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Remarks on Recent Results for Generalized F-Contractions

Huaping Huang [1,*], Vesna Todorčević [2] and Stojan Radenović [3]

1 School of Mathematics and Statistics, Chongqing Three Gorges University, Wanzhou 404020, China
2 Department of Mathematics, Faculty of Organizational Sciences, University of Belgrade, Jove Ilića 154, 11000 Belgrade, Serbia; vesna.todorcevic@fon.bg.ac.rs
3 Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Belgrade, Serbia; radens@beotel.net
* Correspondence: huaping@sanxiau.edu.cn

Abstract: In this paper, we give some remarks on the recent papers on generalized F-contractions. Our results unify and generalize the previous results in the existing literature. Moreover, we give an example to support our results. As an application, we give the existence and uniqueness of a solution to a class of differential equations.

Keywords: strictly increasing function; generalized F-contraction; fixed point; integral equation

MSC: 47H09; 47H10; 54H25

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1. Introduction and Preliminaries

Since Banach invented the Banach contraction principle (for short, BCP) [1], fixed point theory has shown rapid development (see [2–5]). In recent years, by using different methods, BCP has been generalized greatly. One of the significant generalizations of BCP is due to Wardowski [6], who introduced a new contraction based on an auxiliary function F satisfying certain conditions, called F-contraction, and presented a fixed point theorem. Subsequently, several interesting modifications and extensions dealing with the original results of Wardowski have been carried out in various ways by many researchers. For the details on F-contractions, the reader may refer to [7–10] and references therein. Motivated by the previous work on F-contractions, throughout this paper, we introduce a generalized F-contraction, which embeds a Hardy–Rogers type function (see [3]). Moreover, we obtain a fixed point theorem for this contraction. In our opinion, our results are more general than ones in the literature. We simplify and correct some errors from previous papers. Further, as an application, we utilize our results to study the existence of the unique solution for a class of differential equations.

In what follows, unless otherwise specified, we always denote by $\mathbb{R}$ the set of all real numbers, $\mathbb{N}$, the set of all nonnegative integers, $\mathbb{N}^*$, the set of all positive integers.

Inspired by the works of Wardowski [6,11,12], we denote by $\mathcal{F}\uparrow$ the family of all the functions $F:(0,+\infty)\to\mathbb{R}$ satisfying that F is a strictly increasing function. Clearly, $\lim_{c\to d^-}F(c)=F(d^-)$ and $\lim_{c\to d^+}F(c)=F(d^+)$ hold for all $d\in(0,+\infty)$. Moreover, for all $x\in(0,+\infty)$, we have

$$F(x^-)\le F(x)\le F(x^+).$$

Let $F\in\mathcal{F}\uparrow$, then, there are two possibilities as follows:
(1) $F(0^+)=\lim_{x\to0^+}F(x)=r\in\mathbb{R}$;
(2) $F(0^+)=\lim_{x\to0^+}F(x)=-\infty$ (for more details, see [1,13–15]).
That is to say, each $F\in\mathcal{F}\uparrow$ satisfies either (1) or (2) (see Proposition 1, Section 8 of [1]). Hence, the second and third conditions for the function F from [6,11,12] are superfluous.

Suppose that $\alpha,\beta,\gamma,\delta,\eta$ are nonnegative real constants for which one of the following conditions is satisfied:

(I) $\alpha + \beta + \gamma + \delta + \eta < 1$;

(II) $\delta < \frac{1}{2}$, $\gamma < 1$, $\alpha + \beta + \gamma + 2\delta = 1$ and $0 < \alpha + \delta + \eta \leq 1$.

It is not hard to verify that (**I**) and (**II**) are incompatible. Indeed, if we take $\alpha = 1$, $\beta = \gamma = \delta = \eta = 0$, then (**II**) does not imply (**I**). Further, if we take $\alpha = \frac{1}{3}$, $\beta = \gamma = \delta = \eta = 0$, then (**I**) does not imply (**II**). Note that if $\alpha = \delta = \frac{1}{3}$, $\beta = \gamma = \eta = 0$, then both (**I**) and (**II**) are satisfied.

