}}\right),\right.
\end{array}\right. \\
& \left.\left(\frac{\prod_{i=1}^{e}\left(\beta_{\diamond}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\eta_{\diamond}\right)^{\theta \gamma} \gamma_{i}+\prod_{i=1}^{e}\left(\beta_{\diamond}\right)^{\theta \gamma_{i}}} \times\left(1-\prod_{i=1}^{e}\left(1-\eta_{\diamond}-\beta_{\diamond}\right)^{\vartheta \gamma_{i}}\right)\right\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
& \left\langle\frac{\left(\beta_{\diamond}\right)^{\Sigma_{i=1}^{e}}{ }^{\vartheta \gamma_{i}}}{\left(\eta_{\diamond}\right)^{\sum_{i=1}^{e}{ }^{\vartheta \gamma_{i}}}+\left(\beta_{\diamond}\right)^{\Sigma_{i=1}^{e}}{ }^{\vartheta \gamma_{i}}} \times\left(1-\left(1-\eta_{\diamond}-\beta_{\diamond}\right)^{\sum_{i=1}^{e} \vartheta \gamma_{i}}\right)\right\rangle \\
& =\left(\left\langle\xi_{\diamond}, \rho_{\diamond}\right\rangle,\left\langle\eta_{\diamond}, \beta_{\diamond}\right\rangle\right)=\alpha_{\diamond}^{\gamma}
\end{aligned}
$$

Theorem 5. Let $\alpha^{\gamma}{ }_{i}=\left(\left\langle\xi_{i}, \rho_{i}\right\rangle,\left\langle\eta_{i}, \beta_{i}\right\rangle\right)$ be the accumulation of LDFNs. Then, for $\operatorname{LDFFWA}\left(\alpha^{\gamma}{ }_{1}, \alpha^{\gamma}{ }_{2}, \ldots, \alpha^{\gamma}{ }_{e}\right)=\left(\left\langle\xi_{x}, \rho_{x}\right\rangle,\left\langle\eta_{x}, \beta_{x}\right\rangle\right)$, we have

$$
\min _{i}\left\{\xi_{i}+\rho_{i}\right\}+\min _{i}\left\{\eta_{i}+\beta_{i}\right\} \leq \xi_{x}+\rho_{x}+\eta_{x}+\beta_{x} \leq \max _{i}\left\{\xi_{i}+\rho_{i}\right\}+\max _{i}\left\{\eta_{i}+\beta_{i}\right\}
$$

Proof. Consider the following to start:

$$
\begin{aligned}
\min _{i}\left\{\eta_{i}+\beta_{i}\right\} & =1-\left(1-\min _{i}\left\{\eta_{i}+\beta_{i}\right\}\right) \\
& =1-\left(1-\min _{i}\left\{\eta_{i}+\beta_{i}\right\}\right)^{\sum_{i=1}^{e} \theta \gamma_{i}} \\
& =1-\prod_{i=1}^{e}\left(1-\min _{i}\left\{\eta_{i}+\beta_{i}\right\}\right)^{\vartheta \gamma_{i}} \\
& \leq 1-\prod_{i=1}^{e}\left(1-\left\{\eta_{i}+\beta_{i}\right\}\right)^{\theta \gamma_{i}} \\
& \leq 1-\prod_{i=1}^{e}\left(1-\max _{i}\left\{\eta_{i}+\beta_{i}\right\}\right)^{\vartheta \gamma_{i}} \\
& =1-\left(1-\max _{i}\left\{\eta_{i}+\beta_{i}\right\}\right)^{\sum_{i=1}^{e} \theta \gamma_{i}} \\
& =\max _{i}\left\{\eta_{i}+\beta_{i}\right\}
\end{aligned}
$$

By Theorem 2, we obtain

$$
\begin{aligned}
& \eta_{x}=\frac{\prod_{i=1}^{e}\left(\eta_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\eta_{i}\right)^{\vartheta \gamma_{i}}+\prod_{i=1}^{e}\left(\beta_{i}\right)^{\vartheta \gamma_{i}}} \times\left(1-\prod_{i=1}^{e}\left(1-\eta_{i}-\beta_{i}\right)^{\vartheta \gamma_{i}}\right) \\
& \beta_{x}=\frac{\prod_{i=1}^{e}\left(\beta_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\eta_{i}\right)^{\vartheta \gamma_{i}}+\prod_{i=1}^{e}\left(\beta_{i}\right)^{\vartheta \gamma_{i}}} \times\left(1-\prod_{i=1}^{e}\left(1-\eta_{i}-\beta_{i}\right)^{\vartheta \gamma_{i}}\right)
\end{aligned}
$$

From this, we obtain

$$
\eta_{x}+\beta_{x}=\left(1-\prod_{i=1}^{e}\left(1-\eta_{i}-\beta_{i}\right)^{\vartheta \gamma_{i}}\right)
$$

Consequently,

$$
\begin{equation*}
\min _{i}\left\{\eta_{i}+\beta_{i}\right\} \leq \eta_{x}+\beta_{x} \leq \max _{i}\left\{\eta_{i}+\beta_{i}\right\} . \tag{3}
\end{equation*}
$$

On the same way we also obtain

$$
\begin{equation*}
\min _{i}\left\{\xi_{i}+\rho_{i}\right\} \leq \xi_{x}+\rho_{x} \leq \max _{i}\left\{\xi_{i}+\rho_{i}\right\} \tag{4}
\end{equation*}
$$

By Equations (3) and (4), we obtain the desire result as,

$$
\min _{i}\left\{\xi_{i}+\rho_{i}\right\}+\min _{i}\left\{\eta_{i}+\beta_{i}\right\} \leq \xi_{x}+\rho_{x}+\eta_{x}+\beta_{x} \leq \max _{i}\left\{\xi_{i}+\rho_{i}\right\}+\max _{i}\left\{\eta_{i}+\beta_{i}\right\} .
$$

Theorem 6. Assume that $\alpha^{\gamma}{ }_{i}=\left(\left\langle\xi_{i}, \rho_{i}\right\rangle,\left\langle\eta_{i}, \beta_{i}\right\rangle\right)$ and $\alpha^{\gamma}{ }_{i^{*}}=\left(\left\langle\xi_{i^{*}}, \rho_{i^{*}}\right\rangle,\left\langle\eta_{i^{*}}, \beta_{i^{*}}\right\rangle\right)$ are the families of LDFNs, and also consider

$$
\operatorname{LDFFWA}\left(\alpha^{\gamma}{ }_{1}, \alpha^{\gamma}{ }_{2}, \ldots \alpha^{\gamma}{ }_{e}\right)=\alpha^{\gamma}=(\langle\xi, \rho\rangle,\langle\eta, \beta\rangle)
$$

and

$$
\operatorname{LDFFWA}\left(\alpha^{\gamma} 1_{1^{*}}, \alpha^{\gamma} 2_{2^{*}}, \ldots \alpha^{\gamma}{e^{*}}\right)=\alpha^{\gamma}=\left(\left\langle\zeta^{*}, \rho_{*}\right\rangle,\left\langle\eta_{*}, \beta_{*}\right\rangle\right)
$$

Then,

$$
\xi+\rho \leq \xi^{*}+\rho_{*}, \quad \text { if } \xi_{i}+\rho_{i} \leq \xi_{i^{*}}+\rho_{i^{*}}
$$

and

$$
\eta+\beta \leq \eta^{*}+\beta_{*}^{*}, \text { if } \eta_{i}+\beta_{i} \leq \eta_{i^{*}}+\beta_{i^{*}}
$$

