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# Distributivity between 2-uninorms and Mayor's aggregation operators

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**Abstract**

The issue of distributivity of aggregation operators is crucial for many different areas such as the utility theory and the integral theory. The topic of this paper is distributivity between 2-uninorms and Mayor's aggregation operators. The presented research is an extension and upgrade of the previously obtained results. The full characterization of distributive pairs of operators is given.

*Keywords:* GM aggregation operators, uninorms, t-norms, t-conorms, 2-uninorms, distributivity equations.

## 1 Introduction

The applicability of aggregation operators is recognized in many research areas, both theoretical and practical, from natural sciences and mathematics to economics and social sciences. Due to this great quantity of applications, there exists an increasing interest in a theoretical study of aggregation operators, which can lead to new possibilities in applications. Of special interest is the study of additional properties of aggregation operators, that are usually derived from the process of solving functional equations involving these kind of functions.

The investigation of distributivity equations comes from [1]. Lately, the focus is on t-norms and t-conorms [14], as well as on their generalizations such as uninorms, nullnorms, semi-nullnorms, semi-uninorms, semi-t-operators, uni-nullnorms, 2-uninorms, Mayor's aggregation operators etc (see [3, 4, 5, 6, 10, 11, 13, 20, 22, 23, 25, 26, 28, 29, 30]). Researchers have also been investigating the problem of distributivity on the restricted domain, i.e., the conditional distributivity, because this particular approach ensures a larger variety of solutions (see [9, 16]). The significance of the considered contemporary topic (see [9, 10, 16, 25, 29, 30]) follows not only from the theoretical point of view, but also from its applicability in the utility theory [8, 12] and integral theory [14, 15, 21]. The aim of this paper is to continue the research from [6, 30] towards aggregation operators defined in the sense of G. Mayor which generalize t-norms and t-conorms. On the other hand, since 2-uninorms are generalizations of uninorms and nullonrms, this research also upgrades some results from [11, 28]. Therefore, the main focus of this paper is the study of distributivity equations on the whole domain between Mayor's aggregation operators and special classes of 2-uninorms.

This paper is organized as follows. Section 2 of this paper contains preliminary notions concerning the aggregation operators in general, then the preliminary notions on GM aggregation operators, uninorms, 2-uninorms and the distributivity equations. Distributivity of the Mayor's aggregation operators over 2-uninorms is considered in the third section. Section 4 contains results of distributivity of 2-uninorms over Mayor's aggregation operators. Finally, Section 5 contains some concluding remarks and discussions on the possible future work.

## 2 Preliminaries

As mentioned above, this section presents a short overview of notions essential as a background for this research (see [1, 2, 4, 7, 8, 17, 27, 31]).

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**Definition 2.1.** [8] *A binary aggregation operator* $A$ *on* $[0,1]$ *is a mapping* $A:[0,1]^2 \to [0,1]$, *that is nondecreasing in each variable and for which the following holds*

$$A(0,0)=0 \quad \text{and} \quad A(1,1)=1.$$

Of course, the previous definition is in its binary form and it can be easily extended to an n-ary aggregation operator on an arbitrary nonempty real interval I (bounded or not). The aggregation operators of interest for this research are operators introduced by G. Mayor in [19] that, for the sake of simplicity, will be refereed to as *the GM aggregation operators*, uninorms and 2-uninorms.

## 2.1 GM aggregation operators, uninorms

**Definition 2.2.** [19] *A GM aggregation operator* $G:[0,1]^2 \to [0,1]$ *is a commutative binary aggregation operator that satisfy the following boundary conditions for all* $x \in [0,1]$ :

$$G(0,x)=G(0,1)x \quad \text{and} \quad G(x,1)=(1-G(0,1))x+G(0,1).$$

The following properties of the GM aggregation operators are essential for the further characterizations.

**Theorem 2.3.** [19] *Let* $G$ *be a GM aggregation operator. Then, the following holds:*

(*i*) $G$ *is associative if and only if* $G$ *is a t-norm or t-conorm;*

(*ii*) $G=\min$ *or* $G=\max$ *if and only if* $G(0,1)=0$ *or* $G(0,1)=1$, *and* $G(x,x)=x$ *for all* $x\in[0,1]$*;*

(*iii*) $G$ *is idempotent if and only if* $\min \le G \le \max$.

As seen from the previous theorem, the GM aggregation operators generalize both t-norms and t-conorms.

**Definition 2.4.** [27] *A uninorm* $U:[0,1]^2 \to [0,1]$ *is an aggregation operator that is commutative, associative, and for which there exists a neutral element* $e\in[0,1]$, *that is* $U(x,e)=x$ *for all* $x\in[0,1]$.

If $e=1$, the uninorm $U$ becomes a t-norm, and if $e=0$ it is a t-conorm. With omitting commutativity and associativity from definitions of t-norms and t-conorms, t-seminorms and t-semiconorms are obtained, respectively. For $e\in(0,1)$ the following result holds.

**Theorem 2.5.** [7] *Let* $U$ *be a uninorm with a neutral element* $e\in(0,1)$. *Then there exists a t-norm* $T_U$, *a t-conorm* $S_U$, *and an increasing operator* $C:[0,e)\times(e,1]\cup(e,1]\times[0,e)\to[0,1]$ *that fulfils* $\min\le C\le\max$, *such that* $U$ *is given by*

$$U(x,y)=\begin{cases} eT_U\left(\frac{x}{e},\frac{y}{e}\right) & \text{if } (x,y)\in[0,e]^2,\\ e+(1-e)S_U\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y)\in[e,1]^2,\\ C(x,y) & \text{otherwise.}\end{cases} \tag{1}$$

The t-norm $T_U$ and the t-conorm $S_U$ from the previous theorem are called the underlying t-norm and the underlying t-conorm of $U$. For all uninorms it holds that $U(0,1)\in\{0,1\}$. If $U(0,1)=0$, the uninorm $U$ is a conjunctive one, and, if $U(0,1)=1$, the uninorm $U$ is a disjunctive one. Unfortunately, the previous result gives only the necessary condition for the representation of uninorms. For given t-norm $T_U$, t-conorm $S_U$ and neutral element $e\in(0,1)$ finding the corresponding uninorm $U$ is equivalent to finding an operation $C$ so that (1) holds. In general, it is still an open problem. Under additional assumptions this problem is solved, completely or partially, and thus some well known classes of uninorms are obtained (see [7, 17]). Some of those classes are:

- Uninorms in $\mathcal{U}_{\min}$ ($\mathcal{U}_{\max}$), i.e., uninorms given by minimum (maximum) in $A(e)=[0,e)\times(e,1]\cup(e,1]\times[0,e)$.
- Representable uninorms, i.e., uninorms that have an additive generator.
- Uninorms continuous in the open unit square $(0,1)^2$.
- Uninorms with continuous underlying t-norms and t-conorms.
- Idempotent uninorms, i.e., uninorms that satisfy $U(x,x)=x$ for all $x\in[0,1]$.
- Locally internal uninorms, i.e., uninorms such that $U(x,y)\in\{x,y\}$ for all $(x,y)\in A(e)$.