In the statement of our results in the sequel we will use the following lemmas.

Lemma 1. *If the nonnegative real numbers $\alpha, \beta, \gamma, \delta$ and η satisfy Condition (**II**), then $\gamma + \delta < 1$.*

Proof. Suppose the opposite, i.e., that $\gamma + \delta \geq 1$, then

$$1 = \alpha + \beta + \gamma + 2\delta \geq \alpha + \beta + 1 + \delta,$$

so $\alpha + \beta + \delta \leq 0$, which follows $\alpha = \beta = \delta = 0$. Thus, by $\alpha + \beta + \gamma + 2\delta = 1$, it means $\gamma = 1$. This leads to a contradiction with $\gamma < 1$. □

Lemma 2. *([5]) Suppose that $\{x_n\}_{n \in \mathbb{N}}$, which belongs to a metric space (X, d) and satisfies $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, is not a Cauchy sequence. Then, there exist $\varepsilon > 0$ and sequences of positive integers $\{n_k\}, \{m_k\}$, $n_k > m_k > k$ such that each of the next sequences*

$$\{d(x_{n_k}, x_{m_k})\},\ \{d(x_{n_k+1}, x_{m_k})\},\ \{d(x_{n_k}, x_{m_k-1})\},$$

$$\{d(x_{n_k+1}, x_{m_k-1})\},\ \{d(x_{n_k+1}, x_{m_k+1})\}$$

tends to ε^+ as $k \to \infty$.

Lemma 3. *Let (X, d) be a metric space and $T : X \to X$ a mapping. If there exists some $n \in \mathbb{N}$ such that T^n has a unique fixed point p in X, then T admits a unique fixed point p in X.*

Proof. By the hypothesis, it is valid that $T^n p = p$, then

$$T^n(Tp) = T^{n+1}p = T(T^n p) = Tp.$$

Hence, Tp is also a fixed point of T^n. Since T^n has a unique fixed point, then $Tp = p$, i.e., p is a fixed point of T.

We will prove the uniqueness of the fixed point of T. Actually, assume that T has another fixed point q, then

$$T^n p = T^{n-1}(Tp) = T^{n-1}p = \cdots = Tp = p,$$
$$T^n q = T^{n-1}(Tq) = T^{n-1}q = \cdots = Tq = q.$$

Accordingly, p and q are fixed points of T^n. By the uniqueness of the fixed point of T^n, it follows that $p = q$. □

2. Main Results

Let (X, d) be a metric space and $T : X \to X$ a mapping. We introduce a Hardy–Rogers type function as follows:

$$H_{\alpha,\beta,\gamma,\delta,\eta}(x, y) = \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + \eta d(y, Tx),$$

where $x, y \in X$, $\alpha, \beta, \gamma, \delta, \eta$ satisfy Condition (**I**) or Condition (**II**).

We give the following theorem.

Theorem 1. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a generalized F-contraction, that is,*

$$\tau + F(d(Tx, Ty)) \leq F(H_{\alpha,\beta,\gamma,\delta,\eta}(x, y)) \tag{1}$$

holds for all $x, y \in X$ with $Tx \neq Ty$, where $\tau > 0$ is a constant and $F \in \mathcal{F}\uparrow$, then T has a unique fixed point $x^* \in X$, and for each $x \in X$, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.

Proof. By (1) and the monotonicity of the function F, we have

$$d(Tx, Ty) < H_{\alpha,\beta,\gamma,\delta,\eta}(x, y). \tag{2}$$

We will first prove the existence of fixed point of T. Indeed, we take an arbitrary point $x_0 \in X$ and form the corresponding Picard sequence: $\{x_n\} = \{Tx_{n-1}\}$, where $n \in \mathbb{N}^*$.

