Double generalized majorization*

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Abstract

In this paper, we give a complete, explicit and constructive solution to the double generalized majorization problem. Apart from purely combinatorial interest, double generalized majorization problem has strong impact in Matrix and Matrix Pencils Completion Problems, Bounded Rank Perturbation Problems, and it has additional nice interpretation in Representation Theory of Kronecker Quivers.

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1 Introduction

By a partition we mean a finite non-increasing sequence of integers. For any integers $a_1 \geqslant \ldots \geqslant a_s$ we define the corresponding partition $\mathbf{a} = (a_1, \ldots, a_s)$. In the literature there are many ways of comparing two or more partitions [1, 18, 21]. The classical majorization in Hardy-Littlewood-Polya sense [18] that compares two partitions, is one of the best studied and known:

Definition 1. Let $\mathbf{g} = (g_1, \ldots, g_k)$ and $\mathbf{b} = (b_1, \ldots, b_k)$ be two partitions. If

$$\sum_{i=1}^{k} g_i = \sum_{i=1}^{k} b_i$$

and

$$\sum_{i=1}^{j} g_i \leqslant \sum_{i=1}^{j} b_i, \quad j = 1, \ldots, k-1,$$

then we say that $\mathbf{g}$ is *majorized* by $\mathbf{b}$ and write $\mathbf{g} \prec \mathbf{b}$.

The following notation will be used throughout the paper:

Let n, m, s and k be nonnegative integers, such that

$$n + k = m + s. \tag{1}$$

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be the following partitions:

$$\mathbf{a} = (a_1, \ldots, a_s), \tag{2}$$
$$\mathbf{b} = (b_1, \ldots, b_k), \tag{3}$$
$$\mathbf{c} = (c_1, \ldots, c_n), \tag{4}$$
$$\mathbf{d} = (d_1, \ldots, d_m). \tag{5}$$

For any partition $\mathbf{w} = (w_1, \ldots, w_l)$ we shall assume that $w_i := +\infty$, for $i \leqslant 0$, and $w_i := -\infty$, for $i > l$. If $a > b$ are nonnegative integers, then we assume $\sum_{i=a}^{b} w_i := 0$.

In this paper we deal with a generalization of the classical majorization given in [2, 5, 7], that compares three partitions of the appropriate size:

Definition 2. (Generalized majorization) Let $\mathbf{b}$ and $\mathbf{c}$ be partitions as in (3) and (4), respectively. Let $\mathbf{g} = (g_1, \ldots, g_{n+k})$ be a partition. If

$$c_i \geqslant g_{i+k}, \qquad i = 1, \ldots, n, \tag{6}$$
$$\sum_{i=1}^{h_j} g_i - \sum_{i=1}^{h_j - j} c_i \leqslant \sum_{i=1}^{j} b_i, \qquad j = 1, \ldots, k, \tag{7}$$

$$\sum_{i=1}^{n+k} g_i = \sum_{i=1}^{n} c_i + \sum_{i=1}^{k} b_i, \tag{8}$$

where

$$h_j := \min\{i | c_{i-j+1} < g_i\}, \quad j = 1, \dots, k, \tag{9}$$

then we say that $\mathbf{g}$ is *majorized* by $\mathbf{c}$ and $\mathbf{b}$. This type of majorization we call *the generalized majorization*, and we write

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b}).$$

Notice that, if (8) is satisfied, then (7) is equivalent to the following:

$$\sum_{i=h_j+1}^{n+k} g_i \geqslant \sum_{i=h_j-j+1}^{n} c_i + \sum_{i=j+1}^{k} b_i, \quad j = 1, \dots, k. \tag{10}$$

Also, notice that if $n = 0$, generalized majorization gives the classical one.

Definition 3. (Weak generalized majorization) If partitions $\mathbf{g}$, $\mathbf{b}$ and $\mathbf{c}$ in Definition 2 satisfy (6), (10) and

$$\sum_{i=1}^{n+k} g_i \geqslant \sum_{i=1}^{n} c_i + \sum_{i=1}^{k} b_i,$$

then we say that $\mathbf{g}$ is *weakly majorized* by $\mathbf{c}$ and $\mathbf{b}$, and we write

$$\mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}).$$

During the last decade many interesting, purely combinatorial properties of the generalized majorization, including some generalizations of well-known properties of the classical majorization, have been obtained. For the most interesting combinatorial results, see e.g. [4, 5, 8, 11]. These results demonstrate rich structure of the generalized majorization as a combinatorial object, as well as its importance and potential in applications. Indeed, apart from purely combinatorial interest, the generalized majorization has strong impact in Matrix and Matrix Pencils Completion Problems [7, 10, 12, 13], where it appears naturally by studying properties of the Kronecker Invariants of the involved Matrix Pencils [16]. Also, some of its properties are of great importance in solving Bounded rank perturbation problems [4, 9, 22, 23, 28]. Finally, the generalized majorization has additional nice interpretation in Representation Theory of Quivers by using Kronecker quivers [17, 24, 25, 26], as well as exciting diagrammatics introduced in [11].

The problem of particular interest involving the generalized majorization which connects all above mentioned fields is so-called *Double Generalized Majorization Problem*. It appears in Matrix and Matrix Pencils completion problems [2, 7, 10, 12, 13], as well as in Representation Theory of Quivers [24, 25, 26], and Perturbation Theory [4, 9]. It is the central problem of this paper:

Problem 4. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ be partitions given as in (2)-(5). Find necessary and sufficient conditions for the existence of a partition $\mathbf{g} = (g_1, \ldots, g_{m+k+s})$, such that

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}). \tag{11}$$

The solution to Problem 4 given in Theorem 11 is the main result of the paper. In addition, we also solve a stronger version to Problem 4 given by:

Problem 5. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ be partitions given as in (2)-(5). Find necessary and sufficient conditions for the existence of a partition $\mathbf{g} = (g_1, \ldots, g_{m+k+s})$, such that

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}). \tag{12}$$

Both of these problems have been solved in [4] in the case $s = k = 1$. Also, Problem 5 has been studied in [5]. However, it has been realised that the solution in [5, Theorem 5.1] does not cover all of the possible cases, and therefore does not provide a complete solution to Problem 5. In this paper, we improve the main result from [5]. We introduce novel definition of the sets S and Δ, and consequently we give novel, explicit necessary and sufficient conditions, which solve completely and constructively Problems 4 and 5, without any restrictions. This is the main result of the paper given in Theorem 11.

Moreover, in Section 11 we apply the main result obtained in Theorem 11, and obtain simple and elegant necessary and sufficient conditions which do not require the sets S and Δ, for Problems 4 and 5 in some special cases. In particular, we consider these problems when $s = 0$ (or dually when $k = 0$), and also when $n = 0$ (and dually $m = 0$).

Similar combinatorial problems have appeared in different applications in matrix theory, together with a quest for the explicit solution involving inequalities. Among the most famous ones are the Carlson problem [3, 14] and the eigenvalue problem for the sums of Hermitian matrices [15], both of which were shown to be equivalent to the condition that the Littlewood-Richardson coefficient of certain three partitions is non-zero. The last is a purely combinatorial condition involving integer partitions, but yet still an implicit one: it is equivalent to the existence of a certain sequence of partitions (LR sequence). So, one of the main problems was to find an explicit form of this condition which involves only inequalities between the three initial partitions. This was famously solved in [15, 19].

A somewhat similar framework motivates the main problem of this paper. The General Matrix Pencil Completion Problem (GMPCP) reduces to certain combinatorial condition involving several partitions of integers, and among others there is an implicit condition like the one in Problem 4. Therefore, to get the fully explicit necessary and sufficient conditions to GMPCP, it is of high importance to get explicit necessary and sufficient conditions in Problem 4, which depend only on certain inequalities involving the partitions $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{d}$. This is the main result of the present paper.

2 Sets S and Δ

Lemma 6. *[5, Proposition 2.6] Let x be an integer such that there exist $w \in \{1, \dots, n\}$ and $u \in \{1, \dots, m\}$ such that $c_w = d_u = x$. Let*

$$\mathbf{d}' := (d_1, \dots, d_{u-1}, d_{u+1}, \dots, d_m),$$

and

$$\mathbf{c}' := (c_1, \dots, c_{w-1}, c_{w+1}, \dots, c_n).$$

Then there exists a partition $\mathbf{g} = (g_1, \dots, g_{m+s})$ such that

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}),$$

if and only if there exists a partition $\mathbf{g}' = (g'_1, \dots, g'_{m+s-1})$ such that

$$\mathbf{g}' \prec' (\mathbf{d}', \mathbf{a}) \quad \text{and} \quad \mathbf{g}' \prec' (\mathbf{c}', \mathbf{b}).$$

Remark 7. In fact, in the course of proving [5, Proposition 2.6] we have also obtained the corresponding result for the weak generalized majorisation. So, by the notation from Lemma 6, we also have that there exists a partition $\mathbf{g} = (g_1, \dots, g_{m+s})$ such that

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}),$$

if and only if there exists a partition $\mathbf{g}' = (g'_1, \dots, g'_{m+s-1})$ such that

$$\mathbf{g}' \prec'' (\mathbf{d}', \mathbf{a}) \quad \text{and} \quad \mathbf{g}' \prec'' (\mathbf{c}', \mathbf{b}).$$

Now, by Lemma 6 and Remark 7, when solving Problems 4 and 5, i.e. from now on throughout the paper, without loss of generality, we assume that $c_i \neq d_j$ for all $i = 1, \dots, n$, and all $j = 1, \dots, m$.

Let $\mathbf{u}$ be the union of the partitions $\mathbf{c}$ and $\mathbf{d}$. Let $\mathbf{e}$ be the union of the partitions $\mathbf{d}$ and $\mathbf{a}$, and let $\mathbf{e}'$ be the union of the partitions $\mathbf{c}$ and $\mathbf{b}$. Thus, we have

$$\mathbf{u} = (u_1, \dots, u_{n+m}) := (d_1, \dots, d_m) \cup (c_1, \dots, c_n), \tag{13}$$

$$\mathbf{e} = (e_1, \dots, e_{m+s}) := (d_1, \dots, d_m) \cup (a_1, \dots, a_s), \tag{14}$$

$$\mathbf{e}' = (e'_1, \dots, e'_{m+s}) := (c_1, \dots, c_n) \cup (b_1, \dots, b_k). \tag{15}$$

In the definition of e_i's, if $d_i = a_j$, for some $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, s\}$, then let $i_j = \min\{\ell | d_\ell = a_j\}$, and let $v = \min\{\ell | a_\ell = a_j\}$, and $w = \max\{\ell | a_\ell = a_j\}$. Then we put $e_{i_j-1+v} = a_v$, $e_{i_j+v} = a_{v+1}$, $\dots$, $e_{i_j-1+w} = a_w$, $e_{i_j+w} = d_{i_j}$ (i.e. $\mathbf{e} : \dots \geqslant a_v \geqslant \dots \geqslant a_w \geqslant d_{i_j} \geqslant \dots$). Analogously, if $c_i = b_j$, for some $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k\}$, then let $i_j = \min\{\ell | c_\ell = b_j\}$, and let $v = \min\{\ell | b_\ell = b_j\}$, and $w = \max\{\ell | b_\ell = b_j\}$. Then we put $e'_{i_j-1+v} = b_v$, $e'_{i_j+v} = b_{v+1}$, $\dots$, $e'_{i_j-1+w} = b_w$, $e'_{i_j+w} = c_{i_j}$.

Now we can define sets S and Δ for given partitions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$. Sets S and Δ will be subsets of the sets of indices of the partitions $\mathbf{c}$ and $\mathbf{d}$, respectively. More precisely, $S \subset \{1, \dots, n\}$, and $\Delta \subset \{1, \dots, m\}$. Both sets S and Δ, as well as other auxiliary sequences and indices that will be defined in this section, and used throughout the paper, depend solely on the given partitions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$.

Definition 8. (Sets S and Δ)

Definition of the sets S and Δ is given inductively. We start by putting S and Δ to be empty sets, and then we fill them in the following way, step by step:

We shall go through the elements of $\mathbf{u}$ (the union of partitions $\mathbf{c}$ and $\mathbf{d}$), one by one, starting from the smallest one. If there are equals among c_i's or d_i's, we always first choose the element with the largest index (note that we are assuming $c_i \neq d_j$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$). Once we pick one element from $\mathbf{u}$, it is either d_j, for some $j \in \{1, \dots, m\}$, or c_j for some $j \in \{1, \dots, n\}$.

If it is d_j, for some $j \in \{1, \dots, m\}$, since the definition of S and Δ is given inductively, this means that for all i with $j < i \leqslant m$, we have already determined whether $i \in \Delta$ or $i \notin \Delta$. And also for all $i \in \{1, \dots, n\}$ such that $d_j > c_i$, we have already determined whether $i \in S$ or $i \notin S$. Our goal is to determine whether we add the index j to the set Δ or not.

Analogously, if the element from $\mathbf{u}$ that we arrived at is c_j, for some $j \in \{1, \dots, n\}$, inductively we have already determined for all i with $j < i \leqslant n$ whether $i \in S$ os $i \notin S$, as well as for all $i \in \{1, \dots, n\}$ such that $c_j > d_i$, we have already determined whether $i \in \Delta$ or $i \notin \Delta$. Our goal is to determine whether we add the index j to the set Δ or not.

We shall do this by checking couple of inequalities:

– If the element from $\mathbf{u}$ is d_j, for some $j \in \{1, \dots, m\}$, we start with calculating

$$q_j := s - \sharp\{i \in S | c_i < d_j\} + \sharp\{i \in \{j+1, \dots, m\} | i \notin \Delta\} + 1. \tag{16}$$

Next we check the following:

- If $\quad q_j > s \quad \Rightarrow \quad$ then we add j to Δ
- If $\quad q_j \leqslant s \quad \Rightarrow \quad$ then let $l \in S$ be the minimal index such that $d_j > c_l$

  (a) Now, if

$$\sharp\{i \in \{1, \dots, s\} | a_i > c_l\} \geqslant s - \sharp\{i \in S | i > l\} + \sharp\{i \in \{j+1, \dots, m\} | d_i < c_l, i \notin \Delta\}, \tag{17}$$

and if d_j belongs to the smallest

$$\sharp\{i \in \{1, \dots, s\} | a_i > c_l\} - s + \sharp\{i \in S | i > l\} - \sharp\{i \in \{j+1, \dots, m\} | d_i < c_l, i \notin \Delta\} + 1$$

e_i's bigger than c_l, then we do not add j to Δ.[1]

(b) otherwise we check the inequality

$$\sum_{i\in S,\, c_i<d_j} c_i \geqslant \sum_{m\geqslant i>j,\, i\notin\Delta} d_i + d_j + \sum_{i=q_j+1}^{s} a_i. \tag{18}$$

If the equation (18) is satisfied, then we do not add j to Δ, and if the equation (18) is not satisfied then we add j to Δ.