The first uninorms were considered by Yager and Rybalov in [27], and are idempotent uninorms denoted with $U_e^{min}$ and $U_e^{max}$ from classes $\mathcal{U}_{\min}$, and $\mathcal{U}_{\max}$, respectively. $U_e^{min}$ is max on $[e,1]^2$, and min elsewhere. The reverse situation is for $U_e^{max}$, i.e., min on $[0,e]^2$, and max elsewhere. The only idempotent t-norm and t-conorm are operators minimum and maximum, respectively.

## 2.2 2-uninorms

The notion of 2-uninorms generalizes and unifies both uninorms and nullnorms.

**Definition 2.6.** [2] *Let $F$ be a binary commutative operator on $[0,1]$. Then $\{e,f\}_k$ is called the 2-neutral element of $F$ if $F(e,x)=x$ for all $x \leq k$ and $F(f,x)=x$ for all $x \geq k$, where $0<k<1$, $e \in [0,k]$ and $f \in [k,1]$.*

**Definition 2.7.** [2] *A binary aggregation operator $F$ on $[0,1]$ is a 2-uninorm if it is commutative, associative and has a 2-neutral element $\{e,f\}_k$.*

The class of all 2-uninorms with the fixed parameters $e, f, k$ is denoted by $U_{k(e,f)}$

- Let $F \in U_{k(e,f)}$. If $e=k=f$, then $F$ is a uninorm with a neutral element $e$.
- Let $F \in U_{k(e,f)}$. If $e=0$, and $f=1$, then $F$ is a nullnorm with an annihilator $k$.

**Lemma 2.8.** [2] *Let $F \in \mathrm{U}_{k(e,f)}$. Then the following statements hold:*

*(i) The binary operations $U_1$ and $U_2$ on the interval $[0,1]$ defined by*

$$U_1(x,y)=\frac{F(kx,ky)}{k} \quad \text{and} \quad U_2(x,y)=\frac{F(k+(1-k)x,k+(1-k)y)-k}{1-k},$$

*are uninorms with neutral element $v=\frac{e}{k}$ and $w=\frac{f-k}{1-k}$ respectively.*

*(ii) $F(0,1)$ is the annihilator for $F$ and $F(0,1) \in \{0,k,1\}$.*

*(iii) $F(0,k) \in \{0,k\}$ and $F(k,1) \in \{1,k\}$.*

*(iv) $F(1,k)=F(1,e)$ and $F(0,k)=F(0,f)$.*

In general, complete characterization of 2-uninorms, as well as uninorms, still is an open problem. Some partial results are given in [2, 31]. Inspired by the previous lemma the class of 2-uninorms has been divided into the following five mutually exclusive subclasses [31]:

1. The class of 2-uninorms with $F(0,1)=k$;
2. The class of 2-uninorms with $F(0,1)=0$ and $F(1,k)=k$;
3. The class of 2-uninorms with $F(0,1)=1$ and $F(0,k)=k$;
4. The class of 2-uninorms with $F(0,1)=0$ and $F(1,k)=1$;
5. The class of 2-uninorms with $F(0,1)=1$ and $F(0,k)=0$.

Representations of these five subclasses, under additional continuity assumption, are given by the following theorems.

**Theorem 2.9.** [2] *Let $F \in \mathrm{U}_{k(e,f)}$, where $F(.,1)$ is continuous except at points $x=e$ and $x=f$. Then $F(0,1)=0$ and $F(1,k)=k$, for $0<e \leq k<f \leq 1$, if and only if $F$ is given by*

$$F(x,y)=\begin{cases} kU_1\left(\frac{x}{k},\frac{y}{k}\right) & \text{if } (x,y) \in [0,k]^2, \\ k+(1-k)U_2\left(\frac{x-k}{1-k},\frac{y-k}{1-k}\right) & \text{if } (x,y) \in [k,1]^2, \\ \min(x,y) & \text{if } (x,y) \in (k,1] \times [0,e) \cup [0,e) \times (k,1], \\ k & \text{if } (x,y) \in [k,1] \times [e,k] \cup [e,k] \times [k,1], \end{cases} \tag{2}$$

*where $U_1 \in U_{\min}$ and $U_2 \in U_{\min}$.*

**Theorem 2.10.** [2] *Let $F \in \mathrm{U}_{k(e,f)}$, where $F(.,1)$ is continuous except at the point $x=e$, and $F(.,e)$ is continuous except at the point $x=f$. Then $F(0,1)=0$ and $F(1,k)=1$, for $0<e \leq k \leq f<1$, if and only if $F$ is given by*

$$F(x,y)=\begin{cases} kU_1\left(\frac{x}{k},\frac{y}{k}\right) & \text{if } (x,y) \in [0,k]^2, \\ k+(1-k)U_2\left(\frac{x-k}{1-k},\frac{y-k}{1-k}\right) & \text{if } (x,y) \in [k,1]^2, \\ \min(x,y) & \text{if } (x,y) \in (k,1] \times [0,e) \cup [0,e) \times (k,1], \\ \max(x,y) & \text{if } (x,y) \in (f,1] \times [e,k) \cup [e,k) \times (f,1], \\ k & \text{if } (x,y) \in [k,f] \times [e,k] \cup [e,k] \times [k,f], \end{cases} \tag{3}$$

*where $U_1 \in U_{\min}$ and $U_2 \in U_{\max}$.*

**Theorem 2.11.** [2] *Let $F \in \mathrm{U}_{k(e,f)}$, where $F(.,0)$ is continuous except at points $x = e$ and $x = f$. Then $F(0,1) = 1$ and $F(0,k) = k$, for $0 \le e < k \le f < 1$, if and only if $F$ is given by*

$$F(x,y) = \begin{cases} kU_1\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)U_2\left(\frac{x-k}{1-k}, \frac{y-k}{1-k}\right) & \text{if } (x,y) \in [k,1]^2, \\ \max(x,y) & \text{if } (x,y) \in (f,1] \times [0,k) \cup [0,k) \times (f,1], \\ k & \text{if } (x,y) \in [k,f] \times [0,k] \cup [0,k] \times [k,f], \end{cases} \tag{4}$$

*where $U_1 \in U_{\max}$ and $U_2 \in U_{\max}$.*

**Theorem 2.12.** [2] *Let $F \in \mathrm{U}_{k(e,f)}$, where $F(.,0)$ is continuous except at the point $x = f$, and $F(.,f)$ is continuous except at the point $x = e$. Then $F(0,1) = 1$ and $F(0,k) = 0$, for $0 < e \le k \le f < 1$, if and only if $F$ is given by*

$$F(x,y) = \begin{cases} kU_1\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)U_2\left(\frac{x-k}{1-k}, \frac{y-k}{1-k}\right) & \text{if } (x,y) \in [k,1]^2, \\ \min(x,y) & \text{if } (x,y) \in (k,f] \times [0,e) \cup [0,e) \times (k,f], \\ \max(x,y) & \text{if } (x,y) \in (f,1] \times [0,k) \cup [0,k) \times (f,1], \\ k & \text{if } (x,y) \in [k,f] \times [e,k] \cup [e,k] \times [k,f], \end{cases} \tag{5}$$

