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Classification of Left Invariant Riemannian Metrics on Complex Hyperbolic Space

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Abstract. It is well known that $\mathbb{C}H^n$ has the structure of a solvable Lie group with left invariant metric of constant holomorphic sectional curvature. In this paper we give the full classification of all possible left invariant Riemannian metrics on this Lie group. We prove that each of those metrics is of constant negative scalar curvature, only one of them being Einstein (up to isometry and scaling).

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Introduction

Since $\mathbb{C}H^n$ is a symmetric space of negative sectional curvature, in light of the classical results of Heintze [11,12], it can be viewed as a connected solvable real Lie group with a left invariant metric. This group, denoted by $\mathcal{CH}^n$, is the noncompact part of the Iwasawa decomposition $SU(1,n) = KAN$ of the isometry group of the complex hyperbolic space. The compact part is isomorphic to $U(n)$, the nilpotent part N is Heisenberg group H^{2n-1}, and the abelian part is one-dimensional. The semidirect product $\mathcal{CH}^n = AN$ acts simply transitively on $\mathbb{C}H^n$ giving it a structure of a Lie group with the left invariant metric inherited from $\mathbb{C}H^n$. Now an interesting question arises: what are all the other possible left invariant Riemannian metrics on this Lie group?

There are two standard approaches to the classification problem based on the moduli spaces of the left invariant Riemannian metrics on a given Lie group. One is to fix Lie algebra commutators, then consider the space of all possible left invariant metrics and find the simplest representatives under the action of the automorphism group. This approach we use here to classify all non-isometric left invariant Riemannian metrics on $\mathcal{CH}^n$. Since $\mathcal{CH}^n$ is completely solvable and due to Alekseevsky's results [1,2], isometry classes of the left invariant metrics are exactly the orbits of the automorphism group acting on the space of the left invariant metrics. Slightly different but equivalent approach is to fix an orthonormal (or pseudo-orthonormal) base and classify all possible Lie brackets up to the action of the automorphism group. In the space of all bilinear skew-symmetric forms the Jacobi identity defines the hypersurface of admissible Lie brackets. Once again the isometry classes constitute the orbits of the automorphism group. Both approaches are used by Lauret in search for distinguished left invariant metrics (Ricci solitons) on nilpotent Lie groups [19]. The first method is systematically outlined in [14], and the second one in [10]. Tamaru et al. called these methods Milnor-type theorems in reference to the groundwork of Milnor [21] who classified all left invariant Riemannian metrics on three-dimensional unimodular Lie groups. Although Milnor's method relies on the existence of the cross product in dimension three, lots of results have been obtained later for Riemannian and Lorentzian cases in dimensions three and four (see for example [3–7, 18, 23]) as well as for dimension four with neutral signature [24, 25].

The results in an arbitrary dimension are more recent. All the left invariant Riemannian and Lorentzian metrics on Heisenberg group have been classified in [26]. Pseudo-Riemannian metrics of the real hyperbolic space modelled as a Lie group have been explored both by the variation of Lie brackets [16] and by the variation of inner products [27]. It has been shown [17] that the Euclidean space, the real hyperbolic space and $H_3 \times \mathbb{R}^n$ (product of three-dimensional Heisenberg group and Euclidean space) are the only connected and simply-connected Lie groups admitting unique left invariant Riemannian metric up to scaling and isometry. Lorentzian metrics on $H_3 \times \mathbb{R}^n$ have been classified in [15].

Algebraic Ricci solitons on nilpotent and solvable Lie groups are introduced by Lauret [19, 20]. He proved that a solvable Lie group admits at most one left invariant Ricci soliton up to isometry and scaling. Since $\mathcal{CH}^n$ is completely solvable, one can conclude that the Einstein metric of $\mathbb{C}H^n$ is the only Riemannian left invariant Ricci soliton on $\mathcal{CH}^n$. This is an example of codimension one Ricci soliton subgroups of solvable Iwasawa groups, recently classified in [8]. Algebraic Ricci solitons on Heisenberg group have been classified in [22].

This paper is organized as follows.

In Sect. 1 we introduce basic notation and define metric Lie groups and algebras together with isometry classes. We also give a brief review of some concepts of the symplectic linear algebra.

Theorem 2.1 of Sect. 2, the classification of non-isometric Riemannian left invariant metrics on $\mathcal{CH}^n$, is the main result of this paper. We describe the Lie algebra's group of automorphisms $\mathrm{Aut}(ch_n)$ in Lemma 2.1 and Corollary 2.1. It contains the symplectic group $Sp(2(n-1), \mathbb{R})$ that plays an important role in the proof of the classification theorem by permitting a diagonalization with symplectic eigenvalues.

In Sect. 3 we investigate geometrical properties of the Lie group $\mathcal{CH}^n$ equipped with various left invariant metrics $g(p, x, \sigma, \beta)$. This provides a whole class of Riemannian solvmanifolds that could be interesting for further research. The curvature tensor is given both explicitly and on the exterior algebra. None of the metrics is flat but all of them have a constant negative sectional curvature. We prove there is exactly one Einstein metrics up to isometry and scaling and see it is the only Ricci soliton left invariant metric on $\mathcal{CH}^n$.

1. Preliminaries

The complex hyperbolic space is a non-compact rank-one symmetric space of negative sectional curvature:

$$\mathbb{C}H^n = SU(1,n)/S(U(1)\times U(n)).$$

Therefore, it is a solvmanifold, i.e. it can be represented as a connected solvable Lie group with a left invariant metric [11,12]. This group is a semidirect product of the abelian and the nilpotent part (Heisenberg group) of the Iwasawa decomposition of its isometry group:

$$\mathcal{CH}^n = \mathbb{R}\ltimes H^{2n-1}.$$

The Lie algebra ch_n of the Lie group $\mathcal{CH}^n$ is a semidirect product of the abelian and the Heisenberg algebra

$$ch_n = \mathbb{R}\ltimes \mathfrak{h}_{2n-1}.$$

It is spanned by vectors $X, Y_1, \dots, Y_{n-1}, Z_1, \dots, Z_{n-1}, W$ with nonzero commutators:

$$[X,Y_i] = \frac{1}{2}Y_i, \quad [X,Z_i] = \frac{1}{2}Z_i, \quad [X,W] = W, \quad [Z_j,Y_i] = \delta_{ij}W,$$
$$i,j\in\{1,\dots,n-1\}. \tag{1}$$

This algebra is 3-step solvable. The first derived algebra is Heisenberg $\mathfrak{h}_{2n-1}$ and the second is the one-dimensional algebra spanned by the vector W.

Using the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$:

$$(z_1,\dots,z_n)\mapsto(x_1,\dots,x_n,y_1,\dots,y_n),\ \ z_k = x_k + iy_k, \quad k\in\{1,\dots,n\},$$

the multiplication by i on $\mathbb{C}^n$ induces the standard complex structure on $\mathbb{R}^{2n}$ given by

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n is the $n\times n$ identity matrix.
The standard symplectic form in vector space $\mathbb{R}^{2n}$ is

$$\omega(u,v) = u^T J_n v, \quad u,v\in\mathbb{R}^{2n}.$$

The ω-preserving group of all linear transformations of $\mathbb{R}^{2n}$ is the symplectic group

$$\begin{aligned} Sp(2n,\mathbb{R}) &= \{F\in Gl_{2n}(\mathbb{R}) \mid \omega(Fu,Fv) = \omega(u,v)\} \\ &= \{F \mid F^T J_n F = J_n\}. \end{aligned}$$

It is related to the unitary group

$$U(n) = \{U\in GL(n,\mathbb{C}) \mid UU^* = I_n\}$$

by

$$U(n) \cong Sp(2n, \mathbb{R}) \cap SO(2n).$$

The following statements of symplectic geometry will be used for the classification of metrics on the Lie group $\mathcal{CH}^n$. For a more detailed review on the topic see [9] and [28].