– If the chosen element belongs to $\mathbf{c}$, say c_j, for some $j \in \{1,\dots,n\}$, then we have the dual definition, i.e. we consider

$$q'_j := k - \sharp\{i\in\Delta | d_i < c_j\} + \sharp\{i \in \{j+1,\dots,n\} | i \notin S\} + 1. \tag{19}$$

Then we check the following:

- If $q'_j > k \quad \Rightarrow \quad$ then we add j to S
- If $q'_j \leqslant k \quad \Rightarrow \quad$ then let $l \in \Delta$ be the minimal index such that $c_i > d_l$
  (a) Now, if

$$\sharp\{i\in\{1,\dots,k\}|b_i > d_l\} \geqslant k - \sharp\{i\in\Delta|i>l\} + \sharp\{i\in\{j+1,\dots,n\}|c_i<d_l, i\notin S\}, \tag{20}$$

and if c_j belongs to the smallest

$$\sharp\{i\in\{1,\dots,k\}|b_i > d_l\} - k + \sharp\{i\in\Delta|i>l\} - \sharp\{i\in\{j+1,\dots,n\}|c_i<d_l, i\notin S\} + 1$$

e'_i's bigger than d_l, then we do not add j to S

(b) otherwise we check the inequality

$$\sum_{i\in \Delta,\, d_i<c_j} d_i \geqslant \sum_{n\geqslant i>j,\, i\notin S} c_i + c_j + \sum_{i=q'_j+1}^{k} b_i. \tag{21}$$

If the equation (21) is satisfied, then we do not add j to S, and if the equation (21) is not satisfied then we add j to S.

Now choose the next smallest element in $\mathbf{u}$, and proceed until all the elements in $\mathbf{u}$ are checked. This ends our definition of the sets S and Δ.

Moreover, throughout the paper, the complements of the sets S and Δ will be denoted by S^c and Δ^c, respectively, i.e.

$$S^c := \{1,\dots,n\} \setminus S, \quad \text{and} \quad \Delta^c := \{1,\dots,m\}\setminus\Delta.$$

In order to simplify the notation and presentation of the main result and the proof, we also define the following integers related to the sets S and Δ:

[1]More precisely: let $\phi = \max\{i\in\{0,\dots,m+s\}|e_i > c_l\}$, i.e. ϕ is such that $e_\phi > c_l \geqslant e_{\phi+1}$. Also, let $\nu = \sharp\{i\in\{1,\dots,s\}|a_i>c_l\} - s + \sharp\{i\in S|i>l\} - \sharp\{i\in\{j+1,\dots,m\}|d_i<c_l, i\notin\Delta\}+1$. Then we check whether d_j is among $(e_{\phi-\nu+1}, e_{\phi-\nu+2},\dots,e_\phi)$, and if it is one of these, then we do not add j to Δ.

Definition 9. We shall denote the set of d_i's with $i \in \Delta$ by

$$d^1 \geqslant \cdots \geqslant d^h, \quad \text{where} \quad h = \sharp\Delta.$$

Analogously, we denote all c_i's with $i \in S$ by

$$c^1 \geqslant \cdots \geqslant c^{h'}, \quad \text{where} \quad h' = \sharp S.$$

Furthermore, for every d^j, $j = 1, \ldots, h$, we define

$$\begin{aligned} m'_j &:= \sharp\{i \in \{1,\ldots,k\} | b_i > d^j\} \\ t'_j &:= k - (h-j) + \sharp\{i \in S^c | c_i < d^j\} \\ z'_j &:= \sharp\{i \in \{1,\ldots,n\} | c_i > d^j\}, \end{aligned}$$

and for every c^j, $j = 1, \ldots, h'$, we define

$$\begin{aligned} m_j &:= \sharp\{i \in \{1,\ldots,s\} | a_i > c^j\} \\ t_j &:= s - (h'-j) + \sharp\{i \in \Delta^c | d_i < c^j\} \\ z_j &:= \sharp\{i \in \{1,\ldots,m\} | d_i > c^j\}. \end{aligned}$$

Example 10. Let $\mathbf{a} = (9,9,7,6,3)$, $\mathbf{b} = (9,6,2,1)$, $\mathbf{c} = (12,4,3)$ and $\mathbf{d} = (13,5)$ be partitions. Note that the sum of lengths of the partitions $\mathbf{a}$ and $\mathbf{d}$ equals the sum of lengths of the partitions $\mathbf{c}$ and $\mathbf{b}$, and it is 7. Also, by definitions (14), (15) and (13) we have

$$\mathbf{e} = \mathbf{d} \cup \mathbf{a} = (13,9,9,7,6,5,3),$$

$$\mathbf{e}' = \mathbf{c} \cup \mathbf{b} = (12,9,6,4,3,2,1),$$

and

$$\mathbf{u} = \mathbf{d} \cup \mathbf{c} = (13,12,5,4,3).$$

Let us calculate sets S and Δ. By Definition 8 we start with u_5 which is in this case $c_3 = 3$. Next, by (21) we calculate $q'_3 = 5$. Since $q'_3 = 5 > 4 = k$, we put $3 \in S$.

Next we pass to $u_4 = c_2 = 4$, and again since $q'_2 = 5 > 4$, we also put $2 \in S$.

Then we consider $u_3 = d_2 = 5$. By (16) we have that $q_2 = 4 \leqslant 5$, and 2 is the minimal index $l \in S$ such that $d_2 > c_l$. So, by (17) we have $\sharp\{i \in \{1,\ldots,5\} | a_i > c_2\} - 5 + \sharp\{i \in S | i > 2\} = 0 \geqslant 0$. Since $e_6 > c_2 \geqslant e_7$, we have that e_6 is the smallest e_i bigger than c_2. Finally, since e_6 is d_2, by the part (a) of the definition of Δ, we have that $2 \notin \Delta$.

Next we consider $u_2 = c_1 = 12$. By (19) we have $q'_1 = 5 > 4$, and so we put $1 \in S$.

Finally, we consider $u_1 = d_1 = 13$. Then $q_1 = 4$, and 1 is the minimal index $l \in S$ such that $d_1 > c_l$. Thus $\sharp\{i \in \{1,\ldots,5\} | a_i > c_1\} - 5 + \sharp\{i \in S | i > 1\} - \sharp\{i \in \{2\} | d_i < c_1, i \notin \Delta\} = -4 < 0$, and so the part (a) of the definition of Δ is not satisfied. By the part (b) since formula (18) in this case is not satisfied, i.e. since

$$c_1 + c_2 + c_3 = 12 + 4 + 3 < 13 + 5 + 3 = d_1 + d_2 + a_5,$$

we have that $1 \in \Delta$.

Hence
$$S = \{1, 2, 3\} \quad \text{and} \quad \Delta = \{1\},$$
and so $h' = 3$ and $h = 1$. Thus, $d^1 = d_1 = 13$ and $c^1 = c_1 = 12$, $c^2 = c_2 = 4$, $c^3 = c_3 = 3$.

Now, by using the notation given in this section, we can state the main result of the paper. The following theorem resolves Problem 4:

Theorem 11. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c}$ *and* $\mathbf{d}$ *be the partitions as in (2)–(5), respectively. There exists a partition* $\mathbf{g} = (g_1, \ldots, g_{m+s})$*, such that*
$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}) \tag{22}$$
if and only if the following conditions are valid
$$\begin{array}{ll} (i.1) & \textit{if} \quad y \in \{1, \ldots, h'\} \quad \textit{is such that} \quad t_y \leqslant m_y, \quad \textit{then} \\ & \displaystyle\sum_{i=z_y+t_y}^{z_y+m_y} e_i \leqslant \sum_{i=y}^{h'} c^i - \sum_{i \geqslant z_y+1,\, i \in \Delta^c} d_i - \sum_{i=m_y+1}^{s} a_i, \\ (ii.1) & \textit{if} \quad x \in \{1, \ldots, h\} \quad \textit{is such that} \quad t'_x \leqslant m'_x, \quad \textit{then} \\ & \displaystyle\sum_{i=z'_x+t'_x}^{z'_x+m'_x} e'_i \leqslant \sum_{i=x}^{h} d^i - \sum_{i \geqslant z'_x+1,\, i \in S^c} c_i - \sum_{i=m'_x+1}^{k} b_i. \end{array}$$

Remark 12. Both these sets of conditions $(i.1)$ and $(ii.1)$ are complicated, and use the definitions of the sets S and Δ, and corresponding sequences from Definition 9. However, they all depend only on partitions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. Therefore, in applications of this result we usually denote these conditions shortly by $\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b})$.

The following theorem resolves Problem 5:

Theorem 13. *There exists a partition* $\mathbf{g} = (g_1, \ldots, g_{m+s})$*, such that*
$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}) \tag{23}$$
if and only if
$$\sum_{i=1}^{n} c_i + \sum_{i=1}^{k} b_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i, \tag{24}$$
and the condition $\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b})$ *is valid.*

Proofs of Theorems 11 and 13 are given in Sections 9 and 10, respectively. The proofs are very long, combinatorial and technical, and we shall need various auxiliary results in the course. We have split these results into several sections. Section 4 cites some previous results on the generalized majorization that will be used in the sufficiency part of the proof. In Section 5 we give some properties of the sets S and Δ. Section 6 give various results involving bounds for the numbers t_j and q_j, as well as the definition of the special partition $\bar{\mathbf{g}}$, that will be crucial in the rest of the proof. Section 7 consists of Lemmas which are used in the necessity part of the proof, while Section 8 consists of Lemmas which are used in the sufficiency part of the proof.

3 Examples

Example 14. Let us consider data from Example 10, i.e. let $\mathbf{a} = (9, 9, 7, 6, 3)$, $\mathbf{b} = (9, 6, 2, 1)$, $\mathbf{c} = (12, 4, 3)$ and $\mathbf{d} = (13, 5)$ be partitions. As we have calculated in Example 10 then

$$S = \{1, 2, 3\} \quad \text{and} \quad \Delta = \{1\},$$

i.e. $d^1 = d_1 = 13$ and $c^1 = c_1 = 12$, $c^2 = c_2 = 4$, $c^3 = c_3 = 3$. Also, by Definition 9, we have

$$\begin{array}{lll} m'_1 = 0, & t'_1 = 4, & z'_1 = 0 \\ m_1 = 0, & t_1 = 4, & z_1 = 1 \\ m_2 = 4, & t_2 = 4, & z_2 = 2 \\ m_3 = 4, & t_3 = 5, & z_3 = 2. \end{array} \tag{25}$$

Now we can check conditions $(i.1)$ and $(ii.1)$. As we have obtained in (25), $m'_1 < t'_1$, $m_1 < t_1$, $m_2 \geqslant t_2$, and $m_3 < t_3$. Hence, we only need to check condition $(i.1)$ for $y = 2$. Since $e_6 = 5$, $c^2 = 4$, $c^3 = 2$, and $a_5 = 3$, we have that condition $(i.1)$ fails:

$$e_6 > c^2 + c^3 - a_5.$$

Therefore Theorem 11 implies that there is no partition $\mathbf{g}$ such that $\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a})$ and $\mathbf{g} \prec'' (\mathbf{c}, \mathbf{b})$.

Example 15. Let $\mathbf{a} = (11, 9, 1)$, $\mathbf{b} = (12, 3)$, $\mathbf{c} = (8, 7, 5)$ and $\mathbf{d} = (10, 4)$ be partitions. Let us calculate sets S and Δ for the above given partitions, and check the conditions $(i.1)$ and $(ii.1)$ from Theorem 11. Here

$$\begin{array}{rcl} \mathbf{u} & = & (10, 8, 7, 5, 4), \\ \mathbf{e} & = & (11, 10, 9, 4, 1), \\ \mathbf{e}' & = & (12, 8, 7, 5, 3). \end{array}$$

By Definition 8 we start with $u_5 = d_2 = 4$, and we calculate $q_2 = 4 > 3$, and thus we put $2 \in \Delta$.

Next we consider $u_4 = c_3 = 5$. We calculate $q'_3 = 2$, and since the part (a) of the definition is not valid (since $1 < 2$), we pass to the part (b). Since formula (21) in this case becomes $4 < 5$, we conclude that $3 \in S$. Since $u_3 = c_2 = 7$ with $q'_2 = 2$, completely analogously as for c_3, by the part (b) and the formula (21) of Definition 8, we have $2 \in S$. Analogously, by the same reason, for $u_2 = c_1 = 8$ with $q'_1 = 2$, we conclude $1 \in S$.

So we pass to $u_1 = d_1 = 10$. By (16) we obtain that $q_1 = 1$, and the minimal index $l \in S$ such that $d_1 > c_l$ is $l = 1$. Then (17) becomes $2 \geqslant 1$, and the two smallest e_i bigger that $c_1 = 8$ are e_2 and e_3. Since d_1 is e_2, by part (a) of the definition of Δ, we conclude that $1 \notin \Delta$.

Hence

$$S = \{1, 2, 3\} \quad \text{and} \quad \Delta = \{2\},$$

$h' = 3$ and $h = 1$. Thus, $c^1 = c_1 = 8$, $c^2 = c_2 = 7$, $c^3 = c_3 = 5$, and $d^1 = d_2 = 4$. Also, we can calculate the values of m'_1, t'_1 and z'_1, as well as m_i, t_i and z_i, $i = 1, 2, 3$. By Definition 9 we have:

$$\begin{array}{lll} m'_1 = 1 & t'_1 = 2 & z'_1 = 3, \\ m_1 = 2 & t_1 = 1 & z_1 = 1, \\ m_2 = 2 & t_2 = 2 & z_2 = 1, \\ m_3 = 2 & t_3 = 3 & z_3 = 1. \end{array} \tag{26}$$

Hence, $m'_1 < t'_1$, $m_1 \geqslant t_1$, $m_2 \geqslant t_2$, and $m_3 < t_3$. So the only places where we need to check condition $(i.1)$ is for $m_1 \geqslant t_1$ and $m_2 \geqslant t_2$, while $(ii.1)$ is always satisfied.