*where $U_1 \in U_{\min}$ and $U_2 \in U_{\max}$.*

**Theorem 2.13.** [2] *Let $F \in \mathrm{U}_{k(e,f)}$, where $F(.,0)$ and $F(.,1)$ are continuous except at points $x = e$ and $x = f$ respectively. Then $F(0,1) = k$ for $0 \le e < k < f \le 1$ if and only if $F$ is given by*

$$F(x,y) = \begin{cases} kU_1\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)U_2\left(\frac{x-k}{1-k}, \frac{y-k}{1-k}\right) & \text{if } (x,y) \in [k,1]^2, \\ k & \text{if } (x,y) \in [k,1] \times [0,k] \cup [0,k] \times [k,1], \end{cases} \tag{6}$$

*where $U_1 \in U_{\max}$ and $U_2 \in U_{\min}$.*

Of course, 2-uninorms from the previous theorems are idempotent if the underlying uninorms $U_1$ and $U_2$ are idempotent. For more information on 2-uninorms and the other subclasses of 2-uninorms see [2, 31].

**Remark 2.14.** *2-uninorm from the Theorem 2.13 for $0 < e < f = 1$ is a $T$-uninorm in $U_{\max}$, for $0 = e < f < 1$ is a $S$-uninorm in $U_{\min}$, and for $0 < e < f < 1$ is a bi-uninorm in $U_{\max} \cup U_{\min}$ (see* [18]*). Since the distributivity between GM aggregation operators and S,T,bi-uninorms was investigated in* [13]*, 2-uninorms from the Theorem 2.13 will not be considered further in this paper.*

## 2.3 Distributivity equations

Finally, the functional equations that are called the left and the right distributive law ([1], p. 318) are given by the following definition.

**Definition 2.15.** *Let $F$ and $G$ be some binary operators on $[0,1]$. $F$ is distributive over $G$, if the following two laws hold:*

*(LD) $F$ is left distributive over $G$ ,i.e.,*

$$F(x, G(y,z)) = G(F(x,y), F(x,z)), \quad \text{for all } x,y,z \in [0,1],$$

*and*

*(RD) $F$ is right distributive over $G$ ,i.e.,*

$$F(G(y,z), x) = G(F(y,x), F(z,x)), \quad \text{for all } x,y,z \in [0,1].$$

Of course, for the commutative $F$ (LD) and (RD) coincide.

The following two lemmas offer answers to certain questions regarding distributivity and will be used at a later point.

**Lemma 2.16.** [4] *Let $X \neq \emptyset$, $F : X^2 \to X$. Let $e \in Y$, where $Y \subset X$, be the neutral element for the operator $F$ on $Y$ ($\forall_{x \in Y} F(e,x) = F(x,e) = x$). If the operator $F$ is left or right distributive over some operator $G : X^2 \to X$ that fulfils $G(e,e) = e$, then $G$ is idempotent on $Y$.*

**Lemma 2.17.** [4] *All increasing functions $F : [0,1]^2 \to [0,1]$ are distributive over* $\max$ *and* $\min$.

# 3 Distributivity of Mayor's aggregation operators over 2-uninorms

The first result is of a general form it provides only idempotent solution.

**Proposition 3.1.** *Let $G$ be a GM aggregation operator, and $F$ be a 2-uninorm. If $G$ is distributive over $F$, then $F$ is an idempotent 2-uninorm.*

*Proof.* Let us assume that $G(0,1) = a$. First, if $x \in [0,1]$, and if $y = z = 0$, then, from the (LD) condition, follows

$$ax = G(x,0) = G(x, F(0,0)) = F(G(x,0), G(x,0)) = F(ax, ax).$$

Since $ax \in [0,a]$, the conclusion is that $F$ is idempotent on $[0,a]$.

Now, if $x \in [0,1]$, and if $y = z = 1$, then from the (LD) condition follows

$$a + (1-a)x = G(x,1) = G(x, F(1,1)) = F(G(x,1), G(x,1)) = F(a + (1-a)x, a + (1-a)x).$$

Since $a + (1-a)x \in [a,1]$, the conclusion is that $F$ is idempotent on $[a,1]$. Therefore, $F$ is an idempotent 2-uninorm. $\square$

The following subsections provide more complex analysis of this topic.

## 3.1 $F$ is of the form (2) and $e < k$

**Lemma 3.2.** *Let $G$ be a GM aggregation operator, and $F$ be a 2-uninorm from Theorem 2.9 such that $e < k$. If $G$ is distributive over $F$, then $G(0,1) = 0$.*

*Proof.* Let assume that $G(0,1) = a$. There should be shown that $a = 0$. If $x = 0$, $y = e$, and $z = 1$, then from (LD) condition and the structure of $F$ and $G$ follows

$$ka = G(0,k) = G(0, F(e,1)) = F(G(0,e), G(0,1)) = F(ae, a).$$

Now, the following hold.

- If $a \leq e$, then $ka = \min(ae, a) = ae$, i.e., $a(k-e) = 0$. Since $e < k$, there holds $a = 0$.
- Let $a > e$ and $a < 1$. Then $ea < e < a$ and $ka = \min(ae, a) = ae$, i.e., $a = 0$, which is in contradiction with $a > e$.

Therefore, $a \in \{0,1\}$. Now, there should be shown that $a \neq 1$. Let suppose the opposite, i.e., that $a = 1$. This means that for all $x \in [0,1]$ hold $G(x,0) = x$ and $G(x,1) = 1$. If $x \in (e,k)$, $y = 0$, and $z = 1$, then the (LD) condition provides

$$x = G(x,0) = G(x, F(0,1)) = F(G(x,0), G(x,1)) = F(x,1) = k,$$

which is again contradiction, and $a = 0$ must hold. Therefore, GM aggregation operator $G$ is a commutative t-seminorm. $\square$

**Theorem 3.3.** *Let $G$ be a continuous GM aggregation operator and $F$ be a 2-uninorm from Theorem 2.9 such that $e < k$. $G$ is distributive over $F$ if and only if $F$ is an idempotent 2-uninorm, $G(x,y) = G(x,z)$ for all $x \in [0,e]$, $y,z \in [e,k]$, $G(x,y) = G(x,z)$ for all $x \in [0,e]$, $y,z \in [f,1]$ and $G$ is given by*

$$G(x,y) = \begin{cases} eT^*\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (k-e)T_1\left(\frac{x-e}{k-e}, \frac{y-e}{k-e}\right) & \text{if } (x,y) \in [e,k]^2, \\ k + (f-k)T_2\left(\frac{x-k}{f-k}, \frac{y-k}{f-k}\right) & \text{if } (x,y) \in [k,f]^2, \\ f + (1-f)T_3\left(\frac{x-f}{1-f}, \frac{y-f}{1-f}\right) & \text{if } (x,y) \in [f,1]^2, \\ C(x,y) & \text{if } (x,y) \in [0,e) \times (e,1] \cup (e,1] \times [0,e), \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{7}$$