Theorem 1.1. (Williamson diagonal form) *Let S be a positive-definite symmetric real $2n \times 2n$ matrix.*

(i) *There exists $M \in Sp(2n, \mathbb{R})$ such that*

$$D = M^T S M = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \quad \sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n).$$

(ii) *The sequence $\sigma_1, \ldots, \sigma_n$ does not depend, up to a reordering of its terms, on the choice of M diagonalizing S.*

With the ordering convention $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$, the decreasing sequence $(\sigma_1, \ldots, \sigma_n)$ is called the *symplectic spectrum* of S and is denoted by $\mathrm{Spec}(S)$. The elements of $\mathrm{Spec}(S)$ are the *symplectic eigenvalues*. The diagonal matrix D with the ordering convention above is called Williamson diagonal form of the symmetric matrix S. The elements of symplectic spectrum are symplectic eigenvalues of S. The *multiplicity* of the symplectic eigenvalue σ_i is the number of times it is repeated in σ.

Proposition 1.1. [9] *Symplectic spectrum is a symplectic invariant:*

$$\mathrm{Spec}(M^T S M) = \mathrm{Spec}(S), \quad \textit{for every} \quad M \in Sp(2n, \mathbb{R}).$$

Proposition 1.2. [9] *Assume that M and M' are two elements of $Sp(2n, \mathbb{R})$ such that*

$$S = M^T D M = M'^T D M',$$

where D is the Williamson diagonal form of symmetric matrix S. Then $MM'^{-1} \in U(n)$.

Corollary 1.1. *If $M \in Sp(2n, \mathbb{R})$ preserves D, that is $M^T D M = D$, then $M \in U(n)$.*

The following results od Alekseevsky enable us to classify all non-isometric Riemannian metrics on the completely solvable Lie group $\mathcal{CH}^n$.

Definition 1.1. Two metric Lie algebras (i.e. Lie algebras equipped with inner products) are
(a) *isometric* if there exists a homomorphism of vector spaces that preserves curvature tensor and its covariant derivatives.
(b) *isomorphic* if they are isometric and the isometry preserves Lie algebra commutators (i.e. they are isomorphic as Lie algebras as well).

The relation between the isometry of metric Lie algebras and the isometry of corresponding metric Lie groups (Lie group with the left invariant metric) is given by:

Lemma 1.1. [2] *Two metric Lie algebras are isometric if and only if the corresponding Riemannian spaces are isometric.*

Since ch_n is completely solvable, it is possible to classify all non-isometric inner products thanks to the following lemma:

Lemma 1.2. [1,2] *Isometric completely solvable metric Lie algebras are isomorphic.*

2. Classification of Left Invariant Riemannian Metrics on $\mathcal{CH}^n$

Every inner product on a Lie algebra uniquely determines a left invariant metric on a Lie group by left translations. Therefore, the problem of classification of all non-isometric left invariant Riemannian metrics on the Lie group $\mathcal{CH}^n$ is equivalent to the classification of all non-isometric positive definite inner products on the Lie algebra ch_n.

In a fixed basis with commutators (1), inner products are represented by the symmetric positive definite matrices S. A set of orbits of an automorphism group acting of the space of inner products is called *moduli space* of left invariant metrics (see [14]). Every orbit is in one-to-one correspondence with an equivalence class of the action $S \sim F^T SF$, $F \in \mathrm{Aut}(ch_n)$. Since ch_n is completely solvable, Lemmas 1.1 and 1.2 ensure that isometry classes are precisely the orbits of the automorphism group. Thus the classification of non-isometric metrics is equivalent to finding the simplest representatives of these orbits.

Lemma 2.1. *The automorphism group of the Lie algebra ch_n is*

$$\mathrm{Aut}(ch_n) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ u & M & 0 \\ a & v^T & \lambda \end{pmatrix} \mid M \in GL(2(n-1), \mathbb{R}), \quad M^T J_{n-1} M \right.$$

$$\left. = \lambda J_{n-1}, v \in \mathbb{R}^{2(n-1)}, \quad a, \lambda \in \mathbb{R}, \quad \lambda \neq 0, \quad u = \frac{1}{2\lambda} M J_{n-1} v \right\}. \tag{2}$$

Proof. Fix a basis $(X, Y_1, \ldots, Y_{n-1}, Z_1, \ldots, Z_{n-1}, W)$ with commutators of the form (1). Every automorphism $F \in \mathrm{Aut}(ch_n)$ must preserve commutators by definition, i.e.

$$[FX_1, FX_2] = F[X_1, X_2], \quad X_1, X_2 \in ch_n.$$

It also preserves the derived series. In the ch_n case, its first derived algebra is Heisenberg algebra spanned by $Y_1, \ldots, Y_{n-1}, Z_1, \ldots, Z_{n-1}, W$ and its second derived algebra is one-dimensional spanned by W. This directly simplifies automorphisms represented in the block matrix form:

$$F = \begin{pmatrix} 1 & 0 & 0 \\ u & M & 0 \\ a & v^T & \lambda \end{pmatrix}.$$

Furthermore, the commutators

$$[FZ_j, FY_i] = \delta_{ij} FW, \quad [FY_i, FY_j] = 0, \quad [FZ_i, FZ_j] = 0, \quad i, j \in \{1, \dots, n-1\}$$

impose a restriction on M: $\quad M^T J_{n-1} M = \lambda J_{n-1}$.

Finally, after a short calculation the commutators

$$[FX, FY_i] = \frac{1}{2} FY_i, \quad [FX, FZ_i] = \frac{1}{2} FZ_i, \quad i, j \in \{1, \dots, n-1\},$$

imply a relation of vectors u and v: $\quad u = \frac{1}{2\lambda} M J_{n-1} v$. □

Note that $\mathrm{Aut}(ch_n)$ is generated by three types of automorphisms: **diagonal, symplectic** and **generalized translations**.