Proof. If Theorem 2 is applied to the both assemblage of LDFNs, $\alpha^{\gamma}{ }_{i}=\left(\left\langle\xi_{i}, \rho_{i}\right\rangle,\left\langle\eta_{i}, \beta_{i}\right\rangle\right)$ and $\alpha^{\gamma}{ }_{i^{*}}=\left(\left\langle\zeta_{i^{*}}, \rho_{i^{*}}\right\rangle,\left\langle\eta_{i^{*}}, \beta_{i^{*}}\right\rangle\right)$, we obtain

$$
\begin{aligned}
& \xi=\frac{1}{2} \frac{\prod_{i=1}^{e}\left(\xi_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\xi_{i}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{e}\left(\rho_{i}\right)^{\vartheta \gamma_{i}}} \times\left(1+\prod_{i=1}^{e}\left(2-\xi_{i}-\rho_{i}\right)^{\vartheta \gamma_{i}}\right) \\
& \rho=\frac{1}{2} \frac{\prod_{i=1}^{e}\left(\rho_{i}\right)^{\theta \gamma_{i}}}{\prod_{i=1}^{e}\left(\xi_{i}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{e}\left(\rho_{i}\right)^{\theta \gamma_{i}}} \times\left(1+\prod_{i=1}^{e}\left(2-\xi_{i}-\rho_{i}\right)^{\vartheta \gamma_{i}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \xi^{*}=\frac{1}{2} \frac{\prod_{i=1}^{e}\left(\xi_{i^{*}}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\xi_{i^{*}}\right)^{\vartheta \gamma_{i}}+\prod_{i=1}^{e}\left(\rho_{i^{*}}\right)^{\vartheta \gamma_{i}}} \times\left(1+\prod_{i=1}^{e}\left(2-\xi_{i^{*}}-\rho_{i^{*}}\right)^{\vartheta \gamma_{i}}\right) \\
& \rho_{*}=\frac{1}{2} \frac{\prod_{i=1}^{e}\left(\rho_{i^{*}}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\xi_{i^{*}}\right)^{\vartheta \gamma_{i}}+\prod_{i=1}^{e}\left(\rho_{i^{*}}\right)^{\theta \gamma_{i}}} \times\left(1+\prod_{i=1}^{e}\left(2-\xi_{i^{*}}-\rho_{i^{*}}\right)^{\vartheta \gamma_{i}}\right)
\end{aligned}
$$

By this, if $\xi_{i}+\rho_{i} \leq \xi_{i^{*}}+\rho_{i^{*}}$ then we have,

$$
\xi+\rho=1-\prod_{i=1}^{e}\left(1-\left\{\xi_{i}+\rho_{i}\right\}\right)^{\theta \gamma_{i}} \leq 1-\prod_{i=1}^{e}\left(1-\left\{\xi_{i^{*}}+\rho_{i^{*}}\right\}\right)^{\theta \gamma_{i}} \leq \xi^{*}+\rho^{*}
$$

Again,

$$
\begin{aligned}
& \eta=\frac{\prod_{i=1}^{e}\left(\eta_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\eta_{i}\right)^{\vartheta \gamma_{i}}+\prod_{i=1}^{e}\left(\beta_{i}\right)^{\theta \gamma_{i}}} \times\left(1-\prod_{i=1}^{e}\left(1-\eta_{i}-\beta_{i}\right)^{\vartheta \gamma_{i}}\right) \\
& \beta=\frac{\prod_{i=1}^{e}\left(\beta_{i}\right)^{\vartheta \gamma_{i}}}{\prod_{i=1}^{e}\left(\eta_{i}\right)^{\vartheta \gamma_{i}}+\prod_{i=1}^{e}\left(\beta_{i}\right)^{\vartheta \gamma_{i}} \times\left(1-\prod_{i=1}^{e}\left(1-\eta_{i}-\beta_{i}\right)^{\vartheta \gamma_{i}}\right)}
\end{aligned}
$$

and

$$
\begin{aligned}
& \eta_{*}=\frac{\prod_{i=1}^{e}\left(\eta_{i^{*}}\right)^{\theta \gamma_{i}}}{\prod_{i=1}^{e}\left(\eta_{i^{*}}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{e}\left(\beta_{i^{*}}\right)^{\theta \gamma_{i}}} \times\left(1-\prod_{i=1}^{e}\left(1-\eta_{i^{*}}-\beta_{i^{*}}\right)^{\theta \gamma_{i}}\right) \\
& \beta^{*}=\frac{\prod_{i=1}^{e}\left(\beta_{i^{*}}\right)^{\theta \gamma_{i}}}{\prod_{i=1}^{e}\left(\eta_{i^{*}}\right)^{\theta \gamma_{i}}+\prod_{i=1}^{e}\left(\beta_{i^{*}}\right)^{\theta \gamma_{i}}} \times\left(1-\prod_{i=1}^{e}\left(1-\eta_{i^{*}}-\beta_{i^{*}}\right)^{\theta \gamma_{i}}\right)
\end{aligned}
$$

By this, if $\eta_{i}+\beta_{i} \leq \eta_{i^{*}}+\beta_{i^{*}}$ then we have,

$$
\eta+\beta=1-\prod_{i=1}^{e}\left(1-\left\{\eta_{i}+\beta_{i}\right\}\right)^{\theta \gamma_{i}} \leq 1-\prod_{i=1}^{e}\left(1-\left\{\eta_{i^{*}}+\beta_{i^{*}}\right\}\right)^{\theta \gamma_{i}} \leq \eta_{*}+\beta_{*}
$$

5.2. LDFFOWA Operator

Definition 12. Let $\alpha^{\gamma}{ }_{h}=\left(\left\langle\xi_{h}, \rho_{h}\right\rangle,\left\langle\eta_{h}, \beta_{h}\right\rangle\right)$ be the accumulation of LDFNs, and LDFFOWA: $\mathscr{F}^{n} \rightarrow \mathscr{F}$, be a $n$ dimension mapping. If

$$
\begin{equation*}
\operatorname{LDFFOWA}\left(\alpha^{\gamma}{ }_{1}, \alpha^{\gamma}{ }_{2}, \ldots \alpha^{\gamma}{ }_{e}\right)=\left(\vartheta^{\gamma}{ }_{1} * \alpha^{\gamma} \tau_{(1)} \tilde{\oplus} \tilde{\theta}^{\gamma} \gamma_{2} * \alpha^{\gamma} \tau_{(2)} \tilde{\oplus} \ldots, \tilde{\oplus} \vartheta^{\gamma}{ }_{e} * \alpha^{\gamma} \tau_{(e)}\right) \tag{5}
\end{equation*}
$$

then the mapping LDFFOWA is called "linear Diophantine fuzzy fairly ordered weighted averaging (LDFFOWA) operator", here, $\vartheta^{\gamma}{ }_{i}$ is the WV of $\alpha^{\gamma}{ }_{i}$ with $\vartheta^{\gamma}{ }_{i}>0$ and $\sum_{i=1}^{e} \vartheta^{\gamma}{ }_{i}=1$.
$\xi: 1,2,3, \ldots \ldots, n \rightarrow 1,2,3, \ldots \ldots, n$ is a permutation map s.t. $\alpha^{\gamma} \tau_{(i-1)} \geq \alpha^{\gamma} \tau_{(i)}$.
Additionally, as demonstrated in the theorem further below, we can consider LDFFOWA using fairly operational laws.