*where $T^*$ is a continuous, commutative aggregation operator with an annihilator $0$, $T_1$, $T_2$, $T_3$ are continuous commutative t-seminorms, and $C : [0,e) \times (e,1] \cup (e,1] \times [0,e) \to [0,1]$ is a continuous, commutative, increasing operator with a neutral element $1$ (see Figure 1).*

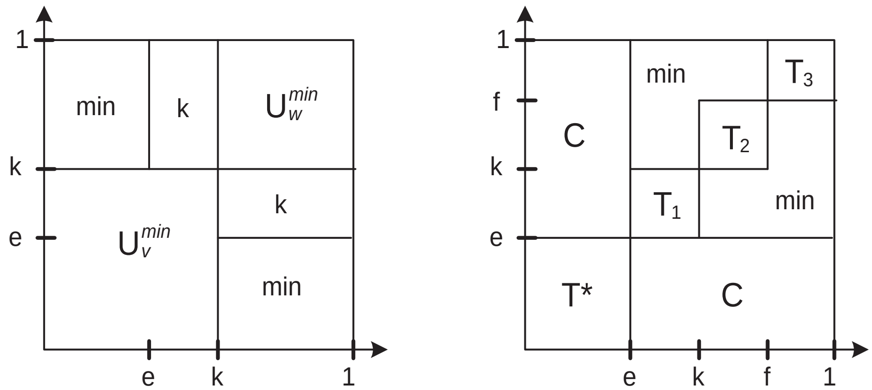


Figure 1. 2-uninorm and GM aggregation operator from Theorem 3.3.

*Proof.* ($\Rightarrow$) Let $G$ be a continuous GM aggregation operator, $F$ be a 2-uninorm from Theorem 2.9 such that $e < k$ and $G$ is distributive over $F$. From Proposition 3.1 follows that $F$ is an idempotent 2-uninorm, and from Lemma 3.2 follows that $G(0,1) = 0$, which means that $G(x,1) = x$, $G(x,0) = 0$ and $G(x,x) \leq x$ for all $x \in [0,1]$.

There should be shown that $G(f,f) = f$. If $x = y = f$, $z = 1$, then the (LD) condition provides the following

$$f = G(f,1) = G(f, F(f,1)) = F(G(f,f), G(f,1)) = F(G(f,f), f).$$

- If $G(f,f) < e$, then $f = F(G(f,f), f) = \min(G(f,f), f) = G(f,f) < e$, which is in a contradiction with $e < f$.
- If $e \leq G(f,f) \leq k$, then $f = F(G(f,f), f) = k$ which is in a contradiction with $k < f$.
- If $G(f,f) > k$, then $f = F(G(f,f), f) = G(f,f)$.

The next step is to show that $G(k,k) = k$

- If $G(k,k) < e$, since $e < G(f,f) = f$, then, because of the continuity of $G$, there exists $x_0 \in (k,f)$ such that $e = G(x_0,x_0)$. If $x = y = x_0$, and $z = 1$, then the (LD) condition provides

  $$e = G(x_0,x_0) = G(x_0, F(x_0,1)) = F(G(x_0,x_0), G(x_0,1)) = F(e,x_0) = k,$$

  which is a contradiction.
- If $G(k,k) = e$, then

  $$e = G(k,k) = G(k, F(k,1)) = F(G(k,k), G(k,1)) = F(e,k) = k,$$

  which is again a contradiction.
- If $G(k,k) > e$, then

  $$G(k,k) = G(k, F(k,1)) = F(G(k,k), G(k,1)) = F(G(k,k), k) = k.$$

Finally, there should be shown that $G(e,e) = e$.

Since $G(e,e) \leq e < k = G(k,k)$, the continuity of $G$ again insures that there exists $x_0 \in [e,k)$ such that $G(x_0,x_0) = e$. The (LD) condition and the structure of $F$ lead to

$$e = G(x_0,x_0) = G(x_0, F(x_0,e)) = F(G(x_0,x_0), G(x_0,e)) = F(e, G(x_0,e)) = G(x_0,e).$$

Therefore,

$$e = G(e,x_0) = G(e, F(x_0,e)) = F(G(e,x_0), G(e,e)) = F(e, G(e,e)) = G(e,e).$$

For $x \geq e$ we have $e = G(e,e) \leq G(x,e) \leq G(1,e) = e$, and thus restriction $G|_{[e,1]^2}$ is isomorphic with a continuous commutative t-seminorm.

Let $x \in [e,1]$, $y = e$, and $z = 1$. Then, the (LD) condition provides

$$G(x,k) = G(x, F(e,1)) = F(G(x,e), G(x,1)) = F(e,x).$$

Now, the structure of 2-uninorm $F$ ensures

$$G(x,k)=\begin{cases} x & \text{for } e\le x\le k,\\ k & \text{for } x\ge k.\end{cases} \tag{8}$$

Therefore, on the square $[e,1]^2$, the GM aggregation operator $G$ is given by

$$G(x,y)=\begin{cases} e+(k-e)T_1\left(\frac{x-e}{k-e},\frac{y-e}{k-e}\right) & \text{if } (x,y)\in[e,k]^2,\\ k+(1-k)T_1'\left(\frac{x-k}{1-k},\frac{y-k}{1-k}\right) & \text{if } (x,y)\in[k,1]^2,\\ \min(x,y) & \text{otherwise,}\end{cases} \tag{9}$$

where $T_1,\ T_1'$ are continuous, commutative t-seminorms.

Now, on the square $[k,1]^2$, Proposition 3.5 from [28] can be applied, and the obtained $T_1'$ is ordinal sum of two continuous commutative t-seminorms $T_2$ and $T_3$.

- Since $G(x,y)\le G(e,e)=e$ for all $(x,y)\in[0,e]^2$ and $G(x,0)=0$ for all $x\in[0,1]$, then the restriction $T^*=G\mid_{[0,e]^2}$ is continuous, commutative aggregation operator with annihilator 0.
- By applying the (LD) condition on $x\in[0,e],\ y=e,\ z=1$, since $G(x,e)\le\min(x,e)=x\le e$, the following is obtained

  $$G(x,k)=G(x,F(e,1))=F(G(x,e),G(x,1))=F(G(x,e),x)=G(x,e).$$

  This implies that for $y\in[e,k]$ holds $G(x,e)=G(x,y)=G(x,k)$.