If $x_p = x_{p-1}$ for some $p \in \mathbb{N}^*$, then x_{p-1} is the fixed point of T. Hence, the proof is finished. Now, suppose that $x_n \neq x_{n-1}$ for each $n \in \mathbb{N}^*$. Putting $x = x_{n-1}$, $y = x_n$ in (2), we obtain

$$0 < d(x_n, x_{n+1}) < H_{\alpha,\beta,\gamma,\delta,\eta}(x_{n-1}, x_n), \tag{3}$$

where

$$\begin{aligned}
&H_{\alpha,\beta,\gamma,\delta,\eta}(x_{n-1}, x_n)\\
&= \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1})\\
&\quad + \delta d(x_{n-1}, x_{n+1}) + \eta d(x_n, x_n)\\
&\leq \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1})\\
&\quad + \delta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \eta \cdot 0\\
&= (\alpha + \beta + \delta)d(x_{n-1}, x_n) + (\gamma + \delta)d(x_n, x_{n+1}).
\end{aligned}$$

Further, putting $x = x_n$, $y = x_{n-1}$ in (2), we obtain

$$0 < d(x_{n+1}, x_n) < H_{\alpha,\beta,\gamma,\delta,\eta}(x_n, x_{n-1}), \tag{4}$$

where

$$\begin{aligned}
&H_{\alpha,\beta,\gamma,\delta,\eta}(x_n, x_{n-1})\\
&= \alpha d(x_n, x_{n-1}) + \beta d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n)\\
&\quad + \delta d(x_n, x_n) + \eta d(x_{n-1}, x_{n+1})\\
&\leq \alpha d(x_n, x_{n-1}) + \beta d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n)\\
&\quad + \delta \cdot 0 + \eta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\\
&= (\alpha + \gamma + \eta)d(x_{n-1}, x_n) + (\beta + \eta)d(x_n, x_{n+1}).
\end{aligned}$$

Adding up (3) and (4) yields

$$d(x_n, x_{n+1}) \leq \mu d(x_{n-1}, x_n), \tag{5}$$

where $\mu = \frac{2\alpha+\beta+\gamma+\delta+\eta}{2-(\beta+\gamma+\delta+\eta)}$.

If Condition (**I**) holds, then $\mu \in [0, 1)$. In view of (5), it is valid that

$$d(x_n, x_{n+1}) \leq \mu d(x_{n-1}, x_n) \leq \mu^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq \mu^n d(x_0, x_1).$$

Accordingly, for any $n, m \in \mathbb{N}$, $n < m$, we speculate

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)\\
&\leq \mu^n d(x_0, x_1) + \mu^{n+1} d(x_0, x_1) + \cdots + \mu^{m-1} d(x_0, x_1)\\
&= \mu^n (1 + \mu + \mu^2 + \cdots + \mu^{m-n-1}) d(x_0, x_1)
\end{aligned}$$

$$\leq \mu^n(1+\mu+\mu^2+\cdots)d(x_0,x_1)$$
$$=\frac{\mu^n}{1-\mu}d(x_0,x_1)\to 0 \ \ (n\to\infty),$$

which means that $d(x_n,x_m)\to 0$ as $n,m\to\infty$. Thus, $\{x_n\}$ is a Cauchy sequence.

If Condition (**II**) holds, then by Lemma 1 and $\alpha+\beta+\gamma+2\delta=1$, (3) becomes

$$0<d(x_n,x_{n+1})<\frac{\alpha+\beta+\delta}{1-(\gamma+\delta)}\cdot d(x_{n-1},x_n)=d(x_{n-1},x_n). \tag{6}$$

Then, by (6), the sequence $\{d(x_{n-1},x_n)\}$ has a limit $\ell\geq 0$ as $n\to\infty$. In the following, we will prove $\ell=0$.

As a matter of fact, suppose that $\ell>0$, then (1) becomes

$$\tau+F(d(x_n,x_{n+1}))\leq F((\alpha+\beta+\delta)d(x_{n-1},x_n)+(\gamma+\delta)d(x_n,x_{n+1})). \tag{7}$$

Taking the limit as $n\to\infty$ in (7), we obtain $\tau+F(\ell^+)\leq F(\ell^+)$ which is a contradiction with $\tau>0$. Consequently, we obtain $\ell=0$, i.e., $\lim_{n\to\infty}d(x_{n-1},x_n)=0$.