Theorem 7. Let $\alpha^{\gamma}{ }_{h}=\left(\left\langle\xi_{h}, \rho_{h}\right\rangle,\left\langle\eta_{h}, \beta_{h}\right\rangle\right)$ be the accumulation of LDFNs; we can also find LDFFOWA by

$$
\operatorname{LDFFOWA}\left(\alpha^{\gamma}{ }_{1}, \alpha^{\gamma}{ }_{2}, \ldots, \alpha^{\gamma}{ }_{e}\right)
$$

where $\vartheta^{i}$ is the $W V$ of $\alpha \gamma_{i}$ with $\vartheta^{\gamma}>0$ and $\sum_{i=1}^{e} \vartheta^{\gamma}{ }_{i}=1$.
Proof. The proof is identical with Theorem 2.
Theorem 8. Let $\alpha^{\gamma}{ }_{i}=\left(\left\langle\xi_{i}, \rho_{i}\right\rangle,\left\langle\eta_{i}, \beta_{i}\right\rangle\right)$ be the accumulation of LDFNs and $\alpha^{\gamma}{ }_{\diamond}=\left(\left\langle\xi_{\diamond}, \rho_{\diamond}\right\rangle\right.$, $\left.\left\langle\eta_{\diamond}, \beta_{\diamond}\right\rangle\right)$ be the LDFNs such that, $\alpha^{\gamma}{ }_{i}=\alpha^{\gamma}{ }_{\diamond i}$. Then

$$
\begin{equation*}
\operatorname{LDFFOWA}\left(\alpha^{\gamma}{ }_{1}, \alpha^{\gamma}{ }_{2}, \ldots, \alpha^{\gamma} e\right)=\alpha^{\gamma}{ }_{\diamond} \tag{6}
\end{equation*}
$$

Proof. The proof is identical with Theorem 5.
Theorem 9. Let $\alpha^{\gamma}{ }_{i}=\left(\left\langle\xi_{i}, \rho_{i}\right\rangle,\left\langle\eta_{i}, \beta_{i}\right\rangle\right)$ be the accumulation of LDFNs.
Then, for LDFFOWA $\left(\alpha^{\gamma}{ }_{1}, \alpha^{\gamma}{ }_{2}, \ldots, \alpha^{\gamma} e\right)=\left(\left\langle\xi_{x}, \rho_{x}\right\rangle,\left\langle\eta_{x}, \beta_{x}\right\rangle\right)$, we have
$\min _{\tau_{(i)}}\left\{\xi_{\tau_{(i)}}+\rho_{\tau_{(i)}}\right\}+\min _{\tau_{(i)}}\left\{\eta_{\tau_{(i)}}+\beta_{\tau_{(i)}}\right\} \leq \xi_{x}+\rho_{x}+\eta_{x}+\beta_{x} \leq \max _{\tau_{(i)}}\left\{\xi_{\tau_{(i)}}+\rho_{\tau_{(i)}}\right\}+\max _{\tau_{(i)}}\left\{\eta_{\tau_{(i)}}+\beta_{\tau_{(i)}}\right\}$
Proof. The proof is identical with Theorem 6.
Theorem 10. Assume that $\alpha^{\gamma}{ }_{i}=\left(\left\langle\xi_{i}, \rho_{i}\right\rangle,\left\langle\eta_{i}, \beta_{i}\right\rangle\right)$ and $\alpha^{\gamma}{ }_{i^{*}}=\left(\left\langle\xi_{i^{*}}, \rho_{i^{*}}\right\rangle,\left\langle\eta_{i^{*}}, \beta_{i^{*}}\right\rangle\right)$ are the families of LDFNs, and also consider

$$
\operatorname{LDFFOWA}\left(\alpha^{\gamma}{ }_{1}, \alpha^{\gamma}{ }_{2}, \ldots \alpha_{e}^{\gamma}\right)=\alpha^{\gamma}=(\langle\xi, \rho\rangle,\langle\eta, \beta\rangle)
$$

and

$$
\operatorname{LDFFOWA}\left(\alpha^{\gamma}{ }_{1^{*}}, \alpha^{\gamma}{ }_{2^{*}}, \ldots \alpha^{\gamma}{ }_{e^{*}}\right)=\alpha^{\gamma}=\left(\left\langle\zeta^{*}, \rho_{*}\right\rangle,\left\langle\eta_{*}, \beta_{*}\right\rangle\right) .
$$

Then,

$$
\xi+\rho \leq \xi^{*}+\rho_{*}^{*}, \quad \text { if } \xi_{\tau_{(i)}}+\rho_{\tau_{(i)}} \leq \xi_{\tau_{(i)}^{*}}+\rho_{\tau_{(i)}^{*}}
$$

and

$$
\eta+\beta \leq \eta^{*}+\beta_{*}, \quad \text { if } \quad \eta_{\tau_{(i)}}+\beta_{\tau_{(i)}} \leq \eta_{\tau_{(i)}^{*}}+\beta_{\tau_{(i)}^{*}}
$$

Proof. The proof is identical with Theorem 7.

## 6. Decision-Making Algorithm Using Proposed AOs

In this MCDM methodology, we will look at each of the $n$ possible solutions in terms of $m$ variety of attributes and rank them appropriately. It is critical to provide a team of $p$ extremely competent specialists, each of whose scores must be more than zero in this case but whose overall score must be one.
You should remember that the option $\mathscr{T}^{\psi}{ }_{j}(j=1,2, \ldots, n)$ was picked by the experts $\mathscr{O}^{\mathcal{G}}{ }_{k}(k=1,2, \ldots, p)$ after some discussion, and the parameters $\mathscr{C}_{i}^{\top}(i=1,2, \ldots, m)$ were also chosen; hence, the evaluation result is presented in terms of LDFNs, $\delta \gamma_{j i}^{p}=$ $\left(\left\langle\xi_{j i}^{p} \rho_{j i}^{p}\right\rangle,\left\langle\eta_{j i}^{p}, \beta_{j i}^{p}\right\rangle\right)$. Moreover, $\mathfrak{W}_{t}$ is the WV for the attribute $\mathscr{C}_{i}^{\top}$ satisfying the conditions, $\mathfrak{W}_{t} \geq 0$ and $\sum_{t=1}^{m} \mathfrak{W}_{t}=1$. The endorsed operator is applied with a view to construct an MCDM for the LDF facts, which includes the levels outlined within the following Algorithm 1.

```
Algorithm 1: MCDM approach based on propsoed AOs
    Step 1:
    Utilising "linguistic terms" (LTs), compute the weights of DMs represented as
```

        LDFNs. The LTs are shown in Table 2. Consider \(\Im_{k}=\left(\left\langle\xi_{k}, \rho_{k}\right\rangle,\left\langle\eta_{k}, \beta_{k}\right\rangle\right)\) as the LDFN for \(k\)-th DM. Consequently, the potential value of the \(k\)-th DM, \(\zeta_{k}\), can be calculated as follows:
    $$
\begin{equation*}
\zeta_{k}=\frac{\Im_{k}}{\sum_{k=1}^{p} \Im_{k}}, k=1,2,3, \ldots, p \tag{7}
\end{equation*}
$$

where $\Im_{k}=\xi_{k}+\left(1+\xi_{k}+\rho_{k}+\eta_{k}+\beta_{k}\right)\left(\frac{\rho_{k}}{\xi_{k}+\rho_{k}+\eta_{k}+\beta_{k}}\right)$ and clearly $\sum_{k=1}^{p} \zeta_{k}=1$
Step 2:
Construct the decision matrices $\mathscr{E}^{\mathcal{G}}{ }_{(p)}=\left(\mathfrak{Y}_{j i}^{(p)}\right)_{n \times m}$ utilising the LDFNs obtaining from the DMs.
Step 3:
Set up an included IF judgement matrix. It is crucial to observe that even as producing the aggregated IF choice matrix utilising a collective judgement method, all separate perspectives need to be summed and blanketed to establish a collective perspective. The proposed AO will help the accompanying to this quit:
Assume $H=\left(H_{j i}\right)_{n \times m}$ is the integrated LDF decision matrix, where

$$
H_{j i}=\operatorname{LDFFWA}\left(\mathfrak{Y}_{j i}^{(1)}, \mathfrak{Y}_{j i}^{(2)}, \ldots, \mathfrak{Y}_{j i}^{(p)}\right)
$$

or

$$
H_{j i}=\operatorname{LDFFOWA}\left(\mathfrak{Y}_{j i}^{(1)}, \mathfrak{Y}_{j i}^{(2)}, \ldots, \mathfrak{Y}_{j i}^{(p)}\right)
$$