For $m_1 \geqslant t_1$ we need to check whether

$$e_2 + e_3 \leqslant c^1 + c^2 + c^3 - a_3$$

which gives $19 \leqslant 19$, and so it holds. As for $m_2 \geqslant t_2$ we need to check whether

$$e_3 \leqslant c^2 + c^3 - a_1$$

which gives $9 \leqslant 11$, and so it also holds. Thus, conditions $(i.1)$ and $(ii.1)$ are satisfied in this case, and by Theorem 11 there exists a solution to Problem 4. Also, since $\sum_{i=1}^{3} a_i + \sum_{i=1}^{2} d_i = \sum_{i=1}^{2} b_i + \sum_{i=1}^{3} c_i$, there also exists a solution to Problem 5 in this case. Later on, in Examples 41 and 43 we shall define concrete and explicit solutions for both problems for given partitions.

4 Properties of the generalized majorization

By using the notation from Definition 2, we list some of the well known basic properties of the auxiliary numbers h_j, defined in (9), that will be used throughout the paper:

Since

$$h_j := \min\{i | c_{i-j+1} < g_i\}, \quad j = 1, \ldots, k,$$

we have:

$$n + k + 1 > h_k > \cdots > h_2 > h_1 > 0, \tag{27}$$

and so in particular:

$$h_j \geqslant j, \quad j = 1, \ldots, k. \tag{28}$$

Also, we set

$$h_0 := 0, \quad \text{and} \quad h_{k+1} := n + k + 1.$$

Since $c_i \geqslant g_{i+k}$, $i = 1, \ldots, n$, we have that $c_{i-k} \geqslant g_i$, $i = 1, \ldots, n+k$, and so

$$c_{i-j+1} \geqslant g_i, \quad \text{for} \quad i < h_j, \quad \text{for any} \quad j = 1, \ldots, k+1. \tag{29}$$

Lemma 16. *[7, Lemma 2] Suppose that* $\mathbf{b} = (b_1, \ldots, b_k)$*,* $\mathbf{c} = (c_1, \ldots, c_n)$*, and* $\mathbf{g} = (g_1, \ldots, g_{n+k})$ *are partitions such that*

$$\mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}).$$

Let $u \in \{1, \ldots, n+k\}$ *be an integer, and let* $j \in \{0, \ldots, k\}$ *be such that*

$$h_j < u \leqslant h_{j+1}.$$

Then

$$\sum_{i=u}^{n+k} g_i \geqslant \sum_{i=u-j}^{n} c_i + \sum_{i=j+1}^{k} b_i.$$

Lemma 17. *[5, Lemma 2.4] Let* $\mathbf{b} = (b_1, \ldots, b_k)$*,* $\mathbf{c} = (c_1, \ldots, c_n)$*, and* $\mathbf{g}' = (g'_1, \ldots, g'_{n+k})$ *be partitions such that*

$$\mathbf{g}' \prec'' (\mathbf{c}, \mathbf{b}).$$

Let $f \in \{2, \ldots, n+k\}$*, and let* $\mathbf{g} = (g_1, \ldots, g_{n+k})$ *be a partition such that*

$$g_i = g'_i, \quad i \geqslant f,$$

$$g_i \leqslant g'_i, \quad i < f,$$

$$g'_{f-1} \geqslant g_1 \geqslant g_{f-1} \geqslant g_1 - 1,$$

$$\sum_{i=1}^{n+k} g_i \geqslant \sum_{i=1}^{n} c_i + \sum_{i=1}^{k} b_i.$$

Then

$$\mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}).$$

Lemma 18. *[7, Lemma 9] Let* $u_1 \geqslant \cdots \geqslant u_k$ *and* $v_1 \geqslant \cdots \geqslant v_k$ *be integers. If*

$$\sharp\{i \in \{1, \ldots, k\} | u_i > v_j\} \geqslant j, \quad \textit{for all} \quad j = 1, \ldots, k,$$

then

$$\sum_{i=1}^{k} u_i \geqslant \sum_{i=1}^{k} v_i + k.$$

5 Ordering of the elements of the sets S and Δ

Before going into the details of the proof, we extend formally index sets of the numbers from Definition 9:

We define $d^0 := +\infty$, $d^{h+1} := -\infty$, $t'_{h+1} := k+1$, $z'_{h+1} := n$, and we extend definitions of m'_j, t'_j and z'_j to the case $j = 0$: $m'_0 := \sharp\{i \in \{1, \ldots, k\} | b_i > d^0\} = 0$, $t'_0 := k - h + \sharp\{i \in S^c | c_i < d^0\} = k - h + \sharp S^c = n + k - h - h'$, and $z'_0 := \sharp\{i \in \{1, \ldots, n\} | c_i > d^0\} = 0$.

Analogously, we also formally define $c^0 := +\infty$, $c^{h'+1} := -\infty$, $t_{h'+1} := s+1$, $z_{h'+1} := m$, and we extend definitions of m_j, t_j and z_j to the case $j = 0$: $m_0 := \sharp\{i \in \{1, \ldots, s\} | a_i > c^0\} = 0$, $t_0 := s - h' + \sharp\{i \in \Delta^c | d_i < c^0\} = s - h' + \sharp\{\Delta^c\} = m + s - h - h'$, $z_0 := \sharp\{i \in \{1, \ldots, m\} | d_i > c^0\} = 0$.

Since $m + s = n + k$, we have

$$t_0 = t'_0 = m + s - h - h'. \tag{30}$$

Now, by using Definition 9 we can re-write conditions (17), (18), (20) and (21) in Definition 8 in the following way:

For d_j, $j \in \{1, \ldots, m\}$, let $l \in \{0, \ldots, h'\}$ be such that $c^l > d_j > c^{l+1}$. Then

$$q_j = s - (h' - l) + \sharp\{i \in \Delta^c | i > j\} + 1,$$

and condition (17) becomes

$$m_{l+1} \geqslant t_{l+1},$$

while (18) is equal to

$$\sum_{i=l+1}^{h'} c^i \geqslant \sum_{i \in \Delta^c,\, i > j} d_i + d_j + \sum_{i=q_j+1}^{s} a_i. \tag{31}$$

Analogously, for c_j, $j \in \{1, \ldots, n\}$, let $l' \in \{0, \ldots, h\}$ be such that $d^{l'} > c_j > d^{l'+1}$. Then

$$q'_j = k - (h - l') + \sharp\{i \in S^c | i > j\} + 1.$$

Also, (20) becomes

$$m'_{l'+1} \geqslant t'_{l'+1},$$

and (21) is equal to

$$\sum_{i=l'+1}^{h} d^i \geqslant \sum_{i \in S^c,\, i > j} c_i + c_j + \sum_{i=q'_j+1}^{k} b_i. \tag{32}$$

Remark 19. We note that by Definition 8, for all j such that $c^{h'} > d_j$, we have $q_j > s$ and thus $j \in \Delta$. So we have

$$c^{h'} > d_{z_{h'}+1} \geqslant \cdots \geqslant d_m \Rightarrow z_{h'} + 1, \ldots, m \in \Delta. \tag{33}$$

Also, for all j such that $d^h > c_j$, we have $q'_j > k$ and thus $j \in S$. So we have

$$d^h > c_{z'_h+1} \geqslant \cdots \geqslant c_n \Rightarrow z'_h + 1, \ldots, n \in S. \tag{34}$$

Hence, we have

$$t'_h = k, \quad \text{and} \quad t_{h'} = s. \tag{35}$$

In the rest of the section we give auxiliary lemmas concerning the ordering of c_i's, with $i \in S$ and d_j's with $j \in \Delta$ with respect to the remaining ones c_i's, with $i \in S^c$ and d_j's with $j \in \Delta^c$. These results follow directly from the Definition 8 of the sets S and Δ. All of them are used in the proof of the main result.

Lemma 20. *Let $y \in \{0, \ldots, h'\}$ and let $j \in \{1, \ldots, m-1\}$ be such that $c^y > d_j \geqslant d_{j+1} > c^{y+1}$. Then, if $j+1 \in \Delta$ we have that $j \in \Delta$.*

Proof. By (16) we have $q_j = q_{j+1}$. From the definition of Δ, since $j+1 \in \Delta$, there are two possibilities: either $q_{j+1} > s$, and then $q_j > s$, i.e. $j \in \Delta$, as wanted; either (18) is not valid for d_{j+1}, in which case we trivially obtain that it is not valid for d_j as well. Hence $j \in \Delta$, as wanted. ☐

Completely analogously we have the dual result:

Lemma 21. *Let $x \in \{0, \ldots, h\}$ and let $j \in \{1, \ldots, n-1\}$ be such that $d^x > c_j \geqslant c_{j+1} > d^{x+1}$. Then, if $j+1 \in S$ we have that $j \in S$.*

Now, let us introduce some additional counters, which count the number of the elements not in Δ between two elements with the indices from S, and analogously, the number of the elements not in S between two elements with the indices from Δ:

Definition 22. For $y \in \{0, \ldots, h'\}$ we define:

$$w_y := \sharp\{i \in \Delta^c | c^y > d_i > c^{y+1}\}.$$

For $x \in \{0, \ldots, h\}$ we define:

$$w'_x := \sharp\{j \in S^c | d^x > c_j > d^{x+1}\}.$$

Now, as direct corollaries of Lemmas 20 and 21, we have the following arrangement of the elements from Δ and S:

Let $y \in \{0, \ldots, h\}$. Then by Lemma 20 we have

$$d_{z_y} > c^y > \underbrace{d_{z_y+1} \geqslant d_{z_y+2} \geqslant \cdots \geqslant d_{z_{y+1}-w_y}}_{\in\Delta} \geqslant \underbrace{d_{z_{y+1}-w_y+1} \geqslant \cdots \geqslant d_{z_{y+1}}}_{\notin\Delta} > c^{y+1}. \tag{36}$$

Let $x \in \{0, \ldots, h'\}$. Then by Lemma 21 we have

$$c_{z'_x} > d^x > \underbrace{c_{z'_x+1} \geqslant c_{z'_x+2} \geqslant \cdots \geqslant c_{z'_{x+1}-w'_x}}_{\in S} \geqslant \underbrace{c_{z'_{x+1}-w'_x+1} \geqslant \cdots \geqslant c_{z'_{x+1}}}_{\notin S} > d^{x+1}. \tag{37}$$

From Definitions 9 and 22 we directly obtain relations between t_j's and w_j's:

Lemma 23.

$$\begin{aligned}
t_{x+1} &= t_x + 1 - w_x, \quad x = 0, \ldots, h', && (38)\\
t'_{y+1} &= t'_y + 1 - w'_y, \quad y = 0, \ldots, h, && (39)\\
z_x + t_x &< z_{x+1} + t_{x+1}, \quad x = 0, \ldots, h', && (40)\\
z'_y + t'_y &< z'_{y+1} + t'_{y+1}, \quad y = 0, \ldots, h. && (41)
\end{aligned}$$

Lemma 24. *Let $j \in \Delta$. Let $i \in \{1, \ldots, h\}$ be such that $d_j = d^i$ and let $y \in \{0, \ldots, h'\}$ be such that $c^y > d_j > c^{y+1}$. Then*

$$z'_i + t'_i = j + t_y.$$

Proof. By Definition 9, together with Lemmas 20 and 21, we obtain

$$z'_i + t'_i = \sharp\{l \in \{1, \ldots, n\} | c_l > d^i\} + k - (h - i) + \sharp\{l \in S^c | c_l < d^i\} =$$

$$= k - (h - i) + (n - \sharp\{l \in S | c_l < d^i\}) = k - (h - i) + n - (h' - y) =$$

$$= m + s - (h - i) - (h' - y).$$

On the other hand

$$t_y = s - (h' - y) + \sharp\{l \in \Delta^c | d_l < c^y\} = s - h' + y + (m - h - \sharp\{l \in \Delta^c | d_l > c^y\}) =$$

$$= m + s - (h - i) + (h' - y) - j.$$

Thus,

$$z'_i + t'_i = j + t_y,$$

as wanted. □

Dually, we have:

Lemma 25. *Let $j \in S$. Let $i \in \{1, \ldots, h'\}$ be such that $c_j = c^i$ and let $x \in \{0, \ldots, h\}$ be such that $d^x > c_j > d^{x+1}$. Then*

$$z_i + t_i = j + t'_x.$$

By Lemmas 23, 24 and 25, we obtain

Lemma 26. *The numbers $z_i + t_i$ for $i = 1, \ldots, h'$, and $z'_i + t'_i$ for $i = 1, \ldots, h$, are all distinct. In addition,*

$$\{z_i + t_i | i = 1, \ldots, h'\} \cup \{z'_i + t'_i | i = 1, \ldots, h\} = \{t_0 + 1, t_0 + 2, \ldots, m + s\}.$$

6 Some results valid under the assumption $c^{h'} \geqslant a_s$ (and $d^h \geqslant b_k$)

In this section we give some important results and improvements on the bounds of q_j's and t_j's, obtained under the assumptions $c^{h'} \geqslant a_s$ and/or $d^h \geqslant b_k$.

Lemma 27. *Suppose that $c^{h'} \geqslant a_s$, and let $j \in \{1, \dots, m\}$ be such that $d_j > c^{h'}$. Then $q_j \leqslant s$. In addition, if $j \in \Delta^c$ then $q_j < s$.*

Proof. Before proceeding note that by the definition of q_l all d_l such that $c^{h'} > d_l$, satisfy $l \in \Delta$, see (33).

Since $d_j > c^{h'}$, we have that $1 \leqslant j \leqslant z_{h'}$. Let $p \in \{0, \dots, h'-1\}$ be such that $c^p > d_j > c^{p+1}$. The rest of the proof goes by the induction on j.

Let $j = z_{h'}$. By definition (16), we have $q_{z_{h'}} = s - (h' - p) + 1 \leqslant s$, as wanted.

Now let $1 \leqslant j < z_{h'}$. By induction we suppose that $q_i \leqslant s$, for all $i = j+1, \dots, z_{h'}$. We are left with proving that then $q_j \leqslant s$.

By definition (16), we have that if $q_{j+1} < s$, then $q_j \leqslant s$. So the only case we are left to consider is when $q_{j+1} = s$, $j+1 \notin \Delta$ and $c^p > d_{j+1} > c^{p+1}$. We shall prove that this case is impossible, i.e. that if $c^p > d_{j+1} > c^{p+1}$ and $q_{j+1} = s$, then $j+1 \in \Delta$.