Under Condition (**II**), if $\{x_n\}$ is not a Cauchy sequence, then putting $x=x_{n_k},y=x_{m_k}$ in (1), we obtain

$$\tau+F(d(x_{n_k+1},x_{m_k+1}))\leq F(H_{\alpha,\beta,\gamma,\delta,\eta}(x_{n_k},x_{m_k})), \tag{8}$$

where

$$\begin{aligned}&H_{\alpha,\beta,\gamma,\delta,\eta}(x_{n_k},x_{m_k})\\&=\alpha d(x_{n_k},x_{m_k})+\beta d(x_{n_k},x_{n_k+1})+\gamma d(x_{m_k},x_{m_k+1})\\&\quad+\delta d(x_{n_k},x_{m_k+1})+\eta d(x_{m_k},x_{n_k+1}).\end{aligned}$$

Then, by Lemma 2, there exist $\varepsilon>0$ and sequences of positive integers $\{n_k\},\{m_k\}$, $n_k>m_k>k$ such that

$$H_{\alpha,\beta,\gamma,\delta,\eta}(x_{n_k},x_{m_k})\to(\alpha+\delta+\eta)\varepsilon^+ \ \ (k\to\infty).$$

Therefore, taking the limit as $k\to\infty$ in (8), we obtain

$$\tau+F(\varepsilon^+)\leq F((\alpha+\delta+\eta)\varepsilon^+)\leq F(\varepsilon^+),$$

which leads to a contradiction with $\tau>0$.

To sum up, under Condition (**I**) or Condition (**II**), we claim that the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X,d) is complete, then there exists $p\in X$ such that $x_n\to p$ as $n\to\infty$. In the sequel, we will prove that $Tp=p$.

Suppose that $Tp\neq p$. Putting $x=x_n,y=p$ in (2), we have

$$d(x_{n+1},Tp)<H_{\alpha,\beta,\gamma,\delta,\eta}(x_n,p), \tag{9}$$

where

$$\begin{aligned}&H_{\alpha,\beta,\gamma,\delta,\eta}(x_n,p)\\&=\alpha d(x_n,p)+\beta d(x_n,x_{n+1})+\gamma d(p,Tp)\\&\quad+\delta d(x_n,Tp)+\eta d(p,x_{n+1})\\&\to(\gamma+\delta)d(p,Tp) \ \ (n\to\infty).\end{aligned}$$

Taking the limit as $n\to\infty$ in (9), we obtain

$$0<d(p,Tp)\leq(\gamma+\delta)d(p,Tp),$$

If Condition (**I**) holds, then

$$\begin{aligned} 0 < d(p, Tp) &\le (\gamma + \delta)d(p, Tp) \\ &\le (\alpha + \beta + \gamma + \delta + \eta)d(p, Tp) \\ &< d(p, Tp), \end{aligned}$$

which is a contradiction.

If Condition (**II**) holds, then by Lemma 1, it is easy to say that

$$0 < d(p, Tp) \le (\gamma + \delta)d(p, Tp) < d(p, Tp).$$

This is a contradiction.

Finally, we will prove the uniqueness of the fixed point of T. To this end, we suppose for absurd that T has another fixed point q in X. That is, $Tq = q$, $q \neq p$. On account of (2), then

$$0 < d(p, q) < \alpha d(p, q) + \beta \cdot 0 + \gamma \cdot 0 + \delta d(p, q) + \eta d(q, p),$$

that is,

$$0 < d(p, q) < (\alpha + \delta + \eta)d(p, q) \le (\alpha + \beta + \gamma + \delta + \eta)d(p, q),$$

which establishes $\alpha + \delta + \eta > 1$ and $\alpha + \beta + \gamma + \delta + \eta > 1$. We obtain a contradiction with Condition (**II**) or Condition (**I**). This means that the fixed point of T is unique. □

Remark 1. *The above theorem shows several things. First, it generalizes and corrects the recent results on a fixed point for a so-called F-contraction (see [7–10]. Secondly, various consequences can be obtained by assuming Condition (**I**) or Condition (**II**) if we choose the special values of parameters $\alpha, \beta, \gamma, \delta$, and η.*

Remark 2. *Putting $\alpha \in [0, 1), \beta = \gamma = \delta = \eta = 0$, we obtain the main results from [6].*