Corollary 2.1. *The identity component of the automorphism group of the Lie algebra ch_n is a semidirect product*

$$Aut_0(ch_n) = D \ltimes (Sp \ltimes T),$$

where

$$D = \left\{ F_d(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha I & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} \mid \alpha > 0 \right\},$$

$$Sp = \left\{ F_{S_p}(M) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid M^T J_{n-1} M = J_{n-1} \right\} \cong Sp(2(n-1), \mathbb{R}),$$

$$T = \left\{ F_t(v, a) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} \quad J_{n-1} v & I & 0 \\ a & v^T & 1 \end{pmatrix} \mid a \in \mathbb{R}, v \in \mathbb{R}^{2(n-1)} \right\}.$$

Theorem 2.1. *(Classification theorem) All positive definite inner products on the Lie algebra ch_n in some basis with commutators (1) are represented by the matrices*

$$S(p, x, \sigma, \beta) = \begin{pmatrix} p & x^T & 0 & 0 \\ x & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}, \tag{3}$$

where $\quad p, \beta > 0, \quad x = (x_1, \dots, x_{n-1})^T \in \mathbb{R}^{n-1}, \quad x_i \geq 0,$
$\sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_{n-2}, 1), \quad \sigma_1 \geq \cdots \geq \sigma_{n-2} \geq 1.$

If all eigenvalues of σ are distinct then all inner products (3) are non-isometric. If $m_1, \dots, m_{k+1}$ are the multiplicities of the eigenvalues of the matrix σ, i.e. $\sigma = \mathrm{diag}(\underbrace{\hat{\sigma}_1, \dots, \hat{\sigma}_1}_{m_1}, \dots \underbrace{\hat{\sigma}_k, \dots, \hat{\sigma}_k}_{m_k}, \underbrace{1, \dots, 1}_{m_{k+1}})$, $m_1 + \cdots + m_{k+1} = n-1$, *then non-isometric inner products are represented by (3) with*

$$x = (\hat{x}_1, \underbrace{0, \dots, 0}_{m_1 - 1}, \hat{x}_2, \underbrace{0, \dots, 0}_{m_2 - 1} \dots, \hat{x}_{k+1}, \underbrace{0, \dots, 0}_{m_{k+1} - 1})^T \in \mathbb{R}^{n-1}, \quad \hat{x}_i \geq 0.$$

Proof. We are looking for the simplest representatives of the orbits of Aut (ch_n) acting on the space of positive definite symmetric matrices. Consider an arbitrary positive definite symmetric $2n \times 2n$ matrix S in a block matrix form

$$S = \begin{pmatrix} p_1 & z_0^T & q \\ z_0 & \bar{S} & w \\ q & w^T & \beta_1 \end{pmatrix}.$$

A generalized translation $F_{t_1}(-\frac{1}{\beta_1}w, 0) \in T$ simplifies S to

$$S_1 = F_{t_1}^T S F_{t_1} = \begin{pmatrix} p_2 & z_1^T & q_1 \\ z_1 & \bar{S}_1 & 0 \\ q_1 & 0 & \beta_1 \end{pmatrix}.$$

Since S_1 is positive definite, $\bar{S}_1$ is also a positive definite $(n-2) \times (n-2)$ matrix. Thus $\bar{S}_1$ allows Wiliamson's diagonalization (Theorem 1.1) by a symplectic matrix M_1. We choose $F_{Sp_2}(M_1) \in Sp$ to obtain

$$S_2 = F_{Sp_2}^T S_1 F_{Sp_2} = \begin{pmatrix} p_2 & z_1^T M_1 & q_1 \\ M_1^T z_1 & M_1^T \bar{S}_1 M_1 & 0 \\ q_1 & 0 & \beta_1 \end{pmatrix} = \begin{pmatrix} p_2 & z_2^T & q_1 \\ z_2 & D_{\bar{\sigma}} & 0 \\ q_1 & 0 & \beta_1 \end{pmatrix},$$

where $D_{\bar{\sigma}} = \mathrm{diag}(\bar{\sigma}_1, \ldots, \bar{\sigma}_{n-1}, \bar{\sigma}_1, \ldots, \bar{\sigma}_{n-1})$ and $\bar{\sigma}_1 \geq \cdots \geq \bar{\sigma}_{n-1} > 0$ form the symplectic spectrum of $\bar{S}_1$. Now we choose a simple translation $F_{t_3}(0, -\frac{q_1}{\beta_1}) \in T$ to obtain

$$S_3 = F_{t_3}^T S_2 F_{t_3} = \begin{pmatrix} p_2 - \frac{q_1^2}{\beta_1} & z_2^T & 0 \\ z_2 & D_{\bar{\sigma}} & 0 \\ 0 & 0 & \beta_1 \end{pmatrix} = \begin{pmatrix} p & z_2^T & 0 \\ z_2 & D_{\bar{\sigma}} & 0 \\ 0 & 0 & \beta_1 \end{pmatrix}.$$

This is further simplified by a diagonal automorphism $F_{d_4}((\bar{\sigma}_{n-1})^{-\frac{1}{2}}) \in D$

$$S_4 = F_{d_4}^T S_3 F_{d_4} = \begin{pmatrix} p & \alpha z_2^T & 0 \\ \alpha z_2 & \alpha^2 D_{\bar{\sigma}} & 0 \\ 0 & 0 & \alpha^2 \beta_1 \end{pmatrix} = \begin{pmatrix} p & z^T & 0 \\ z & D & 0 \\ 0 & 0 & \beta \end{pmatrix},$$

where

$$\begin{aligned} D &= \mathrm{diag}(\frac{\bar{\sigma}_1}{\bar{\sigma}_{n-1}}, \ldots, \frac{\bar{\sigma}_{n-2}}{\bar{\sigma}_{n-1}}, 1, \frac{\bar{\sigma}_1}{\bar{\sigma}_{n-1}}, \ldots, \frac{\bar{\sigma}_{n-2}}{\bar{\sigma}_{n-1}}, 1) \\ &= \mathrm{diag}(\sigma_1, \ldots, \sigma_{n-2}, 1, \sigma_1, \ldots, \sigma_{n-2}, 1) \\ &= \mathrm{diag}(\sigma, \sigma). \end{aligned}$$

Now consider a symplectic matrix M that preserves Williamson's diagonal form $M^T D M = D$. Then (Corollary 1.1)

$$M \in U(n-1) \cong Sp(2(n-1), \mathbb{R}) \cap SO(2(n-1)). \tag{4}$$

Since M is orthogonal, i.e. $M^T = M^{-1}$, it follows that $DM = MD$. M and D are both diagonalizable and they commute, therefore they are simultaneously diagonalizable and they have same eigenspaces. Thus M must preserve eigenspaces of each symplectic eigenvalue from $\mathrm{Spec}(D)$.

- Consider first the most general case when all symplectic eigenvalues of D are distinct. Then corresponding eigenspaces are two-dimensional and can be viewed as the complex line $\mathbb{C}$. Since $Sp(2) = SO(2) = U(1)$, the restriction of M on each two-dimensional eigenspace is a rotation in $\mathbb{C}$. When we identify $\mathbb{R}^{2(n-1)} \cong \mathbb{C}^{n-1}$ by
$$z^T = (\bar{x}_1, \dots, \bar{x}_{n-1}, \bar{y}_1, \dots, \bar{y}_{n-1}) \cong (z_1, \dots, z_{n-1}),$$
the transformation
$$(z_1, \dots, z_{n-1}) \mapsto (\mathrm{e}^{i\theta_1} z_1, \dots, \mathrm{e}^{i\theta_{n-1}} z_{n-1})$$
is symplectic for any choice of angles $\theta = (\theta_1, \dots, \theta_{n-1})$. If we choose θ such that
$$\mathrm{e}^{i\theta_k} z_k = x_k > 0, \quad k \in \{1, \dots, n-1\},$$
the transformation becomes
$$\begin{aligned} z^T &= (\bar{x}_1, \dots, \bar{x}_{n-1}, \bar{y}_1, \dots, \bar{y}_{n-1}) \mapsto \hat{x}^T \\ &= (x_1, \dots, x_{n-1}, 0, \dots, 0) = (x^T, 0). \end{aligned}$$
To represent the last transformation by a block matrix we define diagonal matrices
$$A(\theta) = \operatorname{diag}(\cos\theta_1, \dots, \cos\theta_{n-1}), \quad B(\theta) = \operatorname{diag}(\sin\theta_1, \dots, \sin\theta_{n-1}).$$
Thus, the block matrix is
$$M = \begin{pmatrix} A(\theta) & -B(\theta) \\ B(\theta) & A(\theta) \end{pmatrix}.$$
Using $F_{Sp_5}(M) \in Sp$ we get the final form:
$$S_5 = F_{Sp_5}^T S_4 F_{Sp_5} = \begin{pmatrix} p & z^T M & 0 \\ M^T z & M^T D M & 0 \\ 0 & 0 & \beta \end{pmatrix} = \begin{pmatrix} p & \hat{x}^T & 0 \\ \hat{x} & D & 0 \\ 0 & 0 & \beta \end{pmatrix} = \begin{pmatrix} p & x^T & 0 & 0 \\ x & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$$