Let

$$\gamma = \sharp\{i \in \Delta^c | i = j+2, \dots, z_{p+1}\}.$$

Since $t_{p+1} = q_{j+1} - \gamma = s - \gamma$, and $m_{p+1} \leqslant s - 1$ (since $c^{h'} \geqslant a_s$), we have $m_{p+1} - t_{p+1} + 1 \leqslant \gamma$, so by the definition of γ we have that d_{j+1} doesn't satisfy part (a) of the definition of the set Δ. So we are left with checking the condition (b) of the definition of the set Δ, i.e. we are left with checking

$$\sum_{i=p+1}^{h'} c^i < \sum_{i \geqslant j+2,\, i \in \Delta^c} d_i + d_{j+1}. \tag{42}$$

Since $q_{j+1} = s$, we have

$$1 + \sharp\{i \in \Delta^c | j+2 \leqslant i \leqslant m\} = h' - p. \tag{43}$$

Let $u_1 \geqslant \cdots \geqslant u_{h'-p}$ be the non-increasing ordering of d_{j+1} and d_i with $j+2 \leqslant i \leqslant m$, $i \notin \Delta$, and let $v_1 \geqslant \cdots \geqslant v_{h'-p}$ be defined as $v_i := c^{p+i}$, $i = 1, \dots, h'-p$. We claim that

$$u_i > v_i, \quad i = 1, \dots, h'-p. \tag{44}$$

Since $d_{j+1} > c^{p+1}$ we have $u_1 > v_1$. Now let us fix $i_0 \in \{2, \dots, h'-p\}$. Then $u_{i_0} = d_l$ for some $l \notin \Delta$ with $j+2 \leqslant l \leqslant m$, i.e.

$$i_0 = 1 + \sharp\{i \notin \Delta | j+2 \leqslant i \leqslant l\}.$$

Let $r \in \{0, \dots, h'-1\}$ be such that $c^r > d_l > c^{r+1}$. Note that $l \leqslant z_{h'}$ since for all $i > z_{h'}$ we have $i \in \Delta$.

From $q_l \leqslant s$ we get

$$\sharp\{i \in \Delta^c | l < i \leqslant m\} \leqslant h' - r - 1. \tag{45}$$

Then (43) and (45) together give

$$1 + \sharp\{i \notin \Delta | j+2 \leqslant i \leqslant l\} \geqslant r + 1 - p,$$

i.e.

$$i_0 \geqslant r + 1 - p.$$

Therefore

$$u_{i_0} = d_l > c^{r+1} = c^{p+(r+1-p)} \geqslant c^{p+i_0} = v_{i_0}.$$

By Lemma 18 we get (42). Thus, we have proved that $j+1 \in \Delta$, as wanted. □

Dually, we have:

Lemma 28. *Suppose that $d^h \geqslant b_k$, and let $j \in \{1, \dots, n\}$ be such that $c_j > d^h$. Then $q'_j \leqslant k$. In addition, if $j \in S^c$ then $q'_j < k$.*

As a direct corollary of Lemmas 27 and 28, we have

Corollary 29.

$$c^{h'} \geqslant a_s \quad \Longrightarrow \quad t_y < s, \quad \textit{for all} \quad y = 0, \dots, h'-1, \tag{46}$$

$$d^h \geqslant b_k \quad \Longrightarrow \quad t'_x < k, \quad \textit{for all} \quad x = 0, \dots, h-1. \tag{47}$$

Proof. We shall prove (46), and (47) follows dually.

First note that there are no $i \notin \Delta$ such that $c^{h'-1} > d_i > c^{h'}$. Indeed, suppose on the contrary that $j \in \{1, \dots, m\}$ is the largest such index. Since $m_{h'} \leqslant s-1$ and $t_{h'} = s$, $j \notin \Delta$ implies that (18) is satisfied, i.e. $c^{h'} \geqslant d_j$ which is a contradiction. Therefore $t_{h'-1} = s-1$.

Now fix $y \in \{0, \dots, h'-2\}$. If there are no $i \notin \Delta$ such that $c^y > d_i > c^{h'-1}$ then $t_y = t_{h'-1} - (h'-1-y) = s-1-(h'-1-y) < s$. If there exists $i \notin \Delta$ with $c^y > d_i > c^{h'-1}$, then let j be the smallest such index and let $p \in \{y, \dots, h'-2\}$ be such that $c^p > d_j > c^{p+1}$. Then $t_p = q_j$, and so by Lemma 27, $t_y = t_p - (p-y) = q_j - (p-y) < s - (p-y) \leqslant s$, as wanted. □

Lemma 30. *Suppose that $c^{h'} \geqslant a_s$. Let $j \in \{1, \dots, m\}$ be such that $j \in \Delta$. Let $y \in \{0, \dots, h'\}$ be such that $c^y > d_j > c^{y+1}$. Then $t_y \geqslant 0$.*

Proof. If $y = h'$, then $t_y = s \geqslant 0$, as wanted.

If $0 \leqslant y < h'$, then $d_j > c^{h'}$. Suppose that $t_y < 0$ then:

$$m_{y+1} - t_{y+1} + 1 = m_{y+1} - t_y - 1 + w_y + 1 \geqslant m_{y+1} + w_y + 1.$$

The last means that $d_{z_{y+1}-w_y}$ is among the smallest $m_{y+1} - t_{y+1} + 1$ e_i's larger than c^{y+1}. Since, by Lemma 27 we have that $q_{z_{y+1}-w_y} \leqslant s$, by the part (a) of the definition of the set Δ, we conclude $z_{y+1} - w_y \notin \Delta$, which is a contradiction by (36). Hence $t_y \geqslant 0$, as wanted. □

Dually, we have:

Lemma 31. *Suppose that $d^h \geqslant b_k$. Let $j \in \{1, \ldots, n\}$ be such that $j \in S$. Let $x \in \{0, \ldots, h\}$ be such that $d^x > c_j > d^{x+1}$. Then $t'_x \geqslant 0$.*

Lemma 32. *Suppose that $c^{h'} \geqslant a_s$ and $d^h \geqslant b_k$. Then $t_0 = t'_0 \geqslant 0$.*

Proof. If any of the sets S or Δ is empty, we directly get that $t_0 \geqslant 0$. If none of the sets S and Δ is empty, then if $d^1 > c^1$ by Lemma 30 we have that $t_0 \geqslant 0$, and if $c^1 > d^1$ by Lemma 31 we have that $t'_0 \geqslant 0$, as wanted. □

Now, by Lemma 26 we obtain

Lemma 33. *Let $c^{h'} \geqslant a_s$ and $d^h \geqslant b_k$. The numbers $z_i + t_i$ for $i = 1, \ldots, h'$, and $z'_i + t'_i$ for $i = 1, \ldots, h$, are all distinct and satisfy*

$$\{z_i + t_i | i = 1, \ldots, h'\} \cup \{z'_i + t'_i | i = 1, \ldots, h\} = \{t_0 + 1, t_0 + 2, \ldots, m + s\} \subset$$
$$\subset \{1, \ldots, m + s\}$$

6.1 Definition of the partition $\bar{\mathbf{g}}$

Let $c^{h'} \geqslant a_s$ and $d^h \geqslant b_k$. Then by Lemma 32 we have $t_0 \geqslant 0$, and so we can define the following important partition:

Let $\bar{\mathbf{g}} = (\bar{g}_1, \ldots, \bar{g}_{m+s})$ be a partition defined as the following union

$$(c^1, \ldots, c^{h'}) \cup (d^1, \ldots, d^h) \cup \underbrace{(M, \ldots, M)}_{t_0}, \tag{48}$$

where

$$M := \max(a_1, b_1, c_1, d_1) + 1.$$

In other words, the partition $\bar{\mathbf{g}}$ is defined as the non-increasing union of all c_i, with $i \in S$, all d_j, with $j \in \Delta$, with added t_0 elements equal to M.

By Lemmas 24 and 25, we have that the partition $\bar{\mathbf{g}}$ from (48) also satisfies:

$$\bar{g}_i = \max(a_1, b_1, c_1, d_1) + 1, \quad i = 1, \ldots, t_0, \tag{49}$$
$$\bar{g}_j = d_{j-t_x}, \quad \text{for } z_x + t_x < j < z_{x+1} + t_{x+1}, \quad x = 0, \ldots, h', \tag{50}$$
$$\bar{g}_{z_x+t_x} = c^x, \quad x = 1, \ldots, h', \tag{51}$$

or equivalently:

$$\bar{g}_i = \max(a_1, b_1, c_1, d_1) + 1, \quad i = 1, \ldots, t'_0, \tag{52}$$

$$\bar{g}_j = c_{j-t'_x}, \quad \text{for } z'_x + t'_x < j < z'_{x+1} + t'_{x+1}, \quad x = 0, \dots, h, \tag{53}$$

$$\bar{g}_{z'_x+t'_x} = d^x, \quad x = 1, \dots, h. \tag{54}$$

Moreover, by using this partition, we can equivalently re-write conditions $\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b})$ in the following, more concise, way:

$$(o) \qquad c^{h'} \geqslant a_s, \quad \text{and} \quad d^h \geqslant b_k, \tag{55}$$

(i) if $y \in \{1, \dots, h'-1\}$ is such that $t_y \leqslant m_y$, then

$$\sum_{i=z_y+t_y}^{m+s} \bar{g}_i \geqslant \sum_{i=z_y+t_y}^{m+s} e_i \tag{56}$$

(ii) if $x \in \{1, \dots, h-1\}$ is such that $t'_x \leqslant m'_x$, then

$$\sum_{i=z'_x+t'_x}^{m+s} \bar{g}_i \geqslant \sum_{i=z'_x+t'_x}^{m+s} e'_i. \tag{57}$$

Indeed, condition $(i.1)$ for $y = h'$ and condition $(ii.1)$ for $y = h$, together are (o). And by definition of $\bar{\mathbf{g}}$, condition $(i.1)$ for $y \in \{1, \dots, h'-1\}$ is (i), while condition $(ii.1)$ for $y \in \{1, \dots, h-1\}$ is (ii).

7 Auxiliary lemmas used in the necessity part of the proof

Consider the partitions $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ as in (2)–(5), respectively. Main result of this section is Lemma 38 that will be strongly used in proving the necessity of conditions $\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b})$.

Lemma 34. *Let $\mathbf{g} = (g_1, \dots, g_{m+s})$ be a partition which satisfies*

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a}), \quad \textit{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}).$$

Then

$$c^{h'} \geqslant g_{z_{h'}+s} \quad \textit{and} \quad d^h \geqslant g_{z'_h+k}, \tag{58}$$

as well as

$$c^{h'} \geqslant a_s \quad \textit{and} \quad d^h \geqslant b_k. \tag{59}$$

Proof. We shall prove that $c^{h'} \geqslant g_{z_{h'}+s}$ and $c^{h'} \geqslant a_s$. Dually we will have $d^h \geqslant g_{z'_h+k}$ and $d^h \geqslant b_k$.

If suppose that $z_{h'} = m$, i.e. if $d_m > c^{h'}$, then $c^{h'} = c_n$ and since $\mathbf{g} \prec'' (\mathbf{c}, \mathbf{b})$ we have

$$c^{h'} = c_n \geqslant g_{n+k} = g_{m+s} = g_{z_{h'}+s}, \text{ as wanted.}$$

If $z_{h'} < m$, then $c^{h'} = c_{n-\alpha+1}$ for some $1 \leqslant \alpha \leqslant n$, and $z_{h'} = m - \beta$, for some $1 \leqslant \beta \leqslant m$. Then we have that $i \notin S$ for $n - \alpha + 1 < i \leqslant n$, and $j \in \Delta$ for $m - \beta < j \leqslant m$.

If $\beta < \alpha$, we have $c^{h'} = c_{n-\alpha+1} \geqslant g_{n-\alpha+1+k} = g_{m-\alpha+1+s} \geqslant g_{m-\beta+s} = g_{z_{h'}+s}$, as wanted.

If $\beta \geqslant \alpha$, then from the definition of q'_i we have

$$q'_{n-\alpha+1} = k - \beta + \alpha \leqslant k.$$

Since $n - \alpha + 1 \in S$, from the definition of the set S (part (a)) we have that the index $n-\alpha+1$ does not belong to the $m'_{h-\beta+1} - t'_{h-\beta+1} + 1$ smallest e'_i's bigger than $d_{m-\beta+1}(= d_{z_{h'}+1})$. Let

$$\bar{u} = \sharp\{i \in \{1,\ldots,k\} | b_i > c_{n-\alpha+1}\},$$
$$\bar{v} = \sharp\{i \in \{1,\ldots,k\} | c_{n-\alpha+1} \geqslant b_i > d_{m-\beta+1}\},$$
$$\bar{w} = \sharp\{n-\alpha+1 < i \leqslant n | c_i > d_{m-\beta+1}\}$$

and

$$\bar{z} = \sharp\{n-\alpha+1 < i \leqslant n | c_i < d_{m-\beta+1}\}.$$

Then $\bar{z}+\bar{w} = \alpha-1$, $t'_{h-\beta+1} = k-(\beta-1)+\bar{z}$ and $m'_{h-\beta+1} = \bar{u}+\bar{v}$. Since $n-\alpha+1 \in S$ we have $\bar{v}+\bar{w} \geqslant m'_{h-\beta+1}-t'_{h-\beta+1}+1 = \bar{u}+\bar{v}-k+\beta-\bar{z}$, i.e. $\bar{u} \leqslant \bar{w}+\bar{z}+k-\beta = \alpha-1+k-\beta$. Thus,

$$\alpha + k > \beta,$$

and

$$c^{h'} = c_{n-\alpha+1} \geqslant b_{\alpha+k-\beta} \tag{60}$$

Also, since $n-\alpha+1 \in S$ by the part (b) of the definition of the set S (since $q'_{n-\alpha+1} \leqslant k$), we have

$$\sum_{i=m-\beta+1}^{m} d_i < c_{n-\alpha+1} + \sum_{i=n-\alpha+2}^{n} c_i + \sum_{i=k+\alpha-\beta+1}^{k} b_i. \tag{61}$$

Now, let us suppose the opposite from what we need to prove, i.e. that $c^{h'} < g_{z_{h'}+s}$. Last is equivalent to $c_{n-\alpha+1} < g_{m-\beta+s}$. Thus, by definition of $h'_j = \min\{i | c_{i-j+1} < g_i\}$, we have $h'_{m-\beta+s-n+\alpha} \leqslant m+s-\beta$, i.e. $h'_{k+\alpha-\beta} \leqslant m+s-\beta$. Let $u \in \{0,\ldots,k\}$ be such that $h'_u \leqslant m+s-\beta < h'_{u+1}$. Then $u \geqslant k+\alpha-\beta$.