Remark 3. *Putting $\alpha = 1, \beta = \gamma = \delta = \eta = 0$, we obtain Corollary 2 from [16].*

Remark 4. *Let us recall consequence 2 from [16]. Only the strictly increasing function F is assumed and it is proved that the mapping $T : X \to X$ has a unique fixed point if $d(Tx, Ty) > 0$ which implies $\tau + F(d(Tx, Ty)) \le F(d(x, y))$. In the proof, among other things, $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$ was used. Since $d(x_n, x_{n+1}) > 0$, then $\tau + F(d(x_n, x_{n+1})) \le F(d(x_{n-1}, x_n))$. By taking the limit in the obtained inequality as $n \to \infty$, we obtain $\tau + F(0^+) \le F(0^+)$ which is a contradiction if there is not a condition $F(0^+) = -\infty$. So, the assumption that $F(0^+) = -\infty$ is an infinitely mandatory condition for the mentioned results.*

Now, we give the following example to illustrate Theorem 1.

Example 1. *Let $X = [0, 1]$ with $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a complete metric space. Define $T : X \to X$ as*

$$Tx = \frac{1}{4}x^2 + \frac{1}{2}x$$

for all $x \in X$. Define

$$F(t) = \ln t, \qquad t \in (0, +\infty).$$

Then, it is not hard to verify that

$$\ln 2 + F(d(Tx, Ty)) \le F(H_{\alpha,\beta,\gamma,\delta,\eta}(x, y))$$

for all $x, y \in X$, where

$$H_{\alpha,\beta,\gamma,\delta,\eta}(x, y) = \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + \eta d(y, Tx),$$

and $\alpha = \frac{2}{3}, \beta = \frac{1}{6}, \gamma = \frac{1}{8}, \delta = \frac{1}{2}, \eta = \frac{1}{2}$. *Thus, all conditions of Theorem 1 are satisfied. So, T has a unique fixed point 0 in X.*

3. Application

In this section, we will give an application to our main results.

Let X be a Banach space, U a open subset of $\mathbb{R} \times X$, $u_0 = (t_0, x_0) \in U$, $f : U \to X$ a continuous function. The problem is to find a closed interval I such that $t_0 \in I$ and a differentiable function $x : I \to X$ satisfying

$$\begin{cases} x'(t) = f(t, x(t)), \quad t \in I, \\ x(t_0) = x_0. \end{cases} \tag{10}$$

It is easy to see that (10) is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,\mathrm{d}s, \qquad t \in I. \tag{11}$$

Theorem 2. *Assume that the following conditions are satisfied:*

(i) there exists a function $\lambda(t) \in [0, +\infty) \cap L^1((t_0 - a, t_0 + a))$ for some $a > 0$ such that

$$\|f(t, x_1) - f(t, x_2)\|_X \le \lambda(t)\|x_1 - x_2\|_X$$

holds for all $(t, x_1), (t, x_2) \in U$, where $\|\cdot\|_X$ is the norm defined on X;

(ii) there exist a constant $c > 0$ and a closed ball $\overline{B}(u_0, s)$ of U such that $\|f(t, x)\|_X \le c$ for any $(t, x) \in \overline{B}(u_0, s)$, where $\overline{B}(u_0, s)$ is the closure of the ball $B(u_0, s)$.

Then, there exists $\tau_0 > 0$ such that, for each $\tau < \tau_0$, there is a unique solution $x \in C^1(I_\tau, X)$ to (10) with $I_\tau = [t_0 - \tau, t_0 + \tau]$.