- If the multiplicity of an symplectic eigenvalue of D is m_i, then we can identify the corresponding $2m_i$-dimensional eigenspace with $\mathbb{C}^{m_i}$ via
$$(z_1, \dots z_{m_i}) \cong (\bar{x}_1, \dots, \bar{x}_{m_i}, \bar{y}_1, \dots, \bar{y}_{m_i}).$$
Since M is unitary (4), it's action on this eigenspace corresponds to the action of $U(m_i)$ on $\mathbb{C}^{m_i}$. $U(m_i)$ acts isometrically transitively on $\mathbb{C}^{m_i}$, hence it can map any $2m_i$-dimensional vector $(\bar{x}_1, \dots, \bar{x}_{m_i}, \bar{y}_1, \dots, \bar{y}_{m_i})$ to the vector of the same length $(\hat{x}_i, 0, \dots, 0)$, i.e.
$$\begin{aligned} &(z_1, \dots z_{m_i}) \cong (\bar{x}_1, \dots, \bar{x}_{m_i}, \bar{y}_1, \dots, \bar{y}_{m_i}) \mapsto (\hat{x}_i, 0, \dots, 0), \\ &\hat{x}_i^2 = \bar{x}_1^2 + \dots + \bar{x}_{m_i}^2 + \bar{y}_1^2 + \dots + \bar{y}_{m_i}^2. \end{aligned}$$
The same method applied to all the eigenspaces simplifies the vector x to:
$$x = (\hat{x}_1, \underbrace{0, \dots, 0}_{m_1 - 1}, \hat{x}_2, \underbrace{0, \dots, 0}_{m_2 - 1} \dots, \hat{x}_{k+1}, \underbrace{0, \dots, 0}_{m_{k+1} - 1})^T \in \mathbb{R}^{n-1}, \quad \hat{x}_i \geq 0. \qquad (5)$$

Now we check when two different matrices of the form (3) correspond to non-isometric inner products. Let $S(p, x, \sigma, \beta)$ and $S'(p', x', \sigma', \beta')$ be two matrices from the same orbit

$$S' = F^T S F, \quad F = \begin{pmatrix} 1 & 0 & 0 \\ u & \bar{M} & 0 \\ a & v^T & \lambda \end{pmatrix} \in \mathrm{Aut}(ch_n),$$

with x and x' of the form (5). We shall prove that $S = S'$.

If we denote $z = (x, 0)$ and $z' = (x', 0)$ we have

$$\begin{aligned} S' &= \begin{pmatrix} p' & z'^T & 0 \\ z' & D' & 0 \\ 0 & 0 & \beta' \end{pmatrix} \\ &= \begin{pmatrix} p + u^T z + (z^T + u^T D)u + a^2\beta & (z^T + u^T D)\bar{M} + a\beta v^T & a\beta\lambda \\ \bar{M}^T(z + Du) + a\beta v & \bar{M}^T D\bar{M} + \beta v v^T & \beta\lambda v \\ a\beta\lambda & \beta\lambda v^T & \beta\lambda^2 \end{pmatrix} \end{aligned}$$

Since $\beta \neq 0$ and $\lambda \neq 0$, we immediately get $a = 0$, $v = 0$ (thus $u = 0$), so the translatory part of the automorphism F is identity and

$$S' = \begin{pmatrix} p & z^T\bar{M} & 0 \\ \bar{M}^T z & \bar{M}^T D\bar{M} & 0 \\ 0 & 0 & \beta\lambda^2 \end{pmatrix}.$$

Therefore $p' = p$ and $D' = \bar{M}^T D\bar{M}$.

Assume $\lambda > 0$ and choose $\alpha > 0$ such that

$$F = \mathrm{diag}(1, \bar{M}, \lambda) = \mathrm{diag}(1, \alpha M, \alpha^2).$$

If $\lambda < 0$ the procedure is analogous with the same result. The only difference is $-\alpha^2$ as the last entry of the previous matrix. Now we have

$$D' = \bar{M}^T D\bar{M} = \alpha^2 M^T D M.$$

The symplectic spectrum $(\sigma_1, \ldots, \sigma_{n-2}, 1)$ is a symplectic invariant by Proposition 1.1 and $D = \mathrm{diag}(\sigma, \sigma)$ is a sorted Williamson diagonal form. $M^T D M$ has the same spectrum as D and it is proportional to a sorted diagonal matrix $D' = \mathrm{diag}(\sigma', \sigma')$. Hence $M^T D M$ is diagonal, has the same eigenvalues as D, both are sorted, therefore $M^T D M = D$. Thus we have $D' = \alpha^2 D$, i.e.

$$\mathrm{diag}(\sigma'_1, \ldots, \sigma'_{n-2}, 1, \sigma'_1, \ldots, \sigma'_{n-2}, 1) = \alpha^2 \,\mathrm{diag}(\sigma_1, \ldots, \sigma_{n-2}, 1, \sigma_1, \ldots, \sigma_{n-2}, 1).$$

Consequently $\alpha^2 = 1$, so $\alpha = 1$ and

$$S' = \begin{pmatrix} p & z^T M & 0 \\ M^T z & D & 0 \\ 0 & 0 & \beta \end{pmatrix}. \tag{6}$$

Therefore, $\beta' = \beta$ and $\sigma' = \sigma$.

As shown before, M is an unitary matrix that preserves eigenspaces of each symplectic eigenvalue from $\mathrm{Spec}(D)$. If $\tilde{M}$ is restriction of M to the eigenspace $\mathbb{C}^{m_i}$ then (6) implies $(\hat{x}'_i, 0, \ldots, 0) = \tilde{M}(\hat{x}_i, 0, \ldots, 0)$. Since the unitary group preserves the length of a vector, it follows that $\hat{x}'_i = \hat{x}_i$. Therefore we have proven $x = x'$ and consequently $S = S'$. $\square$

Remark 2.1. In the following, we will use the simpler notation $x = (x_1, \ldots, x_n)$. Strictly speaking, the exact relation of indices is
$\hat{x}_1 = x_1, \quad \hat{x}_i = x_{\sum_{j=1}^{i-1} m_j + 1}, \ i \in \{2, \ldots, k\}$, with all the rest x_j being 0.

Remark 2.2. Every inner product on a Lie algebra uniquely determines a left invariant metric on a Lie group. Thus the previous theorem gives us a classification of all non-isometric Riemannian metrics on the Lie group $\mathcal{CH}^n$.