  Therefore, for $x\in[0,e],\ y,z\in[e,k]$ holds $G(x,y)=G(x,z)$.
- Now, by applying the (LD) condition on $x\in[0,e],\ y\in[f,1],\ z=f$, since $G(x,f)\le G(x,y)\le x\le e$, the following is obtained

  $$G(x,y)=G(x,F(y,f))=F(G(x,y),G(x,f))=\min(G(x,y),G(x,f))=G(x,f).$$

  Thus, for $x\in[0,e]$, and $y,z\in[f,1]$, holds $G(x,y)=G(x,z)$.
- Finally, let us denote $G(x,y)=C(x,y)$ for $(x,y)\in[0,e)\times(e,1]\cup(e,1]\times[0,e)$. From the previous follows that $C$ is a continuous, commutative, increasing operator with a neutral element 1.

Therefore, the GM aggregation operator $G$ is given by (7), such that for $x\in[0,e]$, and $y,z\in[e,k]$, and for all $x\in[0,e]$, and $y,z\in[f,1]$, holds $G(x,y)=G(x,z)$.

($\Leftarrow$) Let $G$ be a GM aggregation operator given by (7) such that for all $x\in[0,e],\ y,z\in[e,k]$ and for all $x\in[0,e],\ y,z\in[f,1]$ holds $G(x,y)=G(x,z)$. Let $F$ be an idempotent 2-uninorm. Without any loss of generality, we can suppose that $y\le z$. To prove the distributive law, the following cases have to be considered.

(I) If $(y,z)\in[0,e]^2$, then from $G(x,y)\le G(x,z)\le\min(x,z)\le e$ and $F(y,z)=\min(y,z)$ the distributivity equation holds by Lemma 2.17.

(II) If $y<e<z$, then $F(y,z)=\min(y,z)=y$.

- If $x\le e$, since $G(x,y)\le G(x,z)$ and $G(x,y)\le\min(x,y)<e$, then

  $$G(x,F(y,z))=G(x,y)=\min(G(x,y),G(x,z))=F(G(x,y),G(x,z)).$$
- If $x>e$, then (LD) is obtained analogously to the previous.

(III) If $(y,z)\in[e,k]^2$, then $F(y,z)=\max(y,z)=z$.

- If $x<e$, then $G(x,y)=G(x,z)$ due to $y,z\in[e,k]$. Since $F$ is idempotent, it follows

  $$G(x,F(y,z))=G(x,z)=G(x,y)=F(G(x,y),G(x,z)).$$
- For $e\le x\le k$, distributivity follows from Lemma 2.17.

- If $x > k$, then, since $G(x,y) = y$ and $G(x,z) = z$, it follows

$$G(x, F(y,z)) = G(x,z) = z = F(y,z) = F(G(x,y), G(x,z)).$$

(IV) If $(y,z) \in [k,f]^2$, then $F(y,z) = \min(y,z) = y$.

- If $x < e$, then, since $G(x,y) \leq G(x,z) \leq \min(x,z) < e$, it follows

$$G(x, F(y,z)) = G(x,y) = \min(G(x,y), G(x,z)) = F(G(x,y), G(x,z)).$$

- If $e \leq x \leq k$, then, since $G(x,y) = G(x,z) = x$ it follows

$$G(x, F(y,z)) = G(x,y) = x = F(x,x) = F(G(x,y), G(x,z)).$$

- For $k < x \leq f$ distributivity follows from Lemma 2.17.
- If $x > f$, then, since $G(x,y) = y$, $G(x,z) = z$ it follows

$$G(x, F(y,z)) = G(x,y) = y = F(y,z) = F(G(x,y), G(x,z)).$$

(V) If $e \leq y \leq k \leq z$, then $F(y,z) = k$.

- If $x \leq e$, then, since $G(x,y) = G(x,k)$ because $y \in [e,k]$, and $G(x,y) \leq G(x,z) \leq \min(x,z) \leq e$, it follows

$$G(x, F(y,z)) = G(x,k) = G(x,y) = \min(G(x,y), G(x,z)) = F(G(x,y), G(x,z)).$$

- If $e < x \leq k$, then, since $G(x,z) = x$, $G(x,k) = x$ and $e \leq G(x,y) \leq k$, it follows

$$G(x, F(y,z)) = G(x,k) = x = \max(G(x,y), x) = F(G(x,y), G(x,z)).$$

- If $x > k$, then, since $e = G(k,e) \leq G(x,y) \leq G(x,k) = k \leq G(x,z)$, it follows

$$G(x, F(y,z)) = G(x,k) = k = F(G(x,y), G(x,z)).$$

(VI) If $(y,z) \in [f,1]^2$, then $F(y,z) = \max(y,z) = z$.

- If $x < e$, then, $G(x,y) = G(x,z)$ due to $y, z \in [f,1]$, and since $F$ is idempotent, it follows

$$G(x, F(y,z)) = G(x,z) = G(x,y) = F(G(x,y), G(x,z)).$$

- If $e \leq x \leq f$, then, since $G(x,y) = G(x,z) = x$, it follows

$$G(x, F(y,z)) = G(x,z) = x = F(x,x) = F(G(x,y), G(x,z)).$$

- For $x > f$, distributivity follows from Lemma 2.17.

(VII) If $k < y < f < z$, then $F(y,z) = \min(y,z) = y$.

- If $x < e$, then, since $G(x,y) \leq G(x,z) \leq x < e$, it follows

$$G(x, F(y,z)) = G(x,y) = \min(G(x,y), G(x,z) = F(G(x,y), G(x,z)).$$

- If $e \leq x \leq k$, then, since $G(x,y) = G(x,z) = x$, it follows

$$G(x, F(y,z)) = G(x,y) = x = F(x,x) = F(G(x,y), G(x,z)).$$

- If $k < x \leq f$, then, since $k \leq G(x,y) \leq f$ and $G(x,z) = x$, it follows

$$G(x, F(y,z)) = G(x,y) = \min(G(x,y), x) = F(G(x,y), G(x,z)).$$

- If $x > f$, then, since $G(x,y) = y$ and $G(x,z) \geq f$, it follows

$$G(x, F(y,z)) = G(x,y) = y = \min(y, G(x,z)) = F(G(x,y), G(x,z)).$$

Thus, $G$ is distributive over $F$. $\square$

### 3.2 $F$ is of the form (2) and $e = k,\ f < 1$

**Theorem 3.4.** *Let $G$ be a GM aggregation operator, and $F$ be a 2-uninorm from Theorem 2.9 such that $e = k$ and $f < 1$. $G$ is distributive over $F$ if and only if $F$ is an idempotent uninorm $U_f^{min}$, and $G$ is a commutative t-seminorm given by*

$$G(x,y) = \begin{cases} fT_1\left(\frac{x}{f}, \frac{y}{f}\right) & \text{if } (x,y) \in [0,f]^2, \\ f + (1-f)T_2\left(\frac{x-f}{1-f}, \frac{y-f}{1-f}\right) & \text{if } (x,y) \in [f,1]^2, \\ \min(x,y) & \text{otherwiswe,} \end{cases} \tag{10}$$