Since $\mathbf{g} \prec'' (\mathbf{c},\mathbf{b})$, by the definition of the weak generalized majorization, and by Lemma 16, we have

$$\sum_{i=m+s-\beta+1}^{m+s} g_i \geqslant \sum_{i=m+s-\beta+1-u}^{n} c_i + \sum_{i=u+1}^{k} b_i. \tag{62}$$

Since $\mathbf{g} \prec'' (\mathbf{d},\mathbf{a})$ implies $d_i \geqslant g_{i+s}$, $i = 1,\ldots,m$, by (62) we have

$$\sum_{i=m-\beta+1}^{m} d_i \geqslant \sum_{i=m+s-\beta+1-u}^{n} c_i + \sum_{i=u+1}^{k} b_i. \tag{63}$$

Since $u \geqslant k + \alpha - \beta$, from (60) we have that

$$\sum_{i=m+s-\beta+1-u}^{n} c_i + \sum_{i=u+1}^{k} b_i = \sum_{i=n-\alpha+1}^{n} c_i + \sum_{i=k-\alpha+\beta+1}^{k} b_i +$$

$$+ \left(\sum_{i=m+s-\beta+1-u}^{n-\alpha} c_i - \sum_{i=k+\alpha-\beta+1}^{u} b_i \right) \geqslant \sum_{i=n-\alpha+1}^{n} c_i + \sum_{i=k-\alpha+\beta+1}^{k} b_i,$$

which together with (63) gives

$$\sum_{i=m-\beta+1}^{m} d_i \geqslant \sum_{i=n-\alpha+1}^{n} c_i + \sum_{i=k+\alpha-\beta+1}^{k} b_i, \tag{64}$$

which contradicts (61). Thus, $c^{h'} \geqslant g_{z_{h'}+s}$.

Now, let us prove that $c^{h'} \geqslant a_s$. Let $j \in \{0, \dots, s\}$, be such that $h_j < z_{h'} + s \leqslant h_{j+1}$ ($h_0 = 0$, $h_{s+1} = m + s + 1$).

Then $\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a})$ (by Lemma 16 and the definition of the weak generalized majorization) gives

$$\sum_{i=z_{h'}+s}^{m+s} g_i \geqslant \sum_{i=z_{h'}+s-j}^{m} d_i + \sum_{i=j+1}^{s} a_i. \tag{65}$$

Conditions (6) and (65) together with $c^{h'} \geqslant g_{z_{h'}+s}$ give

$$c^{h'} + \sum_{i=z_{h'}+1}^{m} d_i \geqslant \sum_{i=z_{h'}+s-j}^{m} d_i + \sum_{i=j+1}^{s} a_i. \tag{66}$$

If $j = s$, (66) becomes $c^{h'} \geqslant d_{z_{h'}}$ which is a contradiction by the definition of $z_{h'}$. On the other hand if $j < s$, then (66) gives

$$(s-j)c^{h'} \geqslant c^{h'} + \sum_{i=z_{h'}+1}^{z_{h'}+s-j-1} d_i \geqslant \sum_{i=j+1}^{s} a_i \geqslant (s-j)a_s,$$

i.e. $c^{h'} \geqslant a_s$, as wanted. □

By (59) and by Lemma 32, we have $t_0 \geqslant 0$, and so we can define $\bar{\mathbf{g}}$ as in (49). Since $t_{h'} = s$ and $t'_h = k$ (see (35)), (58) becomes

$$\bar{g}_{z_{h'}+s} \geqslant g_{z_{h'}+s}, \quad \text{and} \quad \bar{g}_{z'_h+k} \geqslant g_{z'_h+k}. \tag{67}$$

Hence in the course of proving the previous lemma we have also proved:

Corollary 35. *Let* $\mathbf{g} = (g_1, \ldots, g_{m+s})$ *be a partition which satisfies* $\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a})$, *and* $\mathbf{g} \prec'' (\mathbf{c}, \mathbf{b})$. *Then*

$$\bar{g}_i \geqslant g_i, \quad i \geqslant z_{h'} + t_{h'},$$

and

$$\bar{g}_i \geqslant g_i, \quad i \geqslant z'_h + t'_h.$$

Lemma 36. *Let* $\mathbf{g} = (g_1, \ldots, g_{m+s})$ *be a partition such that*

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}). \tag{68}$$

Let $y \in \{0, \ldots, h' - 1\}$. *If*

$$\bar{g}_i \geqslant g_i, \quad i \geqslant z_{y+1} + t_{y+1},$$

then

$$\bar{g}_i \geqslant g_i, \quad i > z_y + t_y. \tag{69}$$

Proof. We need to prove that

$$\bar{g}_i \geqslant g_i, \quad z_{y+1} + t_{y+1} > i > z_y + t_y. \tag{70}$$

Those $\bar{g}_i$ are then precisely d_j such that $j \in \Delta$, and $c^y > d_j > c^{y+1}$. And so (70) is

$$d_j \geqslant g_{j+t_y}, \tag{71}$$

for all such $j = z_y + 1, \ldots, z_{y+1} - w_y$.

Since (68), by Lemma 34 we have $c^{h'} \geqslant a_s$. Then by (46) we have $t_y < s$. Also, by Lemma 30 we have that $t_y \geqslant 0$. Therefore, we have $0 \leqslant t_y < s$, i.e. $1 \leqslant t_y + 1 \leqslant s$, and so by the definition $h_{t_y+1} = \min\{u | d_{u-t_y} < g_u\}$.

We shall prove that

$$h_{t_y+1} \geqslant z_{y+1} + t_{y+1}. \tag{72}$$

If (72) is valid then $d_u \geqslant g_{u+t_y}$, for $u + t_y < z_{y+1} + t_{y+1}$, i.e. $u \leqslant z_{y+1} + t_{y+1} - t_y - 1 = z_{y+1} - w_y$, thus proving (71), and consequently the lemma.

Let suppose the opposite to (72), i.e. let $h_{t_y+1} \leqslant z_{y+1} + t_{y+1} - 1$. Consider $u \in \{1, \ldots, s\}$ such that $h_u < z_{y+1} + t_{y+1} \leqslant h_{u+1}$. Then $u \geqslant t_y + 1$ and since $\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a})$, by the definition of the weak generalized majorization, and by Lemma 16, we have:

$$\sum_{i=z_{y+1}+t_{y+1}}^{m+s} g_i \geqslant \sum_{i=z_{y+1}+t_{y+1}-u}^{m} d_i + \sum_{i=u+1}^{s} a_i. \tag{73}$$

By the assumptions of the lemma, we have

$$\sum_{i=z_{y+1}+t_{y+1}}^{m+s} \bar{g}_i \geqslant \sum_{i=z_{y+1}+t_{y+1}}^{m+s} g_i. \tag{74}$$

Inequalities (73) and (74), together with the definition of $\bar{g}_i$, give

$$\sum_{i=y+1}^{h'} c^i + \sum_{j\in\Delta,\, j>z_{y+1}} d_j \geqslant \sum_{i=z_{y+1}+t_{y+1}-u}^{m} d_i + \sum_{i=u+1}^{s} a_i. \tag{75}$$

Since $z_{y+1} - w_y \in \Delta$, and since $q_{z_{y+1}-w_y} = t_y + 1 \leqslant s$, we have that $d_{z_{y+1}-w_y}$ does not satisfy the condition from the part (b) of the definition of the set Δ:

$$\sum_{i=y+1}^{h'} c^i < d_{z_{y+1}-w_y} + \sum_{i>z_{y+1}-w_y,\, i\in\Delta^c} d_i + \sum_{i=t_y+2}^{s} a_i$$

which further gives

$$\sum_{i=y+1}^{h'} c^i + \sum_{i>z_{y+1},\, i\in\Delta} d_i < \sum_{i=z_{y+1}-w_y}^{m} d_i + \sum_{i=t_y+2}^{s} a_i$$

Last equation together with (75) give

$$\sum_{i=z_{y+1}+t_{y+1}-u}^{m} d_i + \sum_{i=u+1}^{s} a_i < \sum_{i=z_{y+1}-w_y}^{m} d_i + \sum_{i=t_y+2}^{s} a_i.$$

Since $u \geqslant t_y + 1$ and $t_y = t_{y+1} - 1 + w_y$, we have

$$\sum_{i=z_{y+1}+t_{y+1}-u}^{z_{y+1}-w_y-1} d_i < \sum_{i=t_y+2}^{u} a_i. \tag{76}$$

Note that there is the same number of summands on the left and the right hand side in (76). Since $z_{y+1} - w_y \in \Delta$, we know that $d_{z_{y+1}-w_y}$ does not belong to the smallest $m_{y+1} - t_{y+1} + 1$ e_i's larger than c^{y+1}. Therefore $m_{y+1} - t_{y+1} + 1 \leqslant w_y + \sharp\{i | d_{z_{y+1}-w_y} > a_i > c^{y+1}\}$, i.e. $\sharp\{i | a_i \geqslant d_{z_{y+1}-w_y}\} \leqslant t_y$. This is equivalent to $d_{z_{y+1}-w_y} > a_{t_y+1}$, and so the smallest summand on the LHS of (76) is larger then the largest summand on the RHS, which gives a contradiction. Thus (72) is valid, and so we have proved our lemma. ☐

Dually, we have:

Lemma 37. *Consider a partition* $\mathbf{g} = (g_1, \ldots, g_{n+k})$, *such that*

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}).$$

Let $y \in \{0, \ldots, h-1\}$. *If*

$$\bar{g}_i \geqslant g_i, \quad i \geqslant z'_{y+1} + t'_{y+1},$$

then

$$\bar{g}_i \geqslant g_i, \quad i > z'_y + t'_y.$$

Next, we shall unify results from Lemmas 34 –37 to prove that if there exists a partition $\mathbf{g}$ satisfying $\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a})$ and $\mathbf{g} \prec'' (\mathbf{c}, \mathbf{b})$, g_i's are bounded above by $\bar{g}_i$'s. More precisely, by Corollary 35, Lemmas 33, 36 and 37, we have

Lemma 38. *Let* $\mathbf{g} = (g_1, \ldots, g_{m+s})$ *be a partition such that*

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a}), \qquad \text{and} \qquad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}). \tag{77}$$

Then

$$\bar{g}_i \geqslant g_i, \quad i = t_0 + 1, \ldots, m + s. \tag{78}$$

8 Auxiliary lemmas used in the sufficiency part of the proof

Lemma 39. *Let* $i \in \{0, \ldots, h' - 1\}$*. Suppose that* $t_0 \geqslant 0$*. Then:*

$$\text{If} \quad m_i - t_i \leqslant 0, \qquad \text{then} \quad c^i \geqslant e_{z_{i+1}+t_{i+1}}. \tag{79}$$

$$\text{If} \quad m_i - t_i > 0, \qquad \text{then} \quad c^i < e_{z_{i+1}+t_{i+1}}. \tag{80}$$

Proof. Claim (79) follows by Definitions 9 and 22, since $z_{i+1}+t_{i+1} \geqslant z_i+w_i+t_i+1-w_i > z_i + m_i$.

On the other hand, if $m_i > t_i$, we have $m_{i+1} - t_{i+1} + 1 = m_i + \sharp\{j \in \{1, \ldots, s\} | c^i \geqslant a_j > c^{i+1}\} - t_i + w_i > \sharp\{j \in \{1, \ldots, s\} | c^i \geqslant a_j > c^{i+1}\} + w_i$. Therefore $m_{i+1} \geqslant t_{i+1}$ and $m_{i+1} - t_{i+1} + 1$ is strictly bigger than the number of a_l's and d_j's with $j \in \Delta^c$, that are between c^i and c^{i+1}. Therefore at least one among $e_{z_{i+1}+t_{i+1}}, \ldots, e_{z_{i+1}+m_{i+1}}$ is bigger than c^i, i.e. $c^i < e_{z_{i+1}+t_{i+1}}$, as wanted. ∎

Lemma 40. *Let conditions* $(i.1)$ *and* $(ii.1)$ *from Theorem 11 be valid. Then*

$$\sum_{i=1}^{h'} c^i \geqslant \sum_{i \in \Delta^c} d_i + \sum_{i=t_0+1}^{s} a_i, \tag{81}$$

and

$$\sum_{i=1}^{h} d^i \geqslant \sum_{i \in S^c} c_i + \sum_{i=t'_0+1}^{k} b_i. \tag{82}$$

Proof. Before proceeding we note that (81) is dual to (82). So it is enough to prove one of them, e.g. (81).

Let us suppose that there are no $i \in \{1, \ldots, m\}$ such that $i \in \Delta^c$. Then by the definition we have $t_0 = s - h'$ and

$$t_i = t_{i-1} + 1 = t_0 + i, \quad i = 1, \ldots, h'.$$

If $m_i < t_i$ for all $i \in \{1, \ldots, h'\}$, then by the definition of m_i we have $c^i \geqslant a_{t_i} = a_{t_0+i}$, and thus

$$\sum_{i=1}^{h'} c^i \geqslant \sum_{i=t_0+1}^{s} a_i,$$

which is precisely (81) in this case.

If there is $i \in \{1, \ldots, h'\}$ for which $m_i \geqslant t_i$, then let $y \in \{1, \ldots, h'\}$ be the minimal such index. Then condition $(i.1)$ for c^y gives

$$\sum_{i=z_y+t_y}^{z_y+m_y} e_i \leqslant \sum_{i=y}^{h'} c^i - \sum_{i=m_y+1}^{s} a_i. \tag{83}$$

Among e_i's on the LHS there can be no d_i, since by the part (a) of the definition of the set Δ, we would have that those i do not belong to Δ, contradicting the assumption that there are no such indices. Therefore those e_i's are precisely $a_{t_y}, \ldots, a_{m_y}$, and so (83) is equivalent to

$$\sum_{i=y}^{h'} c^i \geqslant \sum_{i=t_y}^{s} a_i = \sum_{i=t_0+y}^{s} a_i. \tag{84}$$

Since for all $i = 1, \ldots, y-1$ we have $m_i + 1 \leqslant t_i = t_0 + i$, from the definition of m_i, we have $c^i \geqslant a_{t_0+i}$, for $i = 1, \ldots, y-1$. This together with (84) prove (81) in this case.

Now suppose that there exists $i \in \{1, \ldots, m\}$ such that $i \notin \Delta$. Let j be the minimal such index. By the definition of the set Δ, we have that $q_j \leqslant s$, and thus, by the definition of q_j, we conclude that S is nonempty.

Since all $d_i < c^{h'}$ satisfy $i \in \Delta$, there exists $y \in \{1, \ldots, h'\}$ such that

$$c^{y-1} > d_j > c^y.$$

Then by the definition of j, we have $j = z_y - w_{y-1} + 1$. Also, we have that $t_i = t_0 + i$, for $i = 1, \ldots, y-1$.