Proof. Put

$$r = \min\{a, s\}, \quad \tau_0 = \min\left\{r, \frac{r}{c}\right\}.$$

Let $\tau < \tau_0$ and construct a complete metric space $Y = \overline{B}(x_0, r)$ with the metric d induced by the norm of $C(I_\tau, X)$, where $I_\tau = [t_0 - \tau, t_0 + \tau]$. By virtue of $\tau < r$, if $y \in Y$, then $(t, y(t)) \in \overline{B}(u_0, r) \subset U$ for all $t \in I_\tau$, Hence, for $y \in Y$, define

$$Ty(t) = x_0 + \int_{t_0}^{t} f(s, y(s))\,\mathrm{d}s, \qquad t \in I_\tau$$

and

$$F(z) = \ln z, \qquad z \in (0, +\infty).$$

Then, we can show

$$\ln 2 + F(d(T^n y_1, T^n y_2)) \le F(H_{\alpha,\beta,\gamma,\delta,\eta}(y_1, y_2)), \tag{12}$$

for any $n \in \mathbb{N}$, where

$$\begin{aligned} &H_{\alpha,\beta,\gamma,\delta,\eta}(y_1, y_2) \\ &= \alpha d(y_1, y_2) + \beta d(y_1, Ty_1) + \gamma d(y_2, Ty_2) + \delta d(y_1, Ty_2) + \eta d(y_2, Ty_1), \end{aligned}$$

and $\alpha = \frac{2}{n!}\|\lambda\|^n_{L^1(I_\tau)}, \beta = \gamma = \delta = \eta = 0$.

In fact, it is easy to deduce that (12) is equivalent to the following inequality:

$$d(T^n y_1, T^n y_2) \le \frac{1}{n!}\|\lambda\|^n_{L^1(I_\tau)} d(y_1, y_2),$$

that is,

$$\|T^n y_1 - T^n y_2\|_{C(I_\tau,X)} \leq \frac{1}{n!}\|\lambda\|^n_{L^1(I_\tau)}\|y_1 - y_2\|_{C(I_\tau,X)}. \tag{13}$$

Notice

$$\sup_{t\in I_\tau}\|Ty(t) - x_0\|_X \leq \sup_{t\in I_\tau}\left|\int_{t_0}^t \|f(s,y(s))\|_X \,\mathrm{d}s\right| \leq c\tau \leq r,$$

then T maps Y into Y. In order to finish the proof of (13), by induction on n we need to show that, for every $t \in I_\tau$,

$$\begin{aligned}&\|T^n y_1(t) - T^n y_2(t)\|_X \\ &\leq \frac{1}{n!}\left(\int_{t_0}^t \lambda(s)\,\mathrm{d}s\right)^n \|y_1 - y_2\|_{C(I_\tau,X)}.\end{aligned} \tag{14}$$

For $n = 1$, (14) holds easily. So, assume that (14) is true for $n-1$, $n \geq 2$. Then, taking $t > t_0$ (the argument for $t < t_0$ is analogous), we have

$$\begin{aligned}&\|T^n y_1(t) - T^n y_2(t)\|_X \\ &= \|T(T^{n-1})y_1(t) - T(T^{n-1})y_2(t)\|_X \\ &\leq \int_{t_0}^t \|f(s,T^{n-1}y_1(s)) - f(s,T^{n-1}y_2(s))\|_X\,\mathrm{d}s \\ &\leq \int_{t_0}^t \lambda(s)\|T^{n-1}y_1(s) - T^{n-1}y_2(s)\|_X\,\mathrm{d}s \\ &\leq \frac{1}{(n-1)!}\left[\int_{t_0}^t \lambda(s)\left(\int_{t_0}^s \lambda(u)\,\mathrm{d}u\right)^{n-1}\mathrm{d}s\right]\|y_1 - y_2\|_{C(I_\tau,X)} \\ &= \frac{1}{n!}\left(\int_{t_0}^t \lambda(s)\,\mathrm{d}s\right)^n \|y_1 - y_2\|_{C(I_\tau,X)}.\end{aligned}$$

Accordingly, we have (14), which leads to (13), so we obtain (12). That is to say, (1) of Theorem 1 holds. On account of $\alpha = \dfrac{2}{n!}\|\lambda\|^n_{L^1(I_\tau)}$, then $\lim_{n\to\infty}\alpha = 0$, thus, for a big enough n, $0 < \alpha < 1$. Hence, for a big enough n and by Theorem 1, T^n has a fixed point. Therefore, by Lemma 3, T has a unique fixed point, which is clearly the (unique) solution to the integral Equation (11) and hence to (10). □

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