*where $T_1$, $T_2$ are commutative t-seminorms.*

*Proof.* ($\Rightarrow$) Let $G$ be a GM aggregation operator. Let $F$ be a 2-uninorm from Theorem 2.9 such that $e = k,\ f < 1$. Let $G$ be distributive over $F$. From Proposition 3.1 follows that $F$ is an idempotent 2-uninorm, and since $e = k$, the conclusion is that $F = U_f^{min}$. Now, using the same arguments as in Lemma 3.4 from [28], $G(0,1) \in \{0,1\}$ is obtained. If $G(0,1) = 0$, then, according to Proposition 3.5 from [28], $G$ is given by (10). The final step is to show that the option $G(0,1) = 1$ is not possible. Let suppose the opposite, i.e., that $G(0,1) = 1$. For arbitrary $x \in (f,1),\ y = 0,\ z = 1$, from the (LD) condition follows

$$x = G(x,0) = G(x, F(0,1)) = F(G(x,0), G(x,1)) = F(x,1) = 1,$$

which is a contradiction. Therefore, $G$ is a commutative t-seminorm given by (10) and $F = U_f^{min}$.

($\Leftarrow$) Conversely, distributive law holds from Proposition 3.5 from [28]. $\square$

### 3.3 $F$ is of the form (2) and $e = k,\ f = 1$

**Theorem 3.5.** *Let $G$ be a GM aggregation operator. Let $F$ be a 2-uninorm from Theorem 2.9 such that $e = k$ and $f = 1$. $G$ is distributive over $F$ if and only if $F = \min$.*

*Proof.* ($\Rightarrow$) Let $G$ be a GM aggregation operator, let $F$ be a 2-uninorm from Theorem 2.9 such that $e = k,\ f = 1$, and let $G$ be distributive over $F$. From Proposition 3.1 we know that $F$ is an idempotent 2-uninorm, and, since $e = k$ and $f = 1$, the conclusion is that $F = \min$.

($\Leftarrow$) Conversely, the distributive law holds from Lemma 2.17. $\square$

**Remark 3.6.** *The previous issue for $G$ being a continuous t-norm (continuous associative GM aggregation operator such that $G(0,1) = 0$) was considered in* [30] *(see Theorem 3.2).*

- *Theorem 3.3 is a generalization of Theorem 3.2(iii). It provides a new solutions of distributivity equations, and more precise insides of the structure of Mayor's operators. In Theorem 3.2, since $G$ is a continuous t-norm, we know that $G(x,e) = x$ for all $x \in [0,e]$. Therefore, in Theorem 3.2(iii), $T^*$ is a continuous t-norm and operator $C = \min$.*

- *Theorem 3.4 is a generalization of Theorem 3.2 (i). It holds without assumption of continuity for the GM aggregation operator.*

### 3.4 $F$ is of the form (4)

Similarly to the previous, the following three results can be obtained.

**Theorem 3.7.** *Let $G$ be a continuous GM aggregation operator and $F$ be a 2-uninorm from Theorem 2.11 such that $k < f$. $G$ is distributive over $F$ if and only if $F$ is an idempotent 2-uninorm, $G(x,y) = G(x,z)$ for all $x \in [f,1],\ y,z \in [k,f]$, $G(x,y) = G(x,z)$ for all $x \in [f,1],\ y,z \in [0,e]$ and $G$ is given by*

$$G(x,y) = \begin{cases} eS_1\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (k-e)S_2\left(\frac{x-e}{k-e}, \frac{y-e}{k-e}\right) & \text{if } (x,y) \in [e,k]^2, \\ k + (f-k)S_3\left(\frac{x-k}{f-k}, \frac{y-k}{f-k}\right) & \text{if } (x,y) \in [k,f]^2, \\ f + (1-f)S^*\left(\frac{x-f}{1-f}, \frac{y-f}{1-f}\right) & \text{if } (x,y) \in [f,1]^2, \\ D(x,y) & \text{if } (x,y) \in [0,f) \times (f,1] \cup (f,1] \times [0,f), \\ \max(x,y) & \text{otherwise,} \end{cases} \tag{11}$$

*where $S^*$ is a continuous, commutative aggregation operator with an annihilator* 1, $S_1$, $S_2$, $S_3$ *are continuous commutative t-semiconorms, and* $D : [0, f) \times (f, 1] \cup (f, 1] \times [0, f) \to [0, 1]$ *is a continuous, commutative, increasing operator with a neutral element* 0 *(see Figure 2).*

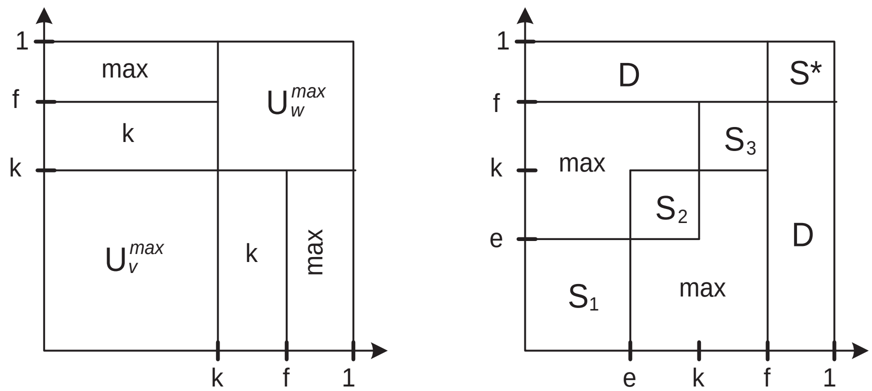


Figure 2. 2-uninorm and GM aggregation operator from Theorem 3.7.

**Theorem 3.8.** *Let $G$ be a GM aggregation operator, and $F$ be a 2-uninorm from Theorem 2.11 such that $f = k$ and $e > 0$. $G$ is distributive over $F$ if and only if $F$ is an idempotent uninorm $U_e^{max}$, and $G$ is a commutative t-semiconorm given by*

$$G(x,y) = \begin{cases} eS_1\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S_2\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \text{otherwiswe,} \end{cases} \tag{12}$$

*where $S_1$, $S_2$ are commutative t-semiconorms.*

**Theorem 3.9.** *Let $G$ be a GM aggregation operator, and $F$ be a 2-uninorm from Theorem 2.11 such that $f = k$ and $e = 0$. $G$ is distributive over $F$ if and only if $F = \max$.*

## 3.5 $F$ is of the form (3) and (5)

The remaining cases is for $F$ being a 2-uninorm from Theorems 2.10 and 2.12.