If there exists $i \in \{1, \ldots, y-1\}$ such that $m_i \geqslant t_i$, denote by x minimal such index. Then in exactly the same way as in the first case (since there are no $i \in \Delta^c$ with $d_i > c^{y-1}$), we obtain that condition $(i.1)$ for c^x implies

$$\sum_{i=x}^{h'} c^i \geqslant \sum_{i \in \Delta^c} d_i + \sum_{i=t_x}^{s} a_i = \sum_{i \in \Delta^c} d_i + \sum_{i=t_0+x}^{s} a_i.$$

Together with $c^i \geqslant a_{t_0+i}$, for $i = 1, \ldots, x-1$, this proves (81).

Thus, suppose that $m_i < t_i$, for all $i = 1, \ldots, y-1$, and therefore

$$c^i \geqslant a_{t_0+i}, \qquad i = 1, \ldots, y-1. \tag{85}$$

Now, since $j \notin \Delta$, we have two possibilities from the definition of Δ. If the part (a) of the definition is satisfied, d_j is among the smallest $m_y - t_y + 1$ e_i's larger than c^y. Thus, $j, j+1, \ldots, z_y \notin \Delta$, as well as $t_y \leqslant m_y$.

Then condition $(i.1)$ for c^y gives:

$$\sum_{i=z_y+t_y}^{z_y+m_y} e_i \leqslant \sum_{i=y}^{h'} c^i - \sum_{i>z_y,\, i\in\Delta^c} d_i - \sum_{i=m_y+1}^{s} a_i. \tag{86}$$

By the above assumptions $(e_{z_y+t_y}, \ldots, e_{z_y+m_y})$ consists of w_{y-1} d_i's, while the remaining $m_y - t_y + 1 - w_{y-1} = m_y - t_{y-1}$ are a_i's, i.e. they are precisely $a_{t_{y-1}+1}, \ldots, a_{m_y}$ (they are all larger than c^y). So, (86) becomes:

$$\sum_{i=y}^{h} c^i \geqslant \sum_{i\in\Delta^c} d_i + \sum_{i=t_{y-1}+1}^{s} a_i = \sum_{i\in\Delta^c} d_i + \sum_{i=t_0+y}^{s} a_i. \tag{87}$$

On the other hand, if $j \notin \Delta$ because of the part (b) of the definition of Δ, then

$$\sum_{i=y}^{h'} c^i \geqslant \sum_{i\in\Delta^c} d_i + \sum_{i=q_j+1}^{s} a_i. \tag{88}$$

Since from the definition of q_i's and t_i's we have that $q_j = t_{y-1}$, the last inequality becomes precisely (87).

Therefore, we have obtained that (87) holds, and together with (85) finally gives the wanted condition (81).

Dually by changing the roles of partitions $\mathbf{c}$ and $\mathbf{b}$ with $\mathbf{d}$ and $\mathbf{a}$, respectively, we obtain (82). $\square$

9 Proof of Theorem 11

9.1 Necessity of conditions $(i.1)$ and $(ii.1)$

Let us assume that there exists a partition $\mathbf{g}$ such that

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a}), \quad \text{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}). \tag{89}$$

Then we shall prove that conditions $(i.1)$ and $(ii.1)$ hold. By (89) and Lemma 34 we obtain

$$c^{h'} \geqslant a_s \quad \text{and} \quad d^h \geqslant b_k, \tag{90}$$

i.e. we have condition $(i.1)$ for $y = h'$, and condition $(ii.1)$ for $x = h$. Also, by Lemma 32, we have $t_0 \geqslant 0$, and thus we can define the partition $\bar{\mathbf{g}}$ as in Section 6.1.

We are left with proving $(i.1)$ for $y \in \{1, \ldots, h'-1\}$, and $(ii.1)$ for $x \in \{1, \ldots, h-1\}$. Let $y \in \{1, \ldots, h'-1\}$ be such that $t_y \leqslant m_y$ (note again that (90) means that $t_{h'} > m_{h'}$).

Let $u \in \{0, \dots, s\}$ be such that $h_u < z_y + t_y \leqslant h_{u+1}$ ($h_0 = 0$, $h_{s+1} = m + s + 1$). From $\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a})$, by the definition of the weak generalized majorization, and by Lemma 16, we have

$$\sum_{i=z_y+t_y}^{m+s} g_i \geqslant \sum_{i=z_y+t_y-u}^{m} d_i + \sum_{i=u+1}^{s} a_i$$

Together with Lemma 38 this gives

$$\sum_{i=z_y+t_y}^{m+s} \bar{g}_i \geqslant \sum_{i=z_y+t_y-u}^{m} d_i + \sum_{i=u+1}^{s} a_i. \tag{91}$$

By the definition of $\mathbf{e}$, we have

$$\sum_{i=z_y+t_y-u}^{m} d_i + \sum_{i=u+1}^{s} a_i \geqslant \sum_{i=z_y+t_y}^{m+s} e_i.$$

The last relation, together with (91) and the definition of the partition $\bar{\mathbf{g}}$ in (48), gives (i.1), as wanted.

Dually we obtain the condition (ii.1) for $x \in \{1, \dots, h-1\}$. This finishes the proof of the necessity of conditions.

9.2 Sufficiency of conditions (*i*.1) and (*ii*.1)

Suppose now that conditions (i.1) and (ii.1) hold. Thus, by Lemma 40 we have that

$$\sum_{i=1}^{h'} c^i \geqslant \sum_{i \in \Delta^c} d_i + \sum_{i=t_0+1}^{s} a_i. \tag{92}$$

and

$$\sum_{i=1}^{h} d^i \geqslant \sum_{i \in S^c} c_i + \sum_{i=t_0'+1}^{k} b_i. \tag{93}$$

Also, condition (i.1) for $y = h'$ and condition (ii.1) for $y = h$, together give

$$c^{h'} \geqslant a_s \quad \text{and} \quad d^h \geqslant b_k. \tag{94}$$

In this section we shall prove that the partition $\bar{\mathbf{g}} = (\bar{g}_1, \dots, \bar{g}_{m+s})$ defined in Section 6.1 satisfies

$$\bar{\mathbf{g}} \quad \prec'' \quad (\mathbf{d}, \mathbf{a}) \tag{95}$$

$$\bar{\mathbf{g}} \quad \prec'' \quad (\mathbf{c}, \mathbf{b}), \tag{96}$$

We start with proving (95). By Definition 3 of the weak majorization we need to prove the following:

$$d_i \geqslant \bar{g}_{i+s}, \quad i = 1, \dots, m, \tag{97}$$

$$\sum_{i=h_j+1}^{m+s} \bar{g}_i \geqslant \sum_{i=h_j-j+1}^{m} d_i + \sum_{i=j+1}^{s} a_i, \quad j = 1, \ldots, s, \tag{98}$$

$$\sum_{i=1}^{m+s} \bar{g}_i \geqslant \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i, \tag{99}$$

where $h_j := \min\{i | d_{i-j+1} < \bar{g}_i\}$, for $j = 1, \ldots, s$.

Before proceeding, we note that (94) and (46), together with (35) give

$$t_x \leqslant s, \quad \text{for all} \quad x = 0, \ldots, h'.$$

Regarding (97), since $t_0 \leqslant s$, we have that $\bar{g}_i$'s appearing in (97) are the ones defined by (50) and (51). Let $i \in \{1, \ldots, m\}$.

If $i \in \Delta$, from (50) we have that $d_i = \bar{g}_{i+t_x}$, for some $x \in \{0, \ldots, h'\}$, and since $t_x \leqslant s$ for any such x, we obtain $d_i \geqslant \bar{g}_{i+s}$, as wanted.

If on the other hand $i \notin \Delta$, then let $y \in \{0, \ldots, h'-1\}$ be such that $c^y > d_i > c^{y+1}$. Then we have that $i \in \{z_{y+1} - w_y + 1, \ldots, z_y\}$, and by (51) we have:

$$d_i > c^{y+1} = \bar{g}_{z_{y+1}+t_{y+1}} = \bar{g}_{z_{y+1}-w_y+1+t_y} \geqslant \bar{g}_{i+s},$$

since $z_{y+1} - w_y + 1 \leqslant i$ and $t_y \leqslant s$. This proves (97).

Now, we pass to (98). First we note that from the definition of $\bar{g}_i$, (49)–(50), we can compute the values of h_j, for $j = 1, \ldots, s$. We have that:

$$h_j = j, \quad j = 1, \ldots, t_0, \tag{100}$$

and

$$h_j = z_x + t_x, \quad j = t_0 + 1, \ldots, s. \tag{101}$$

Here $x = \min\{i \in \{1, \ldots, h'\} | t_i = j\}$.

From (49) we have $\bar{g}_{t_0} \geqslant d_1$, which gives (100).

As for (101), first note that x is well-defined, i.e. the set $\{i \in \{1, \ldots, h'\} | t_i = j\}$ is non-empty, for $j = t_0 + 1, \ldots, s$. Indeed, from the definition of t_x, we have that $t_{x+1} = t_x + 1 - w_x$, and so $t_{x+1} \leqslant t_x + 1$, for $x = 0, \ldots, h'-1$. Since $t_0 \leqslant s = t_{h'}$, for every j, with $t_0 < j \leqslant s$, there must exist at least one $x \in \{1, \ldots, h'\}$ such that $t_x = j$.

Now, we show that for every $j \in \{t_0 + 1, \ldots, s\}$, there exists $i \in \{1, \ldots, h'\}$, such that $h_j = z_i + t_i$.

Indeed, if, on the contrary, there exists $j \in \{t_0 + 1, \ldots, s\}$, for which there are no $i \in \{1, \ldots, h'\}$, such that $h_j = z_i + t_i$, then let $u \in \{0, \ldots, h'\}$ be such that $z_u + t_u < h_j < z_{u+1} + t_{u+1}$. Then by (50) we have $\bar{g}_{h_j} = d_{h_j - t_u}$, and from the definition of h_j, we have $d_{h_j-j+1} < \bar{g}_{h_j} = d_{h_j-t_u}$, which implies $j \leqslant t_u$, and so $u \geqslant 1$. But then, from (51), $\bar{g}_{z_u+t_u} = c^u > d_{z_u+1} \geqslant d_{z_u+t_u-j+1}$, and so $h_j \leqslant z_u + t_u$, which is a contradiction.

Hence we have that there exists $i \in \{1, \ldots, h'\}$ such that $h_j = z_i + t_i$. Then from the definition of h_j we have $d_{z_i} > c^i = \bar{g}_{z_i+t_i} = \bar{g}_{h_j} > d_{h_j-j+1} = d_{z_i+t_i-j+1}$, and so $t_i \geqslant j$. Now, if $t_i > j$, since $t_{x+1} \leqslant t_x + 1$, for $x = 0, \dots, h' - 1$, since and $t_0 < j$, we have that there exists $u \in \{1, \dots, i-1\}$ such that $t_u = j$. Then $\bar{g}_{z_u + t_u} = c^u > d_{z_u+1} = d_{z_u+t_u-j+1}$, which together with $z_u + t_u < z_i + t_i$ (since $u < i$) contradicts the definition of h_j. Therefore $t_i = j$ which finally proves (101).

Now we shall prove (98).

Let $j = 1, \dots, t_0$. By (100), condition (98) becomes

$$\sum_{i=j+1}^{m+s} \bar{g}_i \geqslant \sum_{i=1}^{m} d_i + \sum_{i=j+1}^{s} a_i, \quad j = 1, \dots, t_0. \tag{102}$$

By (49), it is enough to prove (102) for $j = t_0$, i.e.:

$$\sum_{i=t_0+1}^{m+s} \bar{g}_i \geqslant \sum_{i=1}^{m} d_i + \sum_{i=t_0+1}^{s} a_i, \tag{103}$$

which is by the definition of $\bar{g}_{t_0+1}, \dots, \bar{g}_{m+s}$, equivalent to (81).

Now, let $j = t_0 + 1, \dots, s$. Let $x_j = \min\{i \in \{1, \dots, h'\} | t_i = j\}$. Then, by (101), the condition (98) becomes

$$\sum_{i=z_{x_j}+t_{x_j}+1}^{m+s} \bar{g}_i \geqslant \sum_{i=z_{x_j}+1}^{m} d_i + \sum_{i=j+1}^{s} a_i,$$

which is (by the definition of $\bar{g}_i$'s) equivalent to

$$\sum_{i=x_j+1}^{h'} c^i \geqslant \sum_{i \geqslant z_{x_j}+1,\, i \in \Delta^c} d_i + \sum_{i=t_{x_j}+1}^{s} a_i. \tag{104}$$

In order to prove (104) we need to consider the following three possibilities:

$$\begin{gathered} \bullet \quad w_{x_j} > 0, \text{ i.e. } c^{x_j} > d_{z_{x_j+1} - w_{x_j} + 1} > c^{x_j+1}, \\ \text{and } z_{x_j+1} - w_{x_j} + 1 \notin \Delta, \text{ by the part } (b) \text{ of the definition of the set } \Delta \end{gathered} \tag{105}$$

$$\begin{gathered} \bullet \quad w_{x_j} > 0, \text{ i.e. } c^{x_j} > d_{z_{x_j+1} - w_{x_j} + 1} > c^{x_j+1}, \\ \text{and } z_{x_j+1} - w_{x_j} + 1 \notin \Delta, \text{ by the part } (a) \text{ of the definition of the set } \Delta, \end{gathered} \tag{106}$$

$$\bullet \quad w_{x_j} = 0, \text{ i.e. there are no } i \notin \Delta, c^{x_j} > d_i > c^{x_j+1}. \tag{107}$$

First consider the case (105). Suppose that $w_{x_j} > 0$, such that $z_{x_j+1} - w_{x_j} + 1 \notin \Delta$, $c^{x_j} > d_{z_{x_j+1}-w_{x_j}+1} > c^{x_j+1}$, satisfies the following condition (see the part (b) of the definition of the set Δ and note that $q_{z_{x_j+1}-w_{x_j}+1} = t_{x_j}$):

$$\sum_{i=x_j+1}^{h'} c^i \geqslant d_{z_{x_{j+1}}-w_{x_j}+1} + \sum_{i>z_{x_{j+1}}-w_{x_j}+1,\, i\in\Delta^c} d_i + \sum_{i=t_{x_j}+1}^{s} a_i. \tag{108}$$

Condition (108) is equivalent to (104), which finishes our proof in this case.

Next, we consider the case (106). In this case we have that $w_{x_j} > 0$, and $d_{z_{x_j+1}-w_{x_j}+1}$ is among $\sharp\{i \in \{1,\dots,s\}|a_i > c^{x_j+1}\} - s + (h'-x_j) - \sharp\{i \in \Delta^c|d_i < c^{x_j+1}\} + 1$ smallest e_i's larger than c^{x_j+1} (see the part (a) of the definition of the set Δ), i.e.

$$d_{z_{x_j+1}-w_{x_j}+1} \in \{e_{z_{x_j+1}+t_{x_j+1}}, \dots, e_{z_{x_j+1}+m_{x_j+1}}\}.$$

Thus, in this case we have that $t_{x_j+1} \leqslant m_{x_j+1}$.