3. Curvature Properties of Lie Group $\mathcal{CH}^n$

We have classified all non-isometric left invariant Riemannian metrics on $\mathcal{CH}^n$ and now we will show some useful curvature properties of these metrics. Let $(X, Y_1, \ldots, Y_{n-1}, Z_1, \ldots, Z_{n-1}, W)$ be a left invariant basis of ch_n with commutators (1). Denote by $g(p, x, \sigma, \beta)$ a left invariant metric defined by an inner product $S(p, x, \sigma, \beta)$ in the basis given above.

Let ∇ be its Levi–Civita connection. For any left invariant vector fields X_1, X_2, X_3 the Koszul's formula reduces to

$$2g(\nabla_{X_1} X_2, X_3) = g([X_1, X_2], X_3) - g([X_2, X_3], X_1) + g([X_3, X_1], X_2). \quad (7)$$

From Koszul's formula (7) using the fact that ∇ is torsion-free, i.e. $\nabla_{X_1} X_2 - \nabla_{X_2} X_1 = [X_1, X_2]$, we find all non-zero covariant derivatives:

$$\nabla_X X = \frac{1}{2z}\left(\sum_{i=1}^{n-1} \frac{x_i^2}{\sigma_i} X - p \sum_{i=1}^{n-1} \frac{x_i}{\sigma_i} Y_i\right), \quad \nabla_X Y_i = \frac{x_i}{2z}\left(X - \sum_{k=1}^{n-1} \frac{x_k}{\sigma_k} Y_k\right),$$
$$\nabla_{Y_i} X = \frac{x_i}{2z}\left(X - \sum_{k=1}^{n-1} \frac{x_k}{\sigma_k} Y_k\right) - \frac{1}{2} Y_i, \quad \nabla_{Y_i} Y_j = \frac{\delta_{ij}\sigma_i}{2z}\left(X - \sum_{k=1}^{n-1} \frac{x_k}{\sigma_k} Y_k\right),$$
$$\nabla_{Y_i} Z_j = -\frac{\delta_{ij}}{2} W = -\nabla_{Z_j} Y_i, \quad \nabla_{Y_i} W = \frac{\beta}{2\sigma_i} Z_i = \nabla_W Y_i, \quad \nabla_{Z_i} X = -\frac{1}{2} Z_i,$$
$$\nabla_{Z_i} Z_j = \frac{\delta_{ij}\sigma_i}{2z}\left(X - \sum_{k=1}^{n-1} \frac{x_k}{\sigma_k} Y_k\right), \quad \nabla_W W = \frac{\beta}{z}\left(X - \sum_{i=1}^{n-1} \frac{x_i}{\sigma_i} Y_i\right),$$
$$\nabla_W X = -W, \quad \nabla_W Z_i = \frac{\beta}{2z\sigma_i}\left(x_i X - x_i \sum_{k=1}^{n-1} \frac{x_k}{\sigma_k} Y_k - z Y_i\right) = \nabla_{Z_i} W,$$

where $z = p - \sum_{i=1}^{n-1} \frac{x_i^2}{\sigma_i}$ and $\sigma_{n-1} = 1$. The Riemann curvature operator for all $X_1, X_2 \in ch_n$ is defined by

$$R(X_1, X_2) = \nabla_{X_1} \nabla_{X_2} - \nabla_{X_2} \nabla_{X_1} - \nabla_{[X_1, X_2]}.$$

Expressions for the Riemann curvature operator are quite complex so it is convenient to present them in terms of wedge products (see Appendix for the explicit formulas). Since we work with left invariant basis, we have a natural identification $T_p\mathcal{CH}^n \cong ch_n$ for any $p \in \mathcal{CH}^n$. Using the symmetry $g(R(X_1, X_2)X_3, X_4) = -g(R(X_1, X_2)X_4, X_3)$ we know that the Riemann curvature operator belongs to the algebra of skew-symmetric endomorphisms

$so(g)$. Accordingly, we identify the skew-symmetric endomorphisms $so(g)$ with 2-vectors $\Lambda^2 T_p\mathcal{CH}^n$ by

$$(X_1 \wedge X_2)(X_3) := g(X_2, X_3)X_1 - g(X_1, X_3)X_2,$$

for any $X_3 \in T_p\mathcal{CH}^n$. Since $R(X_1, X_2) = -R(X_2, X_1)$ the curvature tensor can be regarded as a skew-symmetric operator on the space of 2-vectors

$$R : \Lambda^2 T_p\mathcal{CH}^n \to \Lambda^2 T_p\mathcal{CH}^n \cong so(g), \qquad R(X_1 \wedge X_2) := R(X_1, X_2).$$

Lemma 3.1. *The Riemann curvature operator $R : \Lambda^2 ch_n \to \Lambda^2 ch_n$ is given by:*

$$\begin{aligned}
R(X, Y_i) &= -\frac{1}{4z} X \wedge Y_i - \frac{1}{4\sigma_i} Z_i \wedge W,\\
R(X, Z_i) &= \frac{1}{4z\sigma_i}\Bigg(-\sigma_i X \wedge Z_i - 2x_i X \wedge W + x_i \sum_l \frac{x_l}{\sigma_l} Y_l \wedge W + zY_i \wedge W\Bigg),\\
R(X, W) &= \frac{1}{2z}\Bigg[-2X \wedge W + \sum_l \frac{x_l}{\sigma_l} Y_l \wedge W\\
&\quad +\beta \sum_m \frac{1}{\sigma_m^2}\Bigg(-\frac{3}{2} x_m X \wedge Z_m + x_m \sum_l \frac{x_l}{\sigma_l} Y_l \wedge Z_m + zY_m \wedge Z_m\Bigg)\Bigg],\\
R(Y_i, Y_j) &= -\frac{1}{4z} Y_i \wedge Y_j - \frac{\beta}{4\sigma_i\sigma_j} Z_i \wedge Z_j,\\
R(Y_i, Z_j) &= \frac{1}{4z\sigma_i\sigma_j}\Bigg[-x_j\sigma_i Y_i \wedge W + 2\delta_{ij}\sigma_i\sigma_j \Bigg(X \wedge W - \sum_l \frac{x_l}{\sigma_l} Y_l \wedge W\Bigg)\\
&\quad -\sigma_i\sigma_j Y_i \wedge Z_j + 2\delta_{ij}\sigma_i\sigma_j\beta \sum_m \Bigg[\frac{1}{\sigma_m^2}\Bigg(x_m X \wedge Z_m\\
&\quad -x_m \sum_l \frac{x_l}{\sigma_l} Y_l \wedge Z_m - zY_m \wedge Z_m\Bigg)\Bigg]\\
&\quad +\beta\Bigg(x_j X \wedge Z_i - x_j \sum_l \frac{x_l}{\sigma_l} Y_l \wedge Z_i - zY_j \wedge Z_i\Bigg)\Bigg],\\
R(Y_i, W) &= \frac{\beta}{4z\sigma_i}\Bigg[\frac{x_i}{\sigma_i}\Bigg(-X \wedge W + \sum_l \frac{x_l}{\sigma_l} Y_l \wedge W\Bigg) + \Bigg(\frac{z}{\sigma_i} - \frac{2\sigma_i}{\beta}\Bigg) Y_i \wedge W\\
&\quad +X \wedge Z_i - \sum_l \frac{x_l}{\sigma_l} Y_l \wedge Z_i - \sigma_i \sum_l \frac{x_l}{\sigma_l^2} Y_i \wedge Z_l\Bigg],\\
R(Z_i, Z_j) &= \frac{1}{4z\sigma_i\sigma_j}\Bigg[(x_i\sigma_j Z_j \wedge W - x_j\sigma_i Z_i \wedge W) - \sigma_i\sigma_j Z_i \wedge Z_j\\
&\quad +\beta\Bigg(x_i X \wedge Y_j - x_j X \wedge Y_i
\end{aligned}$$