**Theorem 3.10.** *Let $G$ be a GM aggregation operator, and $F$ be a 2-uninorm from Theorem 2.10. $G$ is distributive over $F$ if and only if $F = U_e^{\min}$ and $G$ is given by*

$$G(x,y) = \begin{cases} eT_1\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)T_2\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwiswe,} \end{cases} \tag{13}$$

*where $T_1$, $T_2$ are commutative t-seminorms.*

*Proof.* ($\Rightarrow$) Let $G$ be a GM aggregation operator, and $F$ be a 2-uninorm from Theorem 2.10 such that $G$ is distributive over $F$. From Proposition 3.1 follows that $F$ is an idempotent 2-uninorm, and, similarly as in Lemma 3.2, there can be proved that $G(0,1) = 0$. As in Theorem 3.3 from [30] there can be proved that $G(e,e) = e$, and that $k = f$. Thus, $F = U_e^{\min}$, and, according to Proposition 3.5 from [28], we know that $G$ is given by (13).

($\Leftarrow$) Conversely, distributive law holds from Proposition 3.5 from [28]. □

Similarly to the previous theorem the next result can be obtained.

**Theorem 3.11.** *Let $G$ be a GM aggregation operator, and $F$ be a 2-uninorm from Theorem 2.12. $G$ is distributive over $F$ if and only if $F = U_f^{\max}$ and $G$ is given by*

$$G(x,y) = \begin{cases} fS_1\left(\frac{x}{f}, \frac{y}{f}\right) & \text{if } (x,y) \in [0,f]^2, \\ f + (1-f)S_2\left(\frac{x-f}{1-f}, \frac{y-f}{1-f}\right) & \text{if } (x,y) \in [f,1]^2, \\ \max(x,y) & \text{otherwiswe,} \end{cases} \tag{14}$$

*where $S_1$, $S_2$ are commutative t-semiconorms.*

**Remark 3.12.** *Paper* [30] *(see Theorem 3.3 and Theorem 4.3 ) addressed this issue for $G$ being a continuous t-norm or a continuous t-conorm.*

*(i) Theorem 3.10 is a generalization of Theorem 3.3 from [30]. It holds without assumption of continuity for the GM aggregation operator.*

*(ii) Theorem 3.11 is a generalization of Theorem 4.3 from [30]. It also holds without assumption of continuity for the GM aggregation operator.*

# 4 Distributivity of 2-uninorms over Mayor's aggregation operators

**Lemma 4.1.** *Let $G$ be a GM aggregation operator such that $G(0,1) = a$. Let $F$ be*

*(i) a 2-uninorm from Theorem 2.9,*

*(ii) a 2-uninorm from Theorem 2.11,*

*(iii) a 2-uninorm from Theorem 2.10,*

*(iv) a 2-uninorm from the Theorem 2.12.*

*If $F$ is distributive over $G$, then $a \in \{0,1\}$.*

*Proof.* (i) Let assume that a 2-uninorm $F$ from Theorem 2.9 is distributive over a GM aggregation operator $G$ such that $G(0,1) = a$. If $x = e,\ y = 0,\ z = 1$, then, from the (LD) condition and the structure of $F$ and $G$ follows

$$F(e,a) = F(e, G(0,1)) = G(F(e,0), F(e,1)) = G(0,k) = ka.$$

Now, the following holds:

- If $a < k$, then $a = ka$, i.e., $a(1-k) = 0$. Therefore, $a = 0$.
- If $a > k$, then $k = ka$, i.e., $k(1-a) = 0$. Therefore, $a = 1$.
- If $k = a$, then $k = k^2$, i.e., $k(k-1) = 0$, which is in a contradiction with $k \in (0,1)$.

(ii) If $F$ is a 2-uninorm from Theorem 2.11, from $x = f,\ y = 0,\ z = 1$, the (LD) condition, and the similar arguments as above, $G(0,1) \in \{0,1\}$ is obtained. Thus $a \in \{0,1\}$.

(iii) Let assume that a 2-uninorm $F$ from Theorem 2.10 is distributive over a GM aggregation operator $G$ such that $G(0,1) = a$. For an arbitrary $x \geq e, y = 0, z = 1$, based on the (LD) condition, the following is obtained

$$F(x,a) = F(x, G(0,1)) = G(F(x,0), F(x,1)) = G(0,1) = a.$$

- If $e \leq a$, then $a = F(1,a) \geq F(1,e) = 1$. Thus $a = 1$.
- If $a < e$, then $G(0,y) \leq G(0,1) = a < e$, for an arbitrary $y$. Now, by taking $y \in (e,1)$, from (LD) condition and the structure of $F$, it is obtained

  $$G(0,y) = F(1, G(0,y)) = G(F(1,0), F(1,y)) = G(0,1) = a.$$

  Thus, $ay = a$, i.e., $a(1-y) = 0$, and therefore $a = 0$.

(iv) If $F$ is a 2-uninorm from the Theorem 2.12, similarly as in (iii) it can be obtained that $G(0,1) \in \{0,1\}$. Again, $a \in \{0,1\}$. □

Therefore, in the sequel of this paper, two cases are distinguished: $G(0,1) = 0$ and $G(0,1) = 1$.

**Theorem 4.2.** *Let $F$ be a 2-uninorm from Theorem 2.9 and $G$ be a GM aggregation operator such that $G(0,1) = 0$. $F$ is distributive over $G$ if and only if $G = \min$.*

*Proof.* ($\Rightarrow$) Let $F$ be a 2-uninorm distributive over a GM aggregation operator $G$ such that $G(0,1)=0$. For that case we know that $G(x,1)=x$ and $G(x,x)\le x$ for all $x\in[0,1]$.

- For $x<e$ holds
$$x=F(x,1)=F(x,G(1,1))=G(F(x,1),F(x,1))=G(x,x).$$
Hence, $G(e,e)\ge G(x,x)=x$ for all $x<e$ which implies $G(e,e)\ge\sup_{x\in[0,e)}G(x,x)$, i.e., $G(e,e)\ge\sup_{x\in[0,e)}x=e$. Therefore, $G(e,e)=e$, and Lemma 2.16 provides that $G$ is idempotent on $[0,k]$.

- For $k<x<f$, by the same arguments as above, $G(f,f)=f$ is obtained, and from Lemma 2.16 follows that $G$ is idempotent on $[k,1]$.

This means that $G$ is an idempotent GM aggregation operator satisfying $G(0,1)=0$, and, according to Theorem 2.3, $G=\min$.

($\Leftarrow$) Conversely, distributive law holds from Lemma 2.17. □

**Theorem 4.3.** *Let $F$ be a 2-uninorm from Theorem 2.9, and $G$ be a GM aggregation operator such that $G(0,1)=1$ and $g(x)=G(x,x)$ is left-continuous at points $x=e$ and $x=f$. $F$ is distributive over $G$ if and only if $G=\max$.*

*Proof.* ($\Rightarrow$)

- As in Theorem 4.2, there can be proven that $G(x,x)=x$ for all $x\in[0,e)$, and for all $x\in(k,f)$.