Let us consider the differences $m_i - t_i$ for all $i = 0,\dots,x_j+1$. We have that $m_{x_j+1} - t_{x_j+1} \geqslant 0$, and $m_0 - t_0 = -t_0 \leqslant 0$ (because of Lemma 32). Thus, there exists $v := \max\{i \in \{0,\dots,x_j\}|m_i - t_i \leqslant 0\}$. Then $m_{v+1} - t_{v+1} \geqslant 0$ and $v \leqslant x_j$, so we have that condition $(i.1)$ is satisfied for $v+1$. i.e.

$$\sum_{i=z_{v+1}+t_{v+1}}^{z_{v+1}+m_{v+1}} e_i \leqslant \sum_{i=v+1}^{h'} c^i - \sum_{i>z_{v+1},\, i\in\Delta^c} d_i - \sum_{i=m_{v+1}+1}^{s} a_i. \tag{109}$$

Let us suppose that $v = x_j$. Then $m_{x_j} - t_{x_j} \leqslant 0$. This, by Lemma 39 implies that $c^{x_j} \geqslant e_{z_{x_j+1}+t_{x_j+1}}$. Thus, there are exactly w_{x_j} of d_i's among $e_{z_{x_j+1}+t_{x_j+1}}, \dots, e_{z_{x_j+1}+m_{x_j+1}}$, and those are $d_{z_{x_j+1}-w_{x_j}+1}, \dots, d_{z_{x_j+1}}$. The remaining $m_{x_j+1} - t_{x_j+1} + 1 - w_{x_j} = m_{x_j+1} - t_{x_j}$ are a_i's, i.e. $a_{t_{x_j}+1}, \dots, a_{m_{x_j+1}}$. Then (109) becomes (note that we are in the case $v = x_j$)

$$\sum_{i=x_j+1}^{h'} c^i \geqslant \sum_{i>z_{x_j},\, i\in\Delta^c} d_i + \sum_{i=t_{x_j}+1}^{s} a_i, \tag{110}$$

as wanted.

Next, let us suppose that $0 \leqslant v < x_j$. In this case $m_i - t_i > 0$, for all $i = v+1,\dots,x_j$, and so by Lemma 39 we have that $c^i < e_{z_{i+1}+t_{i+1}}$, for all $i = v+1,\dots,x_j$. This implies that there are no $j \in \Delta$ with $c^{v+1} > d_j > c^{x_j+1}$, and so $w_i = z_{i+1} - z_i$ and

$$z_{i+1} + t_{i+1} = z_i + t_{i+1} + w_i = z_i + t_i + 1, \quad i = v+1,\dots,x_j. \tag{111}$$

It also means that (109) can be re-written as :

$$\sum_{i=z_{v+1}+t_{v+1}}^{z_{x_j}+m_{x_j}} e_i \leqslant \sum_{i=v+1}^{h'} c^i - \sum_{i>z_{x_j},\, i\in\Delta^c} d_i - \sum_{i=m_{x_j}+1}^{s} a_i. \tag{112}$$

Since $m_v - t_v \leqslant 0$, by Lemma 39 we have $c^v \geqslant e_{z_{v+1}+t_{v+1}}$, and so $c^v \geqslant e_{z_{v+1}+t_{v+1}} \geqslant \cdots \geqslant e_{z_{x_j}+m_{x_j}} > c^{x_j}$.

From the definition of x_j, we have $t_r < t_{x_j} = j$, for all $r < x_j$, i.e.

$$\sharp\{i \in \Delta^c | c^r > d_i > c^{x_j}\} < x_j - r, \quad \text{for all} \quad r < x_j. \tag{113}$$

Therefore among $e_{z_{v+1}+t_{v+1}}, \ldots, e_{z_{x_j}+m_{x_j}}$ there are at most $x_j - v - 1$ d_i's (note that as we have shown above, all such d_i's satisfy $i \notin \Delta$). Also by (111), $z_{x_j}+t_{x_j} = z_{v+1}+t_{v+1}+x_j-(v+1)$. Thus, among those e_i's there are at least $z_{x_j}+m_{x_j}-(z_{v+1}+t_{v+1})+1-(x_j-v-1) = z_{x_j}+m_{x_j}+1-(z_{x_j}+t_{x_j}) = m_{x_j}-t_{x_j}+1$, a_i's. Thus $a_{t_{x_j}}, \ldots, a_{m_{x_j}}$ surely belong to them. Since $a_{t_{x_j}} \geqslant a_{m_{x_j}} > c^{x_j}$ and since $e_{z_{i+1}+t_{i+1}} > c^i$, $i = v+1, \ldots, x_j$, (113) and (112) give

$$\sum_{i=x_j+1}^{h'} c^i \geqslant \sum_{i>z_{x_j}, i \notin \Delta} d_i + \sum_{i=t_{x_j}+1}^{s} a_i,$$

i.e. we have proved (104).

So, we are left with the case (107). In this case $w_{x_j} = 0$, which means that there are no $i \notin \Delta$, such that $c^{x_j} > d_i > c^{x_j+1}$.

In this case, we are left with two possibilities

$$t_{x_j+1} \leqslant m_{x_j+1} \tag{114}$$

$$t_{x_j+1} > m_{x_j+1} \tag{115}$$

The case (114) is done exactly as in the case (106) when $w_{x_j} > 0$ and $t_{x_j+1} \leqslant m_{x_j+1}$.

Now, consider the case (115). The proof of this case goes by the induction on $j = t_0+1, \ldots, s$.

Let $j = s$. Since $c^{h'} \geqslant a_s$, (46) implies $t_x < s$ for $x < h'$. So since $t_{h'} = s$, we have $x_s = h'$. Hence (104) becomes $0 \geqslant 0$, which is trivially satisfied.

Now, fix $j \in \{t_0+1, \ldots, s-1\}$, and suppose that (104) is satisfied for all $j+1, \ldots, s$. We shall prove that it is then also valid for j.

Since $t_{x_j+1} > m_{x_j+1}$, we have $c^{x_j+1} \geqslant a_{m_{x_j+1}+1} \geqslant a_{t_{x_j+1}}$. Since there are no $i \notin \Delta$ such that $c^{x_j} > d_i > c^{x_j+1}$, we have $t_{x_j+1} = t_{x_j} + 1 = j+1$, and so $x_{j+1} = x_j + 1$. By the induction hypothesis for $j+1$, we have

$$\sum_{i=x_{j+1}+1}^{h'} c^i \geqslant \sum_{i \geqslant z_{x_{j+1}}+1, i \in \Delta^c} d_i + \sum_{i=t_{x_{j+1}}+1}^{s} a_i. \tag{116}$$

Since $c^{x_j+1} \geqslant a_{m_{x_j+1}+1} \geqslant a_{t_{x_j+1}} = a_{t_{x_j}+1}$, then (116) gives (104).

This finishes our proof of (104), and consequently of (98).

Finally, (99) follows from (103) (i.e. (81)), together with (49). Therefore we have shown that

$$\bar{\mathbf{g}} \prec'' (\mathbf{d}, \mathbf{a}).$$

Dually we obtain

$$\bar{\mathbf{g}} \prec'' (\mathbf{c}, \mathbf{b}).$$

This finishes the proof of Theorem 11.

Example 41. Now we can go back to Example 15 and define the wanted partition $\bar{\mathbf{g}}$ such that

$$\bar{\mathbf{g}} \prec'' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \bar{\mathbf{g}} \prec'' (\mathbf{c}, \mathbf{b}),$$

where $\mathbf{a} = (11, 9, 1)$, $\mathbf{b} = (12, 3)$, $\mathbf{c} = (8, 7, 5)$ and $\mathbf{d} = (10, 4)$.

In Example 15 we have shown that the conditions from Theorem 11 are satisfied, implying the existence of the wanted partition $\bar{\mathbf{g}}$. We shall now compute explicitly $\bar{\mathbf{g}}$ as in Section 6.1.

In Example 15, we have computed sets S and Δ for the partitions $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$:

$$S = \{1, 2, 3\} \quad \text{and} \quad \Delta = \{2\}.$$

Thus, $c^1 = c_1 = 8$, $c^2 = c_2 = 7$, $c^3 = c_3 = 5$, and $d^1 = d_2 = 4$. Also, by (30) we have $t_0 = 1$.

We shall define $\bar{\mathbf{g}}$ by formula (48). Hence,

$$(\bar{g}_1, \bar{g}_2, \bar{g}_3, \bar{g}_4, \bar{g}_5) = (c_1, c_2, c_3) \cup (d_2) \cup (M),$$

where $M = \max\{a_1, b_1, c_1, d_1\} + 1 = \max\{12, 11, 10, 8\}+1 = 13$. In such way, we have:

$$\bar{\mathbf{g}} = (13, 8, 7, 5, 4).$$

One can directly check by Definition 3 that the partition $\bar{\mathbf{g}} = (13, 8, 7, 5, 4)$ satisfies

$$(13, 8, 7, 5, 4) \prec'' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad (13, 8, 7, 5, 4) \prec'' (\mathbf{c}, \mathbf{b}),$$

as desired.

10 Proof of Theorem 13

Necessity of conditions (24) and $\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b})$:

Let there exists a partition $\mathbf{g}$ such that

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}) \quad \text{and } \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}). \tag{117}$$

Then (24) follows trivially. Also, such a partition $\mathbf{g}$ also satisfies

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a}) \quad \text{and } \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}), \tag{118}$$

and so by Theorem 11 we obtain condition $\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b})$, as wanted.

Sufficiency of conditions (24) and $\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b})$:

Let us assume that conditions (24) and $\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b})$ are valid. By Theorem 11, condition $\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b})$ implies the existence of a partition $\mathbf{g}'$ such that

$$\mathbf{g}' \prec'' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g}' \prec'' (\mathbf{c}, \mathbf{b}). \tag{119}$$

The rest of this section is completely analogous to [5]. It doesn't depend on the definitions of the sets S and Δ, so it remains completely the same. Indeed, the partition $\mathbf{g}'$ satisfies all the wanted properties for (117) except the total sum, i.e. we have

$$\sum_{i=1}^{m+s} g'_i \geqslant \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i,$$

and not the equality. We shall define the desired partition $\mathbf{g}$ by decreasing some of the largest elements of the partition $\mathbf{g}'$, so that the sum of all elements of $\mathbf{g}$ is correct, and such that all properties of generalized majorization remain valid.

To that end, let

$$\Omega := \sum_{i=1}^{m+s} g'_i - \left(\sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i \right), \tag{120}$$

and let

$$f := \begin{cases} 1 + \max \left\{ i \in \{1, \dots, m+s\} \middle| \sum_{j=1}^{i} g'_j - i g'_i < \Omega \right\}, & \text{if } \Omega > 0, \\ 1, & \text{if } \Omega = 0. \end{cases} \tag{121}$$

Let $X := \sum_{i=1}^{f-1} g'_i - \Omega$, and let u and v be unique non-negative integers, such that $X = u(f-1) + v$, with $v < f - 1$. Then we set:

$$\begin{aligned} g_i &= u + 1, \quad i = 1, \dots, v, && (122) \\ g_i &= u, \qquad i = v+1, \dots, f-1, && (123) \\ g_i &= g'_i, \qquad i = f, \dots, m+s. && (124) \end{aligned}$$

Then $\mathbf{g} = (g_1, g_2, \dots, g_{m+s})$ is a partition of non-negative integers satisfying:

$$\sum_{i=1}^{m+s} g_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i, \tag{125}$$

$$g_i = g'_i, \quad \text{for all} \quad i \geqslant f, \tag{126}$$

$$g'_{f-1} \geqslant g_i \geqslant g'_f, \quad \text{for all} \quad i = 1, \dots, f-1,$$

and

$$g_1 \geqslant g_{f-1} \geqslant g_1 - 1.$$

By Lemma 17 such defined $g_1 \geqslant \cdots \geqslant g_{m+s}$ satisfy

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}).$$

However, since (24) and (125) are valid, by the definition of the generalized majorization we also have

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}),$$

as wanted.

Example 42. Consider a partition $\mathbf{g}' = (10, 9, 8, 5, 3, 1)$, satisfying

$$\mathbf{g}' \prec'' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g}' \prec'' (\mathbf{c}, \mathbf{b})$$

for some partitions **a**, **b**, **c**, and **d** such that the sum of the elements of partitions **a** and **d** is equal to the sum of the elements of partitions **b** and **c**. Also, let the difference between the sums of the elements of the partition $\mathbf{g}'$ and the elements of the partitions **d** and **a** together, be equal to 6.

Then by (120) we have $\Omega = 6$. Let us define a partition **g** such that

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}).$$

Since $\Omega = 6$, by (121) we have that $f = 4$, and hence by (124) we directly obtain $g_4 = 5$, $g_5 = 3$ and $g_6 = 1$. Next, we compute $X = g'_1 + g'_2 + g'_3 - \Omega = 21$, as well as $u = 7$ and $v = 0$. Hence by (123) we have $g_1 = g_2 = g_3 = 7$. Therefore, the wanted partition is

$$\mathbf{g} = (g_1, g_2, g_3, g_4, g_5, g_6) = (7, 7, 7, 5, 3, 1).$$

Example 43. Now we can go back to Examples 15 and 41 and define the wanted partition **g** such that

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}). \tag{127}$$

Recall that $\mathbf{a} = (11, 9, 1)$, $\mathbf{b} = (12, 3)$, $\mathbf{c} = (8, 7, 5)$ and $\mathbf{d} = (10, 4)$. In Example 43, we have obtained the partition $\bar{\mathbf{g}} = (13, 8, 7, 5, 4)$ satisfying

$$(13, 8, 7, 5, 4) \prec'' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad (13, 8, 7, 5, 4) \prec'' (\mathbf{c}, \mathbf{b}).$$

However partition $\bar{\mathbf{g}}$ does not satisfy (127), since $\sum_{i=1}^{5} \bar{g}_i > \sum_{i=1}^{3} a_i + \sum_{i=1}^{2} d_i$. So, we need to decrease its sum by $\Omega = \sum_{i=1}^{5} \bar{g}_i - \sum_{i=1}^{3} a_i - \sum_{i=1}^{2} d_i = 2$, as explained above. We have $f = 1 + \max\{i | \sum_{j=1}^{i} \bar{g}_j - i\bar{g}_i < 2\} = 2$. Hence, we have $g_2 = 8$, $g_3 = 7$, $g_4 = 5$ and $g_5 = 4$. Also, we have that $X = \bar{g}_1 - \Omega = 11$, and so we directly obtain $g_1 = 11$.