$$
-\sum_l \frac{x_l}{\sigma_l}\big(x_i Y_l \wedge Y_j - x_j Y_l \wedge Y_i\big) - z Y_i \wedge Y_j\Bigg)\Bigg],
$$

$$
\begin{aligned}
R(Z_i, W) = \frac{\beta}{4z}\Bigg(&-\frac{1}{\sigma_i} X \wedge Y_i + \frac{1}{\sigma_i}\sum_l \frac{x_l}{\sigma_l} Y_l \wedge Y_i - \sum_l \frac{x_l}{\sigma_l^2} Z_l \wedge Z_i \\
&+ \left(\frac{z}{\sigma_i^2} - \frac{2}{\beta}\right) Z_i \wedge W + \frac{x_i}{\sigma_i}\sum_l \frac{x_l}{\sigma_l^2} Z_l \wedge W\Bigg).
\end{aligned}
$$

Proof. For example, let's prove the formula for $R(Zi, W)$. Start by applying this operator to the vector field X by definition

$$
\begin{aligned}
R(Z_i, W)X &= \nabla_{Z_i}\nabla_W X - \nabla_W \nabla_{Z_i} X - \nabla_{[Z_i, W]} X \\
&= \frac{\beta}{4z\sigma_i}\left(-x_i X + x_i \sum_l \frac{x_l}{\sigma_l} Y_l + z Y_i\right).
\end{aligned}
$$

When we apply all wedge products from the standard basis of $\Lambda^2 T_p \mathcal{CH}^n$ to X we see that the only basis elements that give vector fields X and Y_i in the result are $(X \wedge Y_i)X = x_i X - pY_i$ and $(Y_i \wedge Y_j)X = x_j Y_i - x_i Y_j$. Hence, we have a linear combination $R(Z_i, W)X = \mu(X \wedge Y_i)X + \eta(Y_l \wedge Y_i)X$. Consequently

$$
\frac{\beta}{4z\sigma_i}\left(-x_i X + x_i \sum_l \frac{x_l}{\sigma_l} Y_l + z Y_i\right) = \mu(x_i X - pY_i) + \eta(x_l Y_i - x_i Y_l).
$$

Comparing the coefficients by the vector X we obtain $\mu = -\frac{\beta}{4z\sigma_i}$. When we substitute μ and $p = \sum_l \frac{x_l^2}{\sigma_l} + z$ in the previous equation, it gets simplified to

$$
\frac{\beta}{4z\sigma_i}\left(x_i \sum_l \frac{x_l}{\sigma_l} Y_l + z Y_i\right) = -\frac{\beta}{4z\sigma_i}\Big(-\sum_l \frac{x_l^2}{\sigma_l} + z\Big) Y_i + \eta(x_l Y_i - x_i Y_l).
$$

As a result we find $\eta = \frac{\beta}{4z\sigma_i}\sum_l \frac{x_l}{\sigma_l}$. Therefore we have represented

$$
R(Z_i, W)X = -\frac{\beta}{4z\sigma_i}(X \wedge Y_i)(X) + \frac{\beta}{4z\sigma_i}\sum_l \frac{x_l}{\sigma_l}(Y_l \wedge Y_i)(X)
$$

in terms of wedge product. In this way we have found the first two summands in the expression of $R(Z_i, W)$. Repeating the procedure by applying the operator $R(Z_i, W)$ to vectors Y_j, Z_j, W one will similarly find the coefficients with the remaining summands $Z_l \wedge Z_i$, $Z_i \wedge W$ and $Z_l \wedge W$.

All the remaining curvature operators have been found by analogous lengthy calculations. □

Using the definitions of the Ricci curvature tensor and scalar curvature

$$
\begin{aligned}
&\mathrm{Ric}(X_1, X_2) = \mathrm{Tr}(X \mapsto R(X, X_1)X_2), \quad X_1, X_2 \in ch_n \\
&\tau = \sum g^{ij} r_{ij},
\end{aligned}
$$

we calculate them directly:

Lemma 3.2. *For all left invariant Riemannian metrics $g(p, x, \sigma, \beta)$*

(i) *the Ricci curvature tensor in a basis with commutators (1) is*

$$\mathrm{Ric} = -\frac{1}{2z}\begin{pmatrix} np+z & nx^T & 0 & 0 \\ nx & n\sigma+\beta z\sigma^{-1} & 0 & 0 \\ 0 & 0 & n\sigma+\beta z\sigma^{-1}+\beta vv^T & (2n+1)\frac{\beta}{2}v \\ 0 & 0 & (2n+1)\frac{\beta}{2}v^T & 2n\beta-\beta^2\sum_{k=1}^{n-1}\frac{1}{\sigma_k^2}\left(\frac{x_k^2}{\sigma_k}+z\right) \end{pmatrix}, \quad (8)$$

where
$x = (x_1, \ldots, x_{n-1})^T, \quad \sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_{n-2}, 1), \quad \sigma_{n-1} = 1,$
$v = \sigma^{-1}x = \left(\frac{x_i}{\sigma_i}\right), \quad z = p - \sum_{i=1}^{n-1}\frac{x_i^2}{\sigma_i}.$

(ii) $\mathcal{CH}^n$ *has a constant negative scalar curvature:*

$$\tau = -\frac{1}{2z}\left[2n^2+n+1+\beta\sum_{i=1}^{n-1}\frac{1}{\sigma_i^2}\left(z+\frac{x_i^2}{\sigma_i}\right)\right].$$

Proof. The formulas follow directly from the calculated Levi–Civita connection and Riemann curvature operators. Since $\sigma_i \geq 1, \ i \in \{1, \ldots, n-1\}$, and $\beta, z > 0$, the scalar curvature is strictly negative. □

Remark 3.1. This is consistent with Milnor's result ([21], Theorem 3.1) that a scalar curvature of a non-flat left invariant Riemannian metric on a solvable Lie group is strictly negative.

Remark 3.2. Ricci negative metrics on $\mathcal{CH}^n$ exist. For example, if $x_i = 0$ and $2n - p\beta\sum_{i=1}^{n-1}\frac{1}{\sigma_i^2} > 0$, all eigenvalues are negative. In that case

$$\mathrm{Ric} = -\frac{1}{2p}\,\mathrm{diag}\left((n+1)p, u, u, 2n\beta - \beta^2 p\sum_{k=1}^{n-1}\frac{1}{\sigma_k^2}\right),$$

where $u = \left(n\sigma_1 + \frac{\beta p}{\sigma_1}, \ldots, n\sigma_{n-2} + \frac{\beta p}{\sigma_{n-2}}, n + \beta p\right) \in \mathbb{R}^{n-1}$.

Definition 3.1. A metric is Einstein if its Ricci tensor is proportional to the metric tensor.