For convenience, we take $H_{j i}$ as $H_{j i}=\left(\left\langle\xi_{j i}, \rho_{j i}\right\rangle,\left\langle\eta_{j i}, \beta_{j i}\right\rangle\right)$

## Algorithm 1: Cont. <br> Step 4:

If it is necessary to do so, normalise the LDFNs by applying the following formula to turn the cost category features denoted by $\left(\kappa_{c}\right)$ into benefit category features denoted by $\left(\kappa_{b}\right)$,

$$
\left(\aleph_{j i}^{N}\right)_{n \times m}= \begin{cases}\left(H_{j i}\right)^{c} ; & i \in \kappa_{c}  \tag{8}\\ H_{j i} ; & i \in \kappa_{b} .\end{cases}
$$

where $\left(H_{j i}\right)^{c}$ displays the complement of $\left(H_{j i}\right)$. The normalised decision matrix is given as $\Gamma_{N}=\left(\aleph_{j i}^{N}\right)_{n \times m}$.

## Step 5:

Construct the score matrix by factoring in the score function (SF) of said LDFNs as $\Psi=\left(\breve{\zeta}^{\beth}\left(\aleph_{j i}^{N}\right)\right)_{n \times m}$

Step 6:
In accordance with the idea of a scoring matrix denoted by $\Psi$, the total number of choices and the weighted sum of their respective scores are calculated by $\mathscr{T}^{\psi}{ }_{j}$.

$$
¥\left(\mathscr{T}^{\psi}{ }_{j}\right)=\sum_{i=1}^{m} \mathfrak{W}_{i}^{\zeta} \breve{\zeta}^{\rightrightarrows}\left(\aleph_{j i}^{N}\right), \quad(j=1,2, \ldots, n) .
$$

where $\mathfrak{W}_{1}^{\zeta}, \mathfrak{W}_{2}^{\zeta}, \ldots \mathfrak{W}_{m}^{\zeta}$ are the WV of the given criterion.
Assume that the weights are not specified and that $\overbrace{\square}$ is used to symbolise a selection of them. In order to compute these indeterminate weights, we make use of the mathematical formulation that is provided here:

$$
\operatorname{Max} g=\sum_{i=1}^{m} ¥\left(\mathscr{T}^{\psi}{ }_{j}\right)
$$

under the constraints $\sum_{i=1}^{m} \mathfrak{W}_{i}^{\zeta}=1$. When using this approach, we should probably determine our normalised WV. In this example, the constraints of the situation require that we employ a linear programming paradigm as a good way to compute the weights of the diverse criteria.

## Step 7:

Determine the consolidated weighted LDF decision matrix $\mathfrak{W}^{\zeta}$ by making use of a normalised decision matrix $\Gamma_{N}$ and the WV. We utilised the AOs that are given down below.

$$
\operatorname{LDFFWA}\left(\aleph_{j 1}^{N}, \aleph_{j 2}^{N}, \ldots, \aleph_{j m}^{N}\right)
$$

or

$$
\operatorname{LDFFOWA}\left(\aleph_{j 1}^{N}, \aleph_{j 2}^{N}, \ldots, \aleph_{j m}^{N}\right)
$$

## Step 8:

Employing SF, determine the total score of the overall weighted consolidated product. Evaluate each alternative depending on the SF, and then choose the alternate with the greatest SF.

Table 2. LTs for DMs.

| Linguistic Terms | LDFNs |
| :--- | :--- |
| Very pertinent | $(\langle 0.950,0.100\rangle,\langle 0.95,0.05\rangle)$ |
| Pertinent | $(\langle 0.750,0.150\rangle,\langle 0.75,0.15\rangle)$ |
| Medium pertinent | $(\langle 0.650,0.200\rangle,\langle 0.65,0.20\rangle)$ |
| Un-pertinent | $(\langle 0.300,0.500\rangle,\langle 0.30,0.50\rangle)$ |
| Very un-pertinent | $(\langle 0.100,0.700\rangle,\langle 0.10,0.70\rangle)$ |

## 7. Applications of the Proposed Framework

The lifespan and the existence of people can each be improved with the use of biomaterials. It has become abundantly clear in recent years that the use of bio-materials has become more desirable as a result of the ageing population that occurs in nearly all of the sector's countries. The elderly are at greater risk of difficult tissue insufficiency, which makes them the number one driving force behind this demand. Both the biological and mechanical bio-compatibility of metal bio-substances have significant room for improvement. It is highly desired that the bio-material used in implants has a longer shelf life or that they feature normally in the course of an individual's complete lifestyle without requiring replacement or additional surgical techniques. Bio-materials are required to fulfil a number of stipulations, which include excessive levels of corrosion and wear resistance, advanced bio-compatibility, and appropriate mechanical compatibility.

In its most simple form, a biomaterial is any substance that is evolved mainly to be used in a healthcare context. Bio-materials are materials, whether or not either naturally or artificial, which are useful for the restoration of injured components of the body through contact with dwelling mechanisms [55]. These substances are applied to both function as a replacement for a human frame element or to help the physiological tactics of the body. As an end result, bio-materials are able to interact with human cells, muscle mass, or structures, and they occasionally perform the responsibilities that such entities could generally perform [56]. The practical restorative engineering of various tissues may be accomplished by using bio-materials. Further to this, there is a concept known as "nano biomaterial", which is a combination of the phrases "biomaterial" and "nanotechnology". The capacity to exist in touch with the cells of the dwelling organism without producing an unacceptable degree of harm to the tissues is the most massive feature that separates a biomaterial from some other cloth. This capability is what separates a biomaterial from some other material [57]. Many years in the past, both scientists who examine bio-materials and people who use clinical equipment have been inquisitive about the system of the way a cooperatively appropriate cohabitation of bio-materials and cells can be mounted and maintained. The field of bio-materials science has spent the last fifty years researching the many varieties of bio-materials and makes use of those that can be fabricated from them to replicate or regain the function of organs or tissues that have been damaged or which have degenerated. Most effective within the u.s. are more than 13 million prostheses and different scientific gadgets surgically positioned each year. Numerous areas of the human frame, along with the artificial vessels within the cardiac, the stents inside the blood arteries, the prosthetic implants within the joints, knees, hips, elbows, and ears, in addition to the orthodontics systems, all employ bio-substances [58]. It is essential for a biomaterial to be made of a bio-compatible material, as this is the primary characteristic that must be possessed by the substance before it can be utilised in a biological system. In addition, the material must possess great mechanical qualities, a high resistance to corrosion, osseointegration, and exceptional resistance to wear, as well as ductility and a high degree of hardness. One plausible explanation for the rise in the number of revision procedures is that longer life expectancies are to blame for the phenomenon. As a consequence of this, the situation is different now; as a result of developments in medical technology, people live longer. In addition, the prognosis should be better for those who are physically
traumatised as a result of sports or incorrect or excessive exercise habits as well as as a result of automobile accidents and other types of accidents [59]. As a result, it is anticipated that the implants would function well for a significantly longer period of time, maybe even until the end of life, without requiring any revision surgery [60]. As a result, it is extremely important to work toward the creation of a suitable material that possesses both a high level of durability and outstanding bio-compatibility. Titanium alloys are quickly becoming the material of choice for the majority of applications, despite the fact that numerous materials are now being utilised as bio-materials [61].

The procedure described in the preceding subsection will be followed in order to pick the appropriate material for the femoral component of the hip joint prosthesis. In the collection of research that has been completed, a variety of approaches of selecting this biomedical material have been described. The acetabulum and the femoral head make up the hip joint, which is a significant load-bearing junction in the body. The hip joint is made up of basically only those two elements. The insertion of the femoral head into the socket known as the acetabulum, which is located in the pelvic bone, results in the formation of the hip joint. The femoral head may move in and out of the acetabulum, allowing the leg to rotate, move forward and backward, and move in and out of the center of the body. As a result of the fact that it is responsible for bearing weight, the hip joint is one that must possess both a suitable amount of range of motion and a sufficient amount of stability. Hip arthroplasty is a surgical treatment that replaces or renews the damaged joint in persons whose hip joint is badly calcified (osteoarthritis) or damaged. This process is performed on patients who are candidates for hip replacement surgery. When significant discomfort, mobility restrictions, and shortness of breath prevent patients from engaging in activities of daily life, hip replacement surgery is the most effective therapeutic option. When applying a hip prosthesis, having implants that are made of the right material and have the right design elements might boost the likelihood of success. There are a variety of prostheses available today, each made of a unique material and featuring a distinct design. The design and material features of the chosen implant should make it possible for the prosthesis to be straightforward, easily manufactured, reasonably priced, consistently dependable, and durable. The fact that the design of hip replacement prostheses calls for many different key qualities makes it difficult to come up with solutions using only one material. This adds an additional layer of complexity to the process of selecting materials for the prostheses.

When a biomaterial is introduced into the human body, it is believed that no adverse tissue reactions would occur. In addition, the material must possess a high resistance to corrosion and fatigue, a low elastic modulus, a high mechanical power, and a lower density than bone. The hip prosthesis consists of the femoral component, the acetabular cup, and the acetabular interface. The femoral component serves as a replacement for the femur. A hip prosthesis consists of three basic components: the femoral component, the acetabular cup, and the acetabular interface. The natural femoral head is replaced with the femoral component, a metal pin composed of a strong substance. This pin is put into the femur's hollowed-out shaft. The natural femoral head is replaced with the femoral component, which is a metal pin composed of a strong substance. This pin is put into the femur's hollowed-out shaft. To implant the acetabulum into the pelvis, a soft polymer acetabular cup that is connected to the ilium is utilised. The acetabular interface, which connects the femoral component to the acetabular cup, is available in a number of material configurations. This is performed to reduce the quantity of friction-induced wear debris.

The specific statement about the medical material selection problem is described as follows:

This problem is associated with the hip prosthesis. In this procedure, the femoral head is changed by a stiff pin that is inserted in the shaft of the femur, and the pelvic socket is replaced by either a rigid or a soft cup that is fastened to the ilium. The compression strength of compact bone is around 140 MPa , and its elastic modulus in the longitudinal direction is approximately 14 GPa , whereas the elastic modulus in the radial direction is approximately $1 / 3$ of that value. When compared to the majority of technical alloys, these
values are rather low. Live, healthy bone, on the other hand, is capable of self-healing and has a high resilience to the exhaustion caused by loading. Both the pin and the cup serve distinct purposes, and both are attached to the underlying bone structure by using adhesive cement. In this particular investigation, the material for the pin has been taken into consideration. We consider there are four materials: namely, $\mathscr{T}^{\psi}{ }_{1}=\mathrm{Co}-\mathrm{Cr}$ alloys Cast alloy (1), $\mathscr{T} \psi_{2}=\mathrm{Co}-\mathrm{Cr}$ alloys - Wrought alloy (2), $\mathscr{T} \psi_{3}=\mathrm{Ti}-6 \mathrm{Al}-4 \mathrm{~V}$ (3) and $\mathscr{T} \psi_{4}=$ Unalloyed titanium (4). Three experts will be selected to assess the materials conditions based on the indications listed in Table 3 as per [1].

Table 3. Criterion for land selection.

|  | Criterion |
| :---: | :---: |
| $\mathscr{C}_{1}^{\top}$ | Tissue tolerance |
| $\mathscr{C}_{2}^{\top}$ | Tensile strength (MPa) |
| $\left.\mathscr{C}_{3}\right\rceil$ | Fatigue strength $(\mathrm{MPa})$ |
| $\mathscr{C}_{4}{ }^{\top}$ | Density $\left(\mathrm{g} / \mathrm{cm}^{3}\right)$ |
| $\mathscr{C}_{5}^{\top}$ | Wear resistance |
| $\mathscr{C}_{6}^{\top}$ | Elastic modulus (GPa) |

## Decision-Making Process

## Step 1:

The LTs for each DM (doctor or engineer) are given in Table 4. By the LTs, find the DMs weights by Equation (7); then, the DMs weights are $\zeta_{1}=0.360, \zeta_{2}=0.317$ and $\zeta_{3}=0.322$.

Table 4. Linguistic terms for DMs.

| DM | Linguistic Terms |
| :---: | :---: |
| $\mathscr{O}_{1}$ | Very appropriate |
| $\mathscr{O}_{2}$ | Medium appropriate |
| $\mathscr{O}_{3}$ | Appropriate |

## Step 2:

Now, find out the decision matrix $\mathscr{E}^{\mathcal{G}}{ }_{(p)}=\left(\mathfrak{Y}_{j i}^{(p)}\right)_{n \times m}$ in the format of LDFNs from DMs. The judgement values, given through three DMs, are given in Tables 5-7.

Table 5. Assessment matrix acquired from $\mathscr{O}^{\mathcal{G}}{ }_{1}$.

|  | $\mathscr{C}_{1}{ }^{\top}$ | $\mathscr{C}_{2}^{\top}$ | $\mathscr{C}_{3}{ }^{\top}$ |
| :---: | :---: | :---: | :---: |
| $\mathscr{T} \psi_{1}$ | $(\langle 0.880,0.700\rangle\langle 0.350,0.440\rangle)$ | $(\langle 0.700,0.850\rangle,\langle 0.220,0.210\rangle)$ | $(\langle 0.750,0.450\rangle,\langle 0.250,0.450\rangle)$ |
| $\mathscr{T} \psi_{2}$ | $(\langle 0.950,0.150\rangle,\langle 0.610,0.250\rangle)$ | $(\langle 0.650,0.750\rangle,\langle 0.450,0.500\rangle)$ | $(\langle 0.750,0.500\rangle,\langle 0.550,0.150\rangle)$ |
| $\mathscr{T}^{\psi}{ }_{3}$ | $(\langle 0.750,0.550\rangle,\langle 0.350,0.400\rangle)$ | $(\langle 0.850,0.440\rangle,\langle 0.440,0.450\rangle)$ | $(\langle 0.650,0.800\rangle,\langle 0.550,0.300\rangle)$ |
| $\mathscr{T} \psi_{4}$ | $(\langle 0.650,0.500\rangle,\langle 0.650,0.250\rangle)$ | $(\langle 0.760,0.860\rangle,\langle 0.550,0.200\rangle)$ | $(\langle 0.650,0.350\rangle,\langle 0.150,0.450\rangle)$ |
|  | $\mathscr{C}_{4}^{\top}$ | $\mathscr{C}_{5}{ }^{\top}$ | $\mathscr{C}_{6}{ }^{\top}$ |
| $\mathscr{T}^{\psi}{ }_{1}$ | $(\langle 0.650,0.500\rangle,\langle 0.550,0.120\rangle)$ | $(\langle 0.450,0.600\rangle,\langle 0.350,0.150\rangle)$ | $(\langle 0.560,0.690\rangle,\langle 0.450,0.330\rangle)$ |
| $\mathscr{T} \psi_{2}$ | $(\langle 0.750,0.490\rangle,\langle 0.150,0.650\rangle)$ | $(\langle 0.550,0.650\rangle,\langle 0.450,0.250\rangle)$ | $(\langle 0.560,0.460\rangle,\langle 0.490,0.210\rangle)$ |

Table 5. Cont.

| $\mathscr{T} \psi_{3}$ | $(\langle 0.340,0.650\rangle,\langle 0.150,0.300\rangle)$ | $(\langle 0.550,0.400\rangle,\langle 0.250,0.550\rangle)$ | $(\langle 0.500,0.700\rangle,\langle 0.550,0.150\rangle)$ |
| :---: | :---: | :---: | :---: |
| $\mathscr{T} \psi_{4}$ | $(\langle 0.850,0.650\rangle,\langle 0.150,0.450\rangle)$ | $(\langle 0.560,0.540\rangle,\langle 0.750,0.150\rangle)$ | $(\langle 0.560,0.550\rangle,\langle 0.450,0.450\rangle)$ |

Table 6. Assessment matrix acquired from $\mathscr{O}^{\mathcal{G}}{ }_{2}$.

| $\mathscr{T}_{1}$ | $(\langle 0.450,0.600\rangle,\langle 0.350,0.440\rangle)$ | $(\langle 0.700,0.850\rangle,\langle 0.350,0.550\rangle)$ | $(\langle 0.880,0.850\rangle,\langle 0.350,0.450\rangle)$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathscr{T}_{2} \psi_{2}$ | $(\langle 0.750,0.550\rangle,\langle 0.650,0.250\rangle)$ | $(\langle 0.650,0.250\rangle,\langle 0.450,0.430\rangle)$ | $(\langle 0.650,0.550\rangle,\langle 0.550,0.150\rangle)$ |
| $\mathscr{T}_{3} \psi_{3}$ | $(\langle 0.650,0.450\rangle,\langle 0.350,0.540\rangle)$ | $(\langle 0.550,0.760\rangle,\langle 0.450,0.440\rangle)$ | $(\langle 0.350,0.680\rangle,\langle 0.540,0.350\rangle)$ |
| $\mathscr{T} \psi_{4}$ | $(\langle 0.770,0.650\rangle,\langle 0.250,0.350\rangle)$ | $(\langle 0.540,0.490\rangle,\langle 0.550,0.320\rangle)$ | $(\langle 0.560,0.350\rangle,\langle 0.150,0.450\rangle)$ |
| $\mathscr{T} \psi_{1}$ | $(\langle 0.450,0.540\rangle,\langle 0.550,0.230\rangle)$ | $(\langle 0.540,0.990\rangle,\langle 0.350,0.450\rangle)$ | $(\langle 0.450,0.490\rangle,\langle 0.450,0.340\rangle)$ |
| $\mathscr{T} \psi_{2}$ | $(\langle 0.750,0.860\rangle,\langle 0.450,0.430\rangle)$ | $(\langle 0.570,0.870\rangle,\langle 0.150,0.560\rangle)$ | $(\langle 0.550,0.670\rangle,\langle 0.650,0.250\rangle)$ |
| $\mathscr{T} \psi_{3}$ | $(\langle 0.560,0.540\rangle,\langle 0.570,0.120\rangle)$ | $(\langle 0.540,0.790\rangle,\langle 0.250,0.550\rangle)$ | $(\langle 0.560,0.790\rangle,\langle 0.560,0.350\rangle)$ |
| $\mathscr{T} \psi_{4}$ | $(\langle 0.950,0.550\rangle,\langle 0.560,0.230\rangle)$ | $(\langle 0.550,0.670\rangle,\langle 0.750,0.150\rangle)$ | $(\langle 0.350,0.750\rangle,\langle 0.450,0.540\rangle)$ |

Table 7. Assessment matrix acquired from $\mathscr{O}^{\mathcal{G}}$.

|  | $\mathscr{C}_{1}{ }^{\top}$ | $\mathscr{C}_{2}{ }^{\top}$ | $\mathscr{C b}^{7}$ |
| :---: | :---: | :---: | :---: |
| $\mathscr{T}^{\psi}{ }_{1}$ | $(\langle 0.560,0.780\rangle,\langle 0.350,0.500\rangle)$ | $(\langle 0.650,0.550\rangle,\langle 0.350,0.450\rangle)$ | $(\langle 0.650,0.850\rangle,\langle 0.450,0.150\rangle)$ |
| $\mathscr{T}^{\psi}{ }_{2}$ | $(\langle 0.750,0.550\rangle,\langle 0.650,0.310\rangle)$ | $(\langle 0.650,0.550\rangle,\langle 0.450,0.350\rangle)$ | $(\langle 0.750,0.670\rangle,\langle 0.550,0.250\rangle)$ |
| $\mathscr{T} \psi_{3}$ | $(\langle 0.650,0.250\rangle,\langle 0.350,0.500\rangle)$ | $(\langle 0.650,0.450\rangle,\langle 0.450,0.410\rangle)$ | $(\langle 0.350,0.430\rangle,\langle 0.540,0.340\rangle)$ |
| $\mathscr{T} \psi_{4}$ | $(\langle 0.560,0.870\rangle,\langle 0.450,0.250\rangle)$ | $(\langle 0.690,0.670\rangle,\langle 0.450,0.150\rangle)$ | $(\langle 0.560,0.350\rangle,\langle 0.150,0.450\rangle)$ |
|  | $\mathscr{C}_{4}{ }^{\top}$ | $\mathscr{C}_{5}{ }^{\top}$ | $\mathscr{C}_{6}{ }^{\top}$ |
| $\mathscr{T}{ }^{\psi}{ }_{1}$ | $(\langle 0.550,0.250\rangle,\langle 0.250,0.550\rangle)$ | $(\langle 0.540,0.670\rangle,\langle 0.350,0.250\rangle)$ | $(\langle 0.450,0.670\rangle,\langle 0.570,0.270\rangle)$ |
| $\mathscr{T} \psi_{2}$ | $(\langle 0.750,0.250\rangle,\langle 0.340,0.120\rangle)$ | $(\langle 0.560,0.680\rangle,\langle 0.150,0.340\rangle)$ | $(\langle 0.560,0.670\rangle,\langle 0.410,0.360\rangle)$ |
| $\mathscr{T}^{\psi}{ }_{3}$ | $(\langle 0.350,0.650\rangle,\langle 0.150,0.350\rangle)$ | $(\langle 0.350,0.750\rangle,\langle 0.250,0.550\rangle)$ | $(\langle 0.250,0.690\rangle,\langle 0.560,0.250\rangle)$ |
| $\mathscr{T}{ }^{\psi}{ }_{4}$ | $(\langle 0.360,0.790\rangle,\langle 0.150,0.750\rangle)$ | $(\langle 0.540,0.640\rangle,\langle 0.260,0.580\rangle)$ | $(\langle 0.350,0.750\rangle,\langle 0.450,0.390\rangle)$ |

## Step 3:

To construct the aggregated LDF decision matrix, all individual opinions must be totalled up and integrated to form a group opinion.
$H=\left(H_{j i}\right)$ be the aggregated LDF decision matrix, where
$H_{j i}=\operatorname{LDFFWA}\left(\mathfrak{Y}_{j i}^{(1)}, \mathfrak{Y}_{j i}^{(2)}, \mathfrak{Y}_{j i}^{(3)}\right)=\left(\zeta_{1} * \mathfrak{Y}_{j i}^{(1)} \tilde{\oplus} \zeta_{2} * \mathfrak{Y}_{j i}^{(2)} \tilde{\oplus} \zeta_{3} * \mathfrak{Y}_{j i}^{(3)}\right)$. The aggregated LDF decision matrix given in Table 8.