- Now, the assumption that $G(x,x)$ is left-continuous at points $x=e$ and $x=f$, ensures $G(e,e)=e$ and $G(f,f)=f$. By applying twice Lemma 2.16, $G(x,x)=x$ for all $x\in[0,1]$ is obtained.

This means that $G$ is an idempotent GM aggregation operator satisfying $G(0,1)=1$, and, according to Theorem 2.3, $G=\max$.

($\Leftarrow$) Conversely, distributive law holds from Lemma 2.17. □

**Theorem 4.4.** *Let $F$ be a 2-uninorm from Theorem 2.11 and $G$ be a GM aggregation operator such that $G(0,1)=1$. $F$ is distributive over $G$ if and only if $G=\max$.*

*Proof.* ($\Rightarrow$) Let $F$ be a 2-uninorm distributive over GM aggregation operator $G$ such that $G(0,1)=1$. For this case we know that $G(x,0)=x$ and $G(x,x)\ge x$ for all $x\in[0,1]$.

- For $x>f$ holds
$$x=F(x,0)=F(x,G(0,0))=G(F(x,0),F(x,0))=G(x,x).$$
Hence $G(f,f)\le G(x,x)=x$ for all $x>f$ which implies $G(f,f)\le\inf_{x\in(f,1]}G(x,x)$, i.e., $G(f,f)\le\inf_{x\in(f,1]}x=f$. Therefore, $G(f,f)=f$, and from Lemma 2.16 follows that $G$ is idempotent on $[k,1]$.

- For $e<x<k$, using the same arguments as above, $G(e,e)=e$ is obtained, and from Lemma 2.16 follows that $G$ is idempotent on $[0,k]$.

This means that $G$ is an idempotent GM aggregation operator satisfying $G(0,1)=1$, and, according to Theorem 2.3, $G=\max$.

($\Leftarrow$) Conversely, distributive law holds from Lemma 2.17. □

**Theorem 4.5.** *Let $F$ be a 2-uninorm from Theorem 2.11, and $G$ be a GM aggregation operator such that $G(0,1)=0$, and $g(x)=G(x,x)$ is a right-continuous at points $x=e$ and $x=f$. $F$ is distributive over $G$ if and only if $G=\min$.*

*Proof.* ($\Rightarrow$)

- As in Theorem 4.4, there can be proven that $G(x,x)=x$ for all $x\in(f,1]$, and for all $x\in(e,k)$.

- Now, the assumption that $G(x,x)$ is right-continuous at points $x=e$ and $x=f$, ensures $G(e,e)=e$ and $G(f,f)=f$. By applying twice Lemma 2.16, $G(x,x)=x$ for all $x\in[0,1]$ is obtained.

This means that $G$ is an idempotent GM aggregation operator satisfying $G(0,1)=0$, and, according to Theorem 2.3, $G=\min$.

($\Leftarrow$) Conversely, distributive law holds from Lemma 2.17. □

**Theorem 4.6.** *Let $F$ be a 2-uninorm from Theorem 2.10 and $G$ be a continuous GM aggregation operator. $F$ is distributive over $G$ if and only if $G = \max$ or $G = \min$.*

*Proof.* ($\Rightarrow$) Let $F$ be a 2-uninorm from Theorem 2.10 distributive over a continuous GM aggregation operator $G$. From Lemma 4.1 follows that $G(0,1) \in \{0,1\}$. First, let $G(0,1) = 0$.

- As in Theorem 4.2, there can be proved that $G$ is idempotent on $[0,k]$.
- Now, let suppose that $G(f,f) < f < G(1,1) = 1$. Then, based on the continuity of $G$, there exists $z \in (f,1)$ such that $G(z,z) = f$. For $x \in [k,1]$ the following holds

$$x = F(x,f) = F(x,G(z,z)) = G(F(x,z),F(x,z)).$$

If $x \in (k,f)$, then

$$x = G(\max(x,z),\max(x,z)) = G(z,z) = f,$$

which is a contradiction.

Therefore, $G(f,f) = f$ and $G$ is idempotent on $[k,1]$. Hence, $G$ is an idempotent GM aggregation operator satisfying $G(0,1) = 0$, and, according to Theorem 2.3, $G = \min$.

Now, let $G(0,1) = 1$.

- As in Theorem 4.3, there can be proved that $G$ is idempotent on $[0,k]$.
- Since $G(f,f) \geq f \geq k = G(k,k)$, then by the continuity of $G$ there exists $z \in [k,f]$ such that $G(z,z) = f$. As above, for $x \in [k,1]$ it holds $x = G(F(x,z),F(x,z))$. If $x \in (f,1)$ we obtain

$$x = G(\max(x,z),\max(x,z)) = G(x,x).$$

Due to the continuity of $G$ it follows $G(f,f) = f$

Therefore in this case $G$ is an idempotent GM aggregation operator too, and since $G(0,1) = 1$ holds $G = \max$.

($\Leftarrow$) Conversely, the distributive law holds from Lemma 2.17. ☐

Similarly, the following result can be obtained.

**Theorem 4.7.** *Let $F$ be a 2-uninorm from Theorem 2.12 and $G$ be a continuous GM aggregation operator. $F$ is distributive over $G$ if and only if $G = \max$ or $G = \min$.*

**Remark 4.8.**

- *Theorem 4.1 and Theorem 4.2 from* [6] *are providing results for distributivity of 2-uninorm from the Theorems 2.9-2.12, however only over a continuous t-norm, and over a continuous t-conorm.*
- *Since GM aggregation operators generalize both t-norms and t-conorms, the previous theorems for $G(0,1) = 0$ ($G(0,1) = 1$) also hold for $G$ being a t-norm (t-conorm).*
- *The main difference and advantage of the previous results with respect to Theorem 4.1 and Theorem 4.2 from* [6]*, is that we consider distributivity equations for much wider classes of aggregation operators than t-norms and t-conorms. Also, except Theorem 4.6 and Theorem 4.7, the continuity for GM aggregation operators is not required. Some of the presented results hold without assumption of continuity for operator $G$, and some only with a right (left)-continuity of diagonal section of $G$ in the particular points of the interval* $[0,1]$.

# 5 Conclusion

Distributivity equations on the whole domain for some special classes of 2-uninorms and Mayor's aggregation operators have been studied in this paper. The results presented in Section 3 and Section 4 extend and upgrade the corresponding ones from [6, 30]. Also, some of the presented theorems hold without the assumption of continuity for GM aggregation operators. This again serves to confirm the fact that the distributive law is a rather strong condition, since it simplifies the structure of the inner operator considerably, i.e., it is being reduced to an idempotent 2-uninorm, min or max operator. In the forthcoming work, an analogous study will be done for the other classes of 2-uninorms. Also, since pairs of aggregation operations that are satisfying distributive law play an important role in utility theory (see [8, 12]), investigations will be directed in this research area as well.

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