Theorem 3.1. *The left invariant Riemannian metric* $g(p, x, \sigma, \beta)$ *on* $\mathcal{CH}^n$ *is Einstein if and only if the following conditions are satisfied:*

$$p\beta = 1, \quad x_i = 0, \quad \sigma_i = 1, \quad i \in \{1, \ldots, n-1\},$$

i.e. the corresponding inner product is

$$S = \mathrm{diag}\left(p, 1, \ldots, 1, \frac{1}{p}\right). \quad (9)$$

This is the standard Kähler metric of the complex hyperbolic space $\mathbb{C}H^n$*, unique up to isometry and scaling.*

Proof. Direct comparison of the Ricci tensor (8) with the metric tensor (3) shows that $Ric = kg$ holds for some constant k iff $p\beta = 1$ and $(\forall i)\ x_i = 0, \ \sigma_i = 1$. The resulting metrics (9) is the standard Kähler metric of the complex hyperbolic space with constant holomorphic sectional curvature $-\frac{1}{p}$.

Now we check the uniqueness up to scaling. Consider a scaled metric

$$qS = \operatorname{diag}\left(pq, q, \ldots, q, \frac{q}{p}\right), \quad q > 0.$$

The action of a diagonal automorphism $F = \operatorname{diag}\left(1, \frac{1}{\sqrt{q}}, \ldots, \frac{1}{\sqrt{q}}, \frac{1}{q}\right) \in D$

$$F^T(qS)F = \operatorname{diag}\left(pq, 1, \ldots, 1, \frac{1}{pq}\right)$$

shows that qS is in the orbit of the standard Kähler metric of the complex hyperbolic space with constant holomorphic sectional curvature $-\frac{1}{pq}$. □

Remark 3.3. This is consistent with Herber [13], Theorem E: Any solvable Lie group S admits at most one left invariant standard Einstein metric up to isometry and scaling. If it does, then S does not carry any nonstandard Einstein metric and hence the Einstein metric is essentially unique.

A complete Riemannian metric g on a manifold M is called a *Ricci soliton* if

$$\operatorname{Ric}(g) = cI + L_X g \tag{10}$$

for some smooth vector field X on M and $c \in \mathbb{R}$.

Given a Lie group G, a left invariant metric g (identified with the inner product on a Lie algebra $\mathfrak{g}$) is called an *algebraic Ricci soliton* if

$$\operatorname{Ric}(g) = cI + D \tag{11}$$

for some $c \in \mathbb{R}$, $D \in Der(\mathfrak{g})$. If G is nilpotent, the metric g is called *nilsoliton*; if G is solvable, then g is *solsoliton*.

Any Riemannian algebraic Ricci soliton is automatically a Ricci soliton, i.e. (11) implies (10). For a left invariant Riemannian metric, converse is also true when G is nilpotent or completely solvable (see [19,20]). Furthermore, a given solvable Lie group admits at most one Ricci soliton left invariant metric up to isometry and scaling ([20], Theorem 5.1, Remark 5.2). Since Einstein metric is a trivial example of Ricci soliton and the Lie group $\mathcal{CH}^n$ is completely solvable, we conclude:

Corollary 3.1. *The only Ricci soliton left invariant metric on $\mathcal{CH}^n$ up to isometry and scaling is Einstein metric (9).*

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Appendix A: Explicit Expression of Riemann Curvature Operator

$$
\begin{aligned}
R(X,Y_i)X &= \frac{1}{4z}(-x_iX + pY_i)\\
R(X,Y_i)Y_j &= \frac{1}{4z}(x_jY_i - \delta_{ij}\sigma_iX)\\
R(X,Y_i)Z_j &= \frac{\delta_{ij}}{4}W\\
R(X,Y_i)W &= -\frac{\beta}{4\sigma_i}Z_i
\end{aligned}
$$

$$
\begin{aligned}
R(X,Z_i)X &= \frac{p}{4z}\left(Z_i + \frac{x_i}{\sigma_i}W\right)\\
R(X,Z_i)Y_j &= \frac{x_i}{4z}\left(Z_i + \frac{x_i}{\sigma_i}W - \delta_{ij}\frac{z}{x_j}W\right)\\
R(X,Z_i)Z_j &= -\frac{\sigma_i}{4z}\delta_{ij}X\\
R(X,Z_i)W &= \frac{\beta}{4z\sigma_i}\left(-2x_iX + x_i\sum_l \frac{x_l}{\sigma_l}Y_l + zY_i\right)
\end{aligned}
$$

$$
\begin{aligned}
R(X,W)X &= \frac{p\beta}{4z}\sum_l \frac{x_l}{\sigma_l^2}Z_l + \frac{z+p}{2z}W\\
R(X,W)Y_i &= \frac{x_i\beta}{4z}\sum_l \frac{x_l}{\sigma_l^2}Z_l - \frac{\beta}{2\sigma_i}Z_i + \frac{x_i}{2z}W\\
R(X,W)Z_i &= \frac{\beta}{4z\sigma_i}\left(-3x_iX + 2x_i\sum_l \frac{x_l}{\sigma_l}Y_l + 2zY_i\right)\\
R(X,W)W &= \frac{\beta}{z}\left(-X + \frac{1}{2}\sum_l \frac{x_l}{\sigma_l}Y_l\right)
\end{aligned}
$$

$$
R(Y_i,Y_j)X = \frac{1}{4z}(x_iY_j - x_jY_i)
$$

$$
\begin{aligned}
R(Y_i, Y_j)Y_k &= \frac{1}{4z}(\delta_{ik}\sigma_i Y_j - \delta_{jk}\sigma_j Y_i)\\
R(Y_i, Y_j)Z_k &= \frac{\beta}{4}\left(\frac{\delta_{ik}}{\sigma_j}Z_j - \frac{\delta_{jk}}{\sigma_i}Z_i\right)\\
R(Y_i, Y_j)W &= 0
\end{aligned}
$$

$$
\begin{aligned}
R(Y_i, Z_j)X &= \frac{1}{4z}\left[x_i Z_j + \left(\frac{x_i x_j}{\sigma_j} - 2\delta_{ij} z\right) W\right]\\
R(Y_i, Z_j)Y_k &= \frac{1}{4}\left(\delta_{jk}\frac{\beta}{\sigma_i}Z_i + \delta_{ik}\frac{\sigma_i}{z}Z_j + 2\,\delta_{ij}\frac{\beta}{\sigma_k}Z_k + \delta_{ik}\frac{x_j\sigma_i}{z\sigma_j}W\right)\\
R(Y_i, Z_j)Z_k &= \frac{1}{4z}\left[2\,\delta_{ij}\frac{\beta}{\sigma_k}\left(x_k X - zY_k - x_k\sum_l \frac{x_l}{\sigma_l}Y_l\right)\right.\\
&\quad \left. +\delta_{ik}\frac{\beta}{\sigma_j}\left(x_j X - zY_j - x_j\sum_l \frac{x_l}{\sigma_l}Y_l\right) - \delta_{jk}\sigma_j Y_i\right]\\
R(Y_i, Z_j)W &= \frac{\beta}{4z}\left[-\frac{x_j}{\sigma_j}Y_i + 2\,\delta_{ij}(X - \sum_l \frac{x_l}{\sigma_l}Y_l)\right]\\
R(Y_i, W)X &= \frac{1}{4z}\left(\beta x_i\sum_l \frac{x_l}{\sigma_l^2}Z_l - \frac{\beta z}{\sigma_i}Z_i + 2\,x_i W\right)\\
R(Y_i, W)Y_j &= \frac{\beta}{4z}\delta_{ij}\left[\sigma_i\sum_l \frac{x_l}{\sigma_l^2}Z_l + (2\,\frac{\sigma_i}{\beta} - \frac{z}{\sigma_i})W\right]\\
R(Y_i, W)Z_j &= \frac{\beta}{4z}\left[\delta_{ij}(X - \sum_l \frac{x_l}{\sigma_l}Y_l) - \frac{x_j}{\sigma_j}Y_i\right]\\
R(Y_i, W)W &= \frac{\beta^2}{4z\sigma_i^2}\left[-x_i X + x_i\sum_l \frac{x_l}{\sigma_l}Y_l + (z - 2\frac{\sigma_i^2}{\beta})Y_i\right]
\end{aligned}
$$