One can directly check by Definition 2 that such obtained partition $\mathbf{g} = (11, 8, 7, 5, 4)$ indeed satisfies (127), as desired.

11 Special cases of double generalized majorization

11.1 Case $n = 0$

In this subsection we assume $n = 0$, i.e. we assume that there are no $c_1, \dots, c_n$. In this case the generalized majorization

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b})$$

drops to the classical majorization

$$\mathbf{g} \prec \mathbf{b}.$$

As we have seen Problem 4 for the generalized majorization is very challenging, while the same kind of problem for classical majorization is trivial. Indeed, there always exists a partition simultaneously majorized by two partitions of the same sum. If $n = 0$ (or dually $m = 0$), then Problem 4 becomes something in-between these two problems.

Before solving it, let us consider sets S and Δ in this case. It is straightforward to see that in this case the set S must be empty. Let us check what happens with the set Δ:

By (16) we obtain that, in this case, all $j = 1, \dots, m$ are in Δ. Hence we have $h = m$ and $d_j = d^j$, $j = 1, \dots, m$. We also have:

$$m'_j := \sharp\{i | b_i > d_j\}; \quad t'_j := s + j; \quad z'_j := 0, \quad j = 1, \dots, m.$$

Also, the weak generalized majorization

$$\mathbf{g} \prec'' (\mathbf{c}, \mathbf{b})$$

in this case becomes

$$\sum_{i=j+1}^{m+s} g_i \geqslant \sum_{i=j+1}^{m+s} b_i, \quad j = 0, \dots, m+s-1.$$

As a corollary to Theorem 11 we obtain the following result:

Theorem 44. *Let* $\mathbf{d} = (d_1, \dots, d_m)$, $\mathbf{a} = (a_1, \dots, a_s)$ *and* $\mathbf{b} = (b_1, \dots, b_{m+s})$ *be partitions. There exists a partition* $\mathbf{g} = (g_1, \dots, g_{m+s})$, *such that*

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a})$$

and

$$\sum_{i=j+1}^{m+s} g_i \geqslant \sum_{i=j+1}^{m+s} b_i, \quad j = 0, \dots, m+s-1,$$

if and only if

$$\sum_{i=j}^{m} d_i \geqslant \sum_{i=s+j}^{m+s} b_i, \quad \textit{for all} \quad j = 1, \dots, m. \tag{128}$$

Proof. By Theorem 11 we have that such a $\mathbf{g}$ in this case exists if and only if

$(o.2)$ $d_m \geqslant b_{m+s}$

$(i.2)$ if $x \in \{1,\dots,m-1\}$ is such that $x+s \leqslant m'_x$ then

$$\sum_{i=x}^{m} d_i \geqslant \sum_{i=x+s}^{m+s} b_i.$$

However by the definition of m'_x we have that $x+s \leqslant m'_x$ is equivalent to $d_x \leqslant b_{s+x}$. We are left with proving that $(o.2)$ and $(i.2)$ together are equal to (128).

It is straightforward to see that (128) implies both $(o.2)$ and $(i.2)$. The proof of the contrary goes by the induction on $j \in \{1,\dots,m\}$.

Let $j=m$. Then $(o.2)$ is (128).

Let $j \in \{1,\dots,m-1\}$, and let us suppose that (128) is valid for all $j+1,\dots,m+s$. Then

$$\sum_{i=j+1}^{m} d_i \geqslant \sum_{i=j+s+1}^{m+s} b_i. \tag{129}$$

Now we have to consider two possibilities:

$$d_j \leqslant b_{j+s} \tag{130}$$

$$d_j > b_{j+s} \tag{131}$$

If (130), then $(i.2)$ directly gives

$$\sum_{i=j}^{m} d_i \geqslant \sum_{i=j+s}^{m+s} b_i, \tag{132}$$

as wanted.

If (131), then by (129) we also get (132), which finishes the proof. □

Also, as a corollary to the previous result we have:

Theorem 45. *Let* $\mathbf{d}=(d_1,\dots,d_m)$*,* $\mathbf{a}=(a_1,\dots,a_s)$ *and* $\mathbf{b}=(b_1,\dots,b_{m+s})$ *be partitions. There exists a partition* $\mathbf{g}=(g_1,\dots,g_{m+s})$*, such that*

$$\mathbf{g} \prec' (\mathbf{d},\mathbf{a})$$

and

$$\mathbf{g} \prec \mathbf{b}$$

if and only if

$$\sum_{i=1}^{m+s} b_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i \tag{133}$$

and

$$\sum_{i=j}^{m} d_i \geqslant \sum_{i=s+j}^{m+s} b_i, \qquad \textit{for all } j=1,\dots,m. \tag{134}$$

Remark 46. We note that in Theorems 44 and 45 conditions (128) and (134) are much simpler comparing to (i.1) and (ii.1). Even more surprisingly, (128) and (134) do not depend on partition $\mathbf{a}$.

Example 47. Let $\mathbf{d} = (15, 6, 4, 2)$, $\mathbf{a} = (12, 10)$ and $\mathbf{b} = (20, 11, 9, 5, 3, 1)$. Then

$$\sum_{i=1}^{4} d_i + \sum_{i=1}^{2} a_i = \sum_{i=1}^{6} b_i = 49.$$

Also, then

$$\sum_{i=1}^{4} d_i = 27 \geqslant \sum_{i=3}^{6} b_i = 18,$$

$$\sum_{i=2}^{4} d_i = 12 \geqslant \sum_{i=4}^{6} b_i = 9,$$

$$\sum_{i=3}^{4} d_i = 6 \geqslant \sum_{i=5}^{6} b_i = 4,$$

$$\sum_{i=4}^{4} d_i = 2 \geqslant \sum_{i=6}^{6} b_i = 1,$$

hence by Theorem 45 there exist a partition $\mathbf{g}$ such that

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}) \quad \text{and} \quad \mathbf{g} \prec \mathbf{b}. \tag{135}$$

Moreover, we know how to define such a partition. Since in this case indices of all d_i's belong to Δ, while the set S is empty, by definition (48) we have

$$\bar{\mathbf{g}} = (21, 21, 15, 6, 4, 2).$$

Finally, since $\sum_{i=1}^{6} \bar{g}_i = 69 > 49$, we have to decrease the largest terms of $\bar{\mathbf{g}}$ as in Section 10. To that end, we have $\Omega = \sum_{i=1}^{6} \bar{g}_i - \sum_{i=1}^{6} b_i = 69 - 49 = 20$, and by (121) we have that $f = 4$, and consequently $X = \bar{g}_1 + \bar{g}_2 + \bar{g}_3 - \Omega = 37$, $u = 12$ and $v = 1$. Therefore, we have that

$$\mathbf{g} = (13, 12, 12, 6, 4, 2).$$

By Definitions 1 and 2, it is an easy exercise to check that such defined $\mathbf{g}$ satisfies (135).

11.2 Case $s = 0$

Let us consider another particular case of Problem 4. Let us assume that $s = 0$, i.e. that there is no partition $\mathbf{a}$. By the definition of the weak generalized majorization

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a})$$

in this case becomes

$$d_i \geqslant g_i \quad i = 1, \ldots, m, \tag{136}$$

$$\sum_{i=1}^{m} g_i \geqslant \sum_{i=1}^{m} d_i, \tag{137}$$

i.e. we have

$$g_i = d_i, \quad i = 1, \ldots, m.$$

And so necessary and sufficient condition for the existence of a partition $\mathbf{g}$ satisfying

$$\mathbf{g} \prec'' (\mathbf{c}, \mathbf{b})$$

and (136) and (137) become

$$\mathbf{d} \prec'' (\mathbf{c}, \mathbf{b}).$$

We note that this implies that in Definition 8 we have that $j \in \Delta$ for all $j = 1, \ldots, m$, and that none of c_j is such that $j \in S$, i.e. S is empty.

12 A word or two on non-negative partitions

Since Problems 4 and 5 have already shown their importance in applications (see e.g. [2, 4, 7, 9, 10, 12, 13, 25, 26]), in this section we address the question of non-negativity of involved partitions. This is particularly important when focusing on Matrix Pencil Completion Problems. In fact, both Problems 4 and 5 naturally appear in Matrix Pencil Completion Problems, where partitions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$, are defined by degrees of the Kronecker invariants of the involved matrix pencils [7, 10, 16]. Hence partitions $\mathbf{c}$ and $\mathbf{d}$ must be nonnegative, while entries of partitions $\mathbf{a}$ and $\mathbf{b}$ are always bigger than or equal to -1. Moreover, the wanted partition $\mathbf{g}$ also needs to be non-negative.

In solving Problem 4 we define $\mathbf{g}$ by choosing elements from $\mathbf{c}$ and $\mathbf{d}$: if $\mathbf{c}$ and $\mathbf{d}$ are non-negative partitions, so is $\mathbf{g}$. In fact, in the course of solving Problems 4 and 5 we have also proved the following:

Corollary 48. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c}$ *and* $\mathbf{d}$ *be partitions as in (2)–(5), respectively. Let* $\mathbf{c}$ *and* $\mathbf{d}$ *be partitions of non-negative integers. There exists a partition* $\mathbf{g} = (g_1, \ldots, g_{m+s})$ *of non-negative integers, such that*

$$\mathbf{g} \prec'' (\mathbf{d}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec'' (\mathbf{c}, \mathbf{b}) \tag{138}$$

if and only if

$$\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b})$$

Corollary 49. *Let* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c}$ *and* $\mathbf{d}$ *be partitions as in (2)–(5), respectively. Let* $\mathbf{c}$ *and* $\mathbf{d}$ *be partitions of non-negative integers. There exists a partition* $\mathbf{g} = (g_1, \ldots, g_{m+s})$ *of non-negative integers, such that*

$$\mathbf{g} \prec' (\mathbf{d}, \mathbf{a}) \quad \textit{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}) \tag{139}$$

if and only if

$$\sum_{i=1}^{n} c_i + \sum_{i=1}^{k} b_i = \sum_{i=1}^{m} d_i + \sum_{i=1}^{s} a_i \geqslant 0, \tag{140}$$

and the condition $\bar{\Omega}(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b})$ *is valid.*

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References

[1] G. E. Andrews, K. Eriksson. Integer partitions. Cambridge University Press, 2004.

[2] I. Baragaña, I. Zaballa. Column completion of a pair of matrices. *Linear Multilinear A.* 27 (1990) 243-273.

[3] D. Carlson. Inequalities relating the degrees of elementary divisors within a matrix. *Simon Stevin* 44 (1970) 3-10.

[4] M. Dodig, M. Stošić. Rank one perturbations of matrix pencils. *SIAM J. Matrix Anal. Appl.* Vol. 41, No. 4 (2020) 1889-1911.

[5] M. Dodig, M. Stošić. On properties of the generalized majorization. *Electron. J. Linear Algebra* 26 (2013) 471-509.

[6] M. Dodig. Completion up to a matrix pencil with column minimal indices as the only nontrivial Kronecker invariants. *Linear Algebra Appl.* 438 (2013) 3155-3173.

[7] M. Dodig, M. Stošić. Combinatorics of column minimal indices and matrix pencil completion problems. *SIAM J. Matrix Anal. Appl.* 31 (2010) 2318-2346.

[8] M. Dodig, M. Stošić. On convexity of polynomial paths and generalized majorizations. *Electron. J. Comb.* Vol. 17, No.1:#R61 (2010).

[9] M. Dodig, M. Stošić. Bounded rank perturbations of quasi-regular pencils over arbitrary fields. Preprint.

[10] M. Dodig. Matrix pencils completions under double rank restrictions. *FILOMAT* Vol. 36, issue 4 (2022) 1269–1293.

[11] M. Dodig, M. Stošić. Double generalized majorization and diagrammatics. to appear in *Ars Math. Contemp.* (2022), https://doi.org/10.26493/1855-3974.2691.0b7.

[12] M. Dodig. Complete characterisation of Kronecker invariants of a matrix pencil with a prescribed quasi-regular subpencil. *Electron. J. Linear Algebra* 36 (2020) 430-445.

[13] M. Dodig, M. Stošić. Completion of matrix pencils with a single rank restriction. *Linear Multilinear A.* (2021), https://doi.org/10.1080/03081087.2021.1981810.

[14] M. Dodig, M. Stošić. Explicit solution of the Carlson problem. *Linear Algebra Appl.* 436 (2012) 190–201.

[15] W. Fulton. Eigenvalues, invariant factors, highest weights, and Schubert calculus. *Bull. Amer. Math. Soc.* 37 (3) (2000) 209-249.

[16] F. R. Gantmacher. The Theory of Matrices, Vol.2. Chelsea Publishing Company, New York, 1960.

[17] Y. Han. Subrepresentations of Kronecker representations. *Linear Algebra Appl.* 402 (2005) 150-164.

[18] G. Hardy, J. E. Littlewood, G. Pólya. Inequalities. Cambridge University Press, 1991.

[19] A. Knutson and T. Tao. The honeycomb model of $GL_n(C)$ tensor products I: proof of the saturation conjecture. *J. Amer. Math. Soc.* 12 (1999) 10551090.

[20] J. Loiseau, S. Mondié, I. Zaballa, P. Zagalak. Assigning the Kronecker invariants of a matrix pencil by row or column completion. *Linear Algebra Appl.* 278 (1998) 327-336.

[21] A. W. Marshall, J. Olkin. Inequalities: majorization and its applications. Acad. Press, 1979.

[22] C. Mehl, V. Mehrmann, M. Wojtylak. On the distance to singularity via low rank perturbations. *Oper. Matrices* 9 (2015) 733-772.

[23] S. V. Savchenko. On the change in the spectral properties of a matrix under perturbations of sufficiently low rank. *Funct. Anal. Appl.* 38 (2004) 69-71.

[24] C. Szántó. Submodules of Kronecker modules via extension monoid products. *J. Pure Appl. Algebra* 222 (2018) 3360-3378.

[25] C. Szántó, I. Szöllősi. A short solution to the subpencil problem involving only column minimal indices. *Linear Algebra Appl.* 517 (2017) 99-119.

[26] I. Szöllősi. On the combinatorics of extensions of preinjective Kronecker modules. *Acta Univ. Sapientiae Math.* 6 (1) (2014) 92-106.

[27] I. Szöllősi. private communication.

[28] F. De Terán, F. Dopico. Low rank perturbation of Kronecker structures without full rank. *SIAM J. Matrix Anal. Appl.* 29 (2007) 496-529.

[29] I. Zaballa. Matrices with prescribed rows and invariant factors. *Linear Algebra Appl.* 87 (1987) 113-146.