## Step 4:

Here is no cost type attribute, so the normalised decision matrix will be $\Gamma_{N}=\left(\aleph_{j i}^{N}\right)_{n \times m}{ }^{\prime}$ given in Table 9.
Step 5:
Construct the score matrix by utilising the SF of LDFNs as $\Psi=\left(\breve{\breve{G}}^{J}\left(\aleph_{j i}^{N}\right)\right)_{5 \times 4}$.

|  | $\mathscr{C}_{1}^{\top}$ | $\mathscr{C}_{2}{ }^{\top}$ | $\mathscr{C}_{3}^{\top}$ | $\mathscr{C}_{4}^{\top}$ | $\mathscr{C}_{5}^{\top}$ | $\mathscr{C}_{6}^{\top}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathscr{T} \psi_{1}{ }^{\top}$ | (0.448402 | 0.486930 | 0.536408 | 0.588400 | 0.497964 | 0.516667 |
| $\mathscr{T}^{\psi}{ }_{2}$ | 0.699891 | 0.540463 | 0.619389 | 0.530605 | 0.435513 | 0.557505 |
| $\mathscr{T}^{\psi}{ }_{3}$ | 0.530242 | 0.530864 | 0.513675 | 0.445804 | 0.395927 | 0.523009 |
| $\mathscr{T} \psi_{4}$ | 0.541545 | 0.579266 | 0.490864 | 0.441202 | 0.574909 | 0.442733 |

Table 8. Aggregated LDF decision matrix.

|  | $\mathscr{C}_{1}^{\top}$ | $\mathscr{C}_{2}^{\top}$ |
| :---: | :---: | :---: |
| $\mathscr{T} \psi_{1}$ | $(\langle 0.383944,0.430955\rangle,\langle 0.351249,0.46002\rangle)$ | $(\langle 0.409024,0.382178\rangle,\langle 0.343213,0.422339\rangle)$ |
| $\mathscr{T} \psi_{2}$ | $(\langle 0.621141,0.262138\rangle,\langle 0.643939,0.271831\rangle)$ | $(\langle 0.517585,0.381459\rangle,\langle 0.461147,0.435420\rangle)$ |
| $\mathscr{T} \psi_{3}$ | $(\langle 0.592098,0.346462\rangle,\langle 0.355810,0.480478\rangle)$ | $(\langle 0.495263,0.384580\rangle,\langle 0.446815,0.434040\rangle)$ |
| $\mathscr{T} \psi_{4}$ | $(\langle 0.416858,0.414242\rangle,\langle 0.471004,0.307439\rangle)$ | $(\langle 0.400528,0.402225\rangle,\langle 0.541019,0.222258\rangle)$ |
|  | $\mathscr{C}_{3}^{\top}$ | $\mathscr{C}_{4}^{\top}$ |
| $\mathscr{T} \psi_{1}$ | $(\langle 0.458977,0.335526\rangle,\langle 0.366206,0.344024\rangle)$ | $(\langle 0.573089,0.428591\rangle,\langle 0.480924,0.271821\rangle)$ |
| $\mathscr{T} \psi_{2}$ | $(\langle 0.476127,0.376321\rangle,\langle 0.557076,0.179326\rangle)$ | $(\langle 0.513239,0.322832\rangle,\langle 0.348713,0.416701\rangle)$ |
| $\mathscr{T} \psi_{3}$ | $(\langle 0.382320,0.543392\rangle,\langle 0.544502,0.328730\rangle)$ | $(\langle 0.390435,0.594876\rangle,\langle 0.264717,0.277059\rangle)$ |
| $\mathscr{T} \psi_{4}$ | $(\langle 0.645666,0.382640\rangle,\langle 0.150032,0.449601\rangle)$ | $(\langle 0.401732,0.395055\rangle,\langle 0.274628,0.516498\rangle)$ |
|  | $\mathscr{T} \psi_{1}$ | $(\langle 0.351584,0.506547\rangle,\langle 0.433623,0.286806\rangle)$ |
| $\mathscr{\mathscr { C }}{ }^{\top}$ | $(\langle 0.416422,0.524272\rangle,\langle 0.489285,0.314766\rangle)$ |  |
| $\mathscr{\mathscr { C }} \psi_{2}$ | $(\langle 0.371386,0.480077\rangle,\langle 0.249112,0.398370\rangle)$ | $(\langle 0.449472,0.472278\rangle,\langle 0.529092,0.276265\rangle)$ |
| $\mathscr{T} \psi_{3}$ | $(\langle 0.408280,0.524962\rangle,\langle 0.250035,0.549643\rangle)$ | $(\langle 0.331485,0.578648\rangle,\langle 0.581091,0.241891\rangle)$ |
| $\mathscr{T} \psi_{4}$ | $(\langle 0.434822,0.482565\rangle,\langle 0.615385,0.268007\rangle)$ | $(\langle 0.362350,0.585890\rangle,\langle 0.469136,0.474665\rangle)$ |

Table 9. Normalised LDF decision matrix.

|  | $\mathscr{C}_{1}^{\top}$ | $\mathscr{C}_{2}^{\top}$ |
| :---: | :---: | :---: |
| $\mathscr{T} \psi_{1}$ | $(\langle 0.383944,0.430955\rangle,\langle 0.351249,0.46002\rangle)$ | $(\langle 0.409024,0.382178\rangle,\langle 0.343213,0.422339\rangle)$ |
| $\mathscr{T} \psi_{2}$ | $(\langle 0.621141,0.262138\rangle,\langle 0.643939,0.271831\rangle)$ | $(\langle 0.517585,0.381459\rangle,\langle 0.461147,0.435420\rangle)$ |
| $\mathscr{T} \psi_{3}$ | $(\langle 0.592098,0.346462\rangle,\langle 0.355810,0.480478\rangle)$ | $(\langle 0.495263,0.384580\rangle,\langle 0.446815,0.434040\rangle)$ |
| $\mathscr{T} \psi_{4}$ | $(\langle 0.416858,0.414242\rangle,\langle 0.471004,0.307439\rangle)$ | $(\langle 0.400528,0.402225\rangle,\langle 0.541019,0.222258\rangle)$ |
|  | $\mathscr{C}_{3}{ }^{\top}$ | $\mathscr{C}_{4}{ }^{\top}$ |
| $\mathscr{T} \psi_{1}$ | $(\langle 0.458977,0.335526\rangle,\langle 0.366206,0.344024\rangle)$ | $(\langle 0.573089,0.428591\rangle,\langle 0.480924,0.271821\rangle)$ |
| $\mathscr{T} \psi_{2}$ | $(\langle 0.476127,0.376321\rangle,\langle 0.557076,0.179326\rangle)$ | $(\langle 0.513239,0.322832\rangle,\langle 0.348713,0.416701\rangle)$ |
| $\mathscr{T} \psi_{3}$ | $(\langle 0.382320,0.543392\rangle,\langle 0.544502,0.328730\rangle)$ | $(\langle 0.390435,0.594876\rangle,\langle 0.264717,0.277059\rangle)$ |
| $\mathscr{T} \psi_{4}$ | $(\langle 0.645666,0.382640\rangle,\langle 0.150032,0.449601\rangle)$ | $(\langle 0.401732,0.395055\rangle,\langle 0.274628,0.516498\rangle)$ |
|  | $\mathscr{C}_{5}{ }^{\top}$ | $\mathscr{C}_{6}{ }^{\top}$ |
| $\mathscr{T} \psi_{1}$ | $(\langle 0.351584,0.506547\rangle,\langle 0.433623,0.286806\rangle)$ | $(\langle 0.416422,0.524272\rangle,\langle 0.489285,0.314766\rangle)$ |
| $\mathscr{T} \psi_{2}$ | $(\langle 0.371386,0.480077\rangle,\langle 0.249112,0.398370\rangle)$ | $(\langle 0.449472,0.472278\rangle,\langle 0.529092,0.276265\rangle)$ |
| $\mathscr{T} \psi_{3}$ | $(\langle 0.408280,0.524962\rangle,\langle 0.250035,0.549643\rangle)$ | $(\langle 0.331485,0.578648\rangle,\langle 0.581091,0.241891\rangle)$ |
| $\mathscr{T} \psi_{4}$ | $(\langle 0.434822,0.482565\rangle,\langle 0.615385,0.268007\rangle)$ | $(\langle 0.362350,0.585890\rangle,\langle 0.469136,0.474665\rangle)$ |

## Step 6:

Consider that the DMs provide the following partial weight details about the attribute weights:
$\Psi=0.15 \leq \mathfrak{W}_{1}^{\gamma} \leq 0.40,0.20 \leq \mathfrak{W}_{2}^{\gamma} \leq 0.70,0.30 \leq \mathfrak{W}_{3}^{\gamma} \leq 0.60,0.10 \leq \mathfrak{W}_{4}^{\gamma} \leq 0.80$, $0.05 \leq \mathfrak{W}_{5}^{\gamma} \leq 0.65,0.10 \leq \mathfrak{W}_{6}^{\gamma} \leq 0.50$
Relying on these data, the following optimisation framework can be developed:

$$
\begin{aligned}
\operatorname{Max} g= & 0.448402 \mathfrak{W}_{1}^{\gamma}+0.699891 \mathfrak{W}_{1}^{\gamma}+0.530242 \mathfrak{W}_{1}^{\gamma}+0.541545 \mathfrak{W}_{1}^{\gamma} \\
& 0.486930 \mathfrak{W}_{2}^{\gamma}+0.540463 \mathfrak{W}_{2}^{\gamma}+0.530864 \mathfrak{W}_{2}^{\gamma}+0.579266 \mathfrak{W}_{2}^{\gamma} \\
& 0.536408 \mathfrak{W}_{3}^{\gamma}+0.619389 \mathfrak{W}_{3}^{\gamma}+0.513675 \mathfrak{W}_{3}^{\gamma}+0.490864 \mathfrak{W}_{3}^{\gamma} \\
& 0.588400 \mathfrak{W}_{4}^{\gamma}+0.530605 \mathfrak{W}_{4}^{\gamma}+0.445804 \mathfrak{W}_{4}^{\gamma}+0.441202 \mathfrak{W}_{4}^{\gamma} \\
& 0.497964 \mathfrak{W}_{5}^{\gamma}+0.435513 \mathfrak{W}_{5}^{\gamma}+0.395927 \mathfrak{W}_{5}^{\gamma}+0.574909 \mathfrak{W}_{5}^{\gamma} \\
& 0.516667 \mathfrak{W}_{6}^{\gamma}+0.557505 \mathfrak{W}_{6}^{\gamma}+0.523009 \mathfrak{W}_{6}^{\gamma}+0.442733 \mathfrak{W}_{6}^{\gamma}
\end{aligned}
$$

such that,
$0.15 \leq \mathfrak{W}_{1}^{\gamma} \leq 0.40, \quad 0.20 \leq \mathfrak{W}_{2}^{\gamma} \leq 0.70, \quad 0.30 \leq \mathfrak{W}_{3}^{\gamma} \leq 0.60, \quad 0.10 \leq \mathfrak{W}_{4}^{\gamma} \leq$ 0.80,
$0.05 \leq \mathfrak{W}_{5}^{\gamma} \leq 0.65, \quad 0.10 \leq \mathfrak{W}_{6}^{\gamma} \leq 0.50$
$\mathfrak{W}_{1}^{\gamma}+\mathfrak{W}_{2}^{\gamma}+\mathfrak{W}_{3}^{\gamma}+\mathfrak{W}_{4}^{\gamma}=1, \quad \mathfrak{W}_{1}^{\gamma}, \mathfrak{W}_{2}^{\gamma}, \mathfrak{W}_{3}^{\gamma}, \mathfrak{W}_{4}^{\gamma} \geq 0$.
By solving this model, we obtain $\mathfrak{W}_{1}^{\gamma}=0.25, \mathfrak{W}_{2}^{\gamma}=0.20, \mathfrak{W}_{3}^{\gamma}=0.30, \mathfrak{W}_{4}^{\gamma}=0.10, \mathfrak{W}_{5}^{\gamma}=$ $0.05, \mathfrak{W}_{6}^{\gamma}=0.10$.

## Step 7:

Evaluate the aggregated weighted LDF decision matrix by using the proposed AOs given by Table 10.

Table 10. Aggregated weighted LDF decision matrix.

| $\mathscr{T} \psi_{1}$ | $(\langle 0.558013,0.522136\rangle,\langle 0.388174,0.376082\rangle)$ |
| :---: | :--- |
| $\mathscr{T} \psi_{2}$ | $(\langle 0.629921,0.432568\rangle,\langle 0.540640,0.299443\rangle)$ |
| $\mathscr{T}_{3}$ | $(\langle 0.510652,0.526887\rangle,\langle 0.446086,0.393355\rangle)$ |
| $\mathscr{T} \psi_{4}$ | $(\langle 0.552423,0.496814\rangle,\langle 0.374417,0.400546\rangle)$ |

## Step 8:

Compute the score values of all alternatives.

$$
\begin{aligned}
& \breve{\breve{G}}^{\beth}\left(\mathscr{T}^{\psi}{ }_{1}\right)=0.511992 \\
& \breve{\breve{G}}^{\beth}\left(\mathscr{T}_{2}{ }_{2}\right)=0.609637 \\
& \text { 舀 }^{\beth}\left(\mathscr{T}_{3}{ }_{3}\right)=0.509124 \\
& \breve{\breve{B}}^{\beth}\left(\mathscr{T}^{\psi}{ }_{4}\right)=0.507370
\end{aligned}
$$

At the end, the final ranking will be

$$
\mathscr{T}^{\psi}{ }_{2} \succ \mathscr{T}^{\psi}{ }_{1} \succ \mathscr{T}_{3}{ }_{3} \succ \mathscr{T}^{\psi}{ }_{4} .
$$

Therefore, Co-Cr alloys - Wrought alloy is proposed as the best material.

## 8. Conclusions

The uses of bio-materials in the medical sector that have the qualities of being processable, sterile, and possessing a potent anti-infective capacity have received an increasing amount of attention in recent years. However, certain discrepancies exist between the performance levels of various materials; hence, determining how to select the appropriate bio-materials is also an important issue to discuss. On the basis of this rationale, we represent the challenge of selecting hip joint prosthesis materials as an MCDM problem in an LDF context and offer a new MCDM approach to solve it with incomplete information for characteristics. The studies that have been completed up until now indicate that when assessing items, the aggregate ratings that are connected with them will not be comparable even if a DM delivers an equal quantity of MSDs and NMSDs. In such a scenario, we presented several novel fairness or neutrality operations based on LDFS and proportionate distribution rules for membership and non-membership functions. Furthermore, we placed an emphasis on accuracy and relevance during the decision-making process, which is determined by the DM's disposition. We added to the LDFN information a "linear Diophantine fuzzy fairly weighted averaging (LDFFWA) operator" as well as a "linear Diophantine fuzzy fairly ordered weighted averaging (LDFFOWA) operator", both of which were modelled after fairly operations. An in-depth conversation was had on the many aspects of the suggested AOs. The fundamental benefit of the proposed operators is that they not only make it possible for separate pairs of LDFNs to interact with one another, but they also make it easier to investigate the attitude characteristics of the DMs by permitting a categorical treatment of the degrees of the LDFSs. This is the primary benefit. The suggested technique is validated by testing it against a problem requiring the selection of materials.

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