$$
\begin{aligned}
R(Z_i, Z_j)X &= 0\\
R(Z_i, Z_j)Y_k &= \frac{\beta}{4z}\left[\frac{\delta_{ik}}{\sigma_j}(-x_j X + x_j\sum_l \frac{x_l}{\sigma_l}Y_l + zY_j)\right.\\
&\quad \left. -\frac{\delta_{jk}}{\sigma_i}(-x_i X + x_i\sum_l \frac{x_l}{\sigma_l}Y_l + zY_i)\right]\\
R(Z_i, Z_j)Z_k &= \frac{1}{4z}\left[\delta_{ik}\sigma_i(Z_j + \frac{x_j}{\sigma_j}W) - \delta_{jk}\sigma_j(Z_i + \frac{x_i}{\sigma_i}W)\right]\\
R(Z_i, Z_j)W &= \frac{\beta}{4z}\left(\frac{x_i}{\sigma_i}Z_j - \frac{x_j}{\sigma_j}Z_i\right)
\end{aligned}
$$

$$
R(Z_i, W)X = \frac{\beta}{4z\sigma_i}\left(-x_i X + x_i\sum_l \frac{x_l}{\sigma_l}Y_l + zY_i)\right)
$$

$$
\begin{aligned}
R(Z_i, W)Y_j &= \frac{\beta}{4z}\delta_{ij}\left(-X + \sum_l \frac{x_l}{\sigma_l}Y_l\right)\\
R(Z_i, W)Z_j &= \frac{\beta}{4z}\left[-\frac{x_j}{\sigma_j}Z_i + \delta_{ij}\sigma_i \sum_l \frac{{x_l}^2}{\sigma_l}Z_l + \left(\delta_{ij}\left(2\frac{\sigma_i}{\beta} - \frac{z}{\sigma_j}\right) - \frac{x_i x_j}{\sigma_i\sigma_j}\right)W\right]\\
R(Z_i, W)W &= \frac{\beta^2}{4z}\left[\left(\frac{z}{\sigma_i^2} - \frac{2}{\beta}\right)Z_i + \frac{x_i}{\sigma_i}\sum_l \frac{x_l}{\sigma_l^2}Z_l\right]
\end{aligned}
$$

References

[1] Alekseevsky, D.V.: Conjugacy of polar factorizations of Lie groups. Mat. Sbornik 84 **13**, 14–26 (1971)

[2] Alekseevsky, D.V.: Homogeneous Riemannian spaces of negative curvature. Mat. Sbornik 96 **25**, 87–109 (1975)

[3] Bérard-Bérgery, L.: Homogeneous Riemannian spaces of dimension four. Seminar A. Besse, Four-dimensional Riemannian geometry (1985)

[4] Bokan, N., Šukilović, T., Vukmirović, S.: Lorentz geometry of 4-dimensional nilpotent Lie groups. Geom. Dedic. **177**, 83–102 (2015)

[5] Calvaruso, G., Zaeim, A.: Four-dimensional Lorentzian Lie groups. Differ. Geom. Appl. **31**, 496–509 (2013)

[6] Christodoulakis, T., Papadopoulos, G.O., Dimakis, A.: Automorphisms of real four-dimensional Lie algebras and the invariant characterization of homogeneous 4-spaces. J. Phys. A Math. Gen. **36**, 427–441 (2002)

[7] Cordero, L.A., Parker, P.E.: Left-invariant Lorentz metrics on 3-dimensional Lie groups. Rend. Mat. Appl. **17**, 129–155 (1997)

[8] Domínguez-Vázquez, M., Sanmartín-López, V., Tamaru, H.: Codimension one Ricci soliton subgroups of solvable Iwasawa groups. Journal de Mathématiques Pures et Appliquées **152**, 69–93 (2021)

[9] de Gosson, M.: Symplectic Geometry and Quantum Mechanics. Birkhäuser, Boston (2006)

[10] Hashinaga, T., Tamaru, H., Terada, K.: Milnor-type theorems for left-invariant Riemannian metrics on Lie groups. J. Math. Soc. Jpn. **68**, 669–684 (2016)

[11] Heintze, E.: On homogeneous manifolds of negative curvature. Math. Ann. **211**, 23–34 (1974)

[12] Heintze, E.: Riemannsche Solvmannigfaltigkeiten. Geom. Dedic. **1**, 141–147 (1973)

[13] Heber, J.: Noncompact homogeneous Einstein spaces. Invent. Math. **133**, 279–352 (1998)

[14] Kodama, H., Takahara, A., Tamaru, H.: The space of left-invariant metrics on a Lie group up to isometry and scaling. Manuscr. Math. **135**, 229–243 (2011)

[15] Kondo, Y., Tamaru, H.: A classification of left-invariant Lorentzian metrics on some nilpotent Lie groups. arXiv:2011.09118 [math.DG] (2020)

[16] Kubo, A., Onda, K., Taketomi, Y., Tamaru, H.: On the moduli spaces of left-invariant pseudo-Riemannian metrics on Lie groups. Hiroshima Math. J. **46**, 357–374 (2016)

[17] Lauret, J.: Degenerations of Lie algebras and geometry of Lie groups. Differ. Geom. Appl. **18**, 177–194 (2003)

[18] Lauret, J.: Homogeneous nilmanifolds of dimension 3 and 4. Geom. Dedic. **68**, 145–155 (1997)

[19] Lauret, J.: Ricci soliton homogeneous nilmanifolds. Math. Ann. **319**, 715–733 (2001)

[20] Lauret, J.: Ricci soliton solvmanifolds. Journal für die reine und angewandte Mathematik **2011**, 1–21 (2011)

[21] Milnor, J.: Curvatures of left invariant metrics on Lie groups. Adv. Math. **21**, 293–329 (1976)

[22] Nasehi, M.: On the geometry of higher dimensional Heisenberg groups. Mediterr. J. Math. **16**, 29 (2019)

[23] Rahmani, S.: Metriques de Lorentz sur les groupes de Lie unimodulaires de dimension 3. J. Geom. Phys. **9**, 295–302 (1992)

[24] Šukilović, T.: Classification of left invariant metrics on 4-dimensional solvable Lie groups. Theor. Appl. Mech. **47**, 181–204 (2020)

[25] Šukilović, T.: Geometric properties of neutral signature metrics on 4-dimensional nilpotent Lie groups. Rev. Un. Mat. Argentina **57**, 23–47 (2016)

[26] Vukmirović, S.: Classification of left-invariant metrics on the Heisenberg group. J. Geom. Phys. **94**, 72–80 (2015)

[27] Vukmirović, S., Šukilović, T.: Geodesic completeness of the left-invariant metrics on $\mathbb{R}H^n$. Ukr. Math. J. **72**, 702–711 (2020)

[28] Williamson, J.: On the algebraic problem concerning the normal forms of linear dynamical systems. Am. J. Math. **58**, 141–163 (1